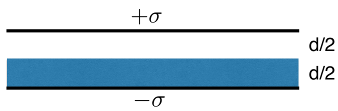
PHYSICS 321:
ELECTROMAGNETISM

10 Dec. 2019 Final exam Solution



1. *Dielectrics and capacitance* Two isolated square, very-large conducting plates, sides of length L and separation d are charged with surface densities $+\sigma > 0$ on the upper plate and $-\sigma$ on the lower plate. A dielectric slab of thickness $d/2$ is then inserted in the bottom half of the region between the plates, as shown in the figure. The permeabilitiy of the dielectric is ϵ. Assume that $d \ll L$.

(a) (6) Write the boundary conditions that $\mathbf{D}$ and $\mathbf{E}$ must satisfy on the boundary between the air and the dielectric.

The normal (up) components of $\mathbf{D}$ are continuous. The tangential components of $\mathbf{E}$ are continuous.

(b) (6) Determine: $\mathbf{E}$ everywhere between the plates.

Both $\mathbf{D}$ and $\mathbf{E}$ point down. $|\mathbf{D}| = \sigma$. Then $|\mathbf{E}| = \frac{\sigma}{\epsilon}$ in the dielectric, and $|\mathbf{E}| = \frac{\sigma}{\epsilon_0}$ in the vacuum.

(c) (5) Determine the bound surface charges on the interface between the dielectric and the vacuum.

The polarization $P = (\epsilon - \epsilon_0)E$ in the dielectric and points up. The bound surface charge density is $\sigma_B = P = (1 - \frac{\epsilon_0}{\epsilon})\sigma$

(d) (5) Determine the capacitance, C, of this system. $C = \frac{Q}{V}$. $V = \frac{\sigma}{\epsilon}d/2 + \frac{\sigma}{\epsilon_0}d/2$. $Q = \sigma L^2$, so $C = \frac{\sigma L^2}{\frac{\sigma}{\epsilon}d/2+\frac{\sigma}{\epsilon_0}d/2} = \frac{2L^2}{d(1/\epsilon+1/\epsilon_0)}$

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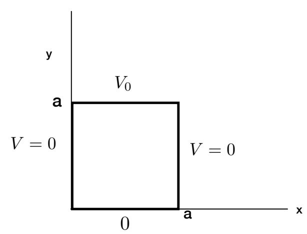
2. *Laplace's equation* Consider the problem of finding the electric potential $V(x, y)$ for two-dimensional situations.

(a) (6) Write ANY non-zero solution of Laplace's equation in two dimensions.

There are many such: $A, Bx, A + Bx, Cy, A + Bx + Cy, A + Cy$ with $A, B, C \neq 0$. Any one of these suffices.

(b) (7) A very, very long square conducting pipe of side length a lies along the z axis. The boundary conditions for $V(x, y)$ are $V(0, y) = 0$, $V(a, y) = 0$, $V(x, 0) = 0\, V(x, a) = V_0$, with V_0 a constant. Write ANY solution of Laplace's equation that satisfies the boundary conditions at $x = 0$ and $x = a$.



Use separation of variables. Must make the function vanish at $x = 0$ and $x = a$. Thus $\sin \frac{n\pi x}{a} e^{\frac{\pm n\pi y}{a}}$, with n an integer or $\sin \frac{n\pi x}{a}(e^{\frac{n\pi y}{a}} + Ae^{\frac{-n\pi y}{a}})$

(c) (9) Find the scalar potential that solves Laplace's equation and satisfies all of the boundary conditions. You may express your answer in terms of *well-defined*, one-dimensional integrals.

The most general solution $V = \sum_{n=1} C_n \sin \frac{n\pi x}{a}(e^{\frac{n\pi y}{a}} + A_n e^{\frac{-n\pi y}{a}})$ T

The boundary condition at $y = 0$ is $0 = \sum_{n=1} C_n \sin \frac{n\pi x}{a}(1 + A_n)$.

The boundary condition at $y = a$ is $V_0 = \sum_{n=1} C_n \sin \frac{n\pi x}{a}(e^{n\pi} + A_n e^{-n\pi})$

Fourier transform (multiply by $\sin \frac{m\pi x}{a}$ and integrate over x from 0 to a.)

This gives $0 = \frac{a}{2} C_m(1 + A_m)$ so $A_m = -1$ and $V_0 \int_0^a dx \sin \frac{m\pi x}{a} = \frac{a}{2}(e^{m\pi} - e^{-m\pi})$. Define $I_m \equiv \int_0^a dx \sin m\pi$. Then $C_m = \frac{2V}{a} \frac{I_m}{e^{m\pi} + e^{-m\pi}}$

SO $V(x, y) = \sum_{m=1} \frac{2V}{a} \frac{I_m}{e^{m\pi} + e^{-m\pi}} \sin \frac{m\pi x}{a}(e^{\frac{n\pi x}{a}} - e^{\frac{-n\pi x}{a}})$

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3. *Azimuthal symmetry* We wish to determine information about an unknown charge distribution of finite extent that is located around an origin. You may assume that the charge distribution obeys azimuthal symmetry. Measurements made outside the charge distribution along the z axis find that the potential is given by $V(z) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{3}\frac{a_n}{z^{n+1}}$, with the coefficients a_n being known (given) values.

(a) (8) Determine the charge of the system.

$Q = a_0$

(b) (7) Suppose that the coefficients $a_1 = 0$, and a_3=0, with a_0, a_2 non-zero. What does this information teach you about the system?

Only Legendre polynomials of even order enter. Thus the potential is the same for a given value of θ and $\pi - \theta$. There is symmetry for reflection about the xy plane.

(c) (7) Suppose instead $a_0 = 0$, and a_2=0, with a_1, a_3 non-zero. Determine the value of the potential along the x-axis.

Here only Legendre polynomials of odd order enter. On the x axis $\theta = \pi/2$ $\cos\pi/2 = 0$ If l is odd $P_l(x = 0) = 0$ and $V = 0$ on the x axis.

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4. *New theory of electrostatics and Gauss's Law* A physicist creates a new theory of electromagnetism in which Poisson's equation is modified to $\nabla^2 V - \frac{1}{L^2}V = -\frac{\rho}{\epsilon_0}$, where L is a length to be determined. For this problem, it is useful to know that in spherical coordinates $\nabla^2 f(r) = \frac{1}{r}\frac{d^2}{dr^2} r f(r)$.

(a) (4) Show that in regions with $\rho = 0$, the function $V(r) = V_0 L \frac{e^{\pm r/L}}{r}$ solves the modified Poisson's equation (to be compared with $V(r) \propto 1/r$ if $L = \infty$.)

$\nabla^2 V(r) = \frac{1}{r}\frac{d^2}{dr^2} r V_0 L \frac{e^{\pm r/L}}{r} = \frac{V_0}{r}\frac{d^2}{dr^2} e^{\frac{\pm r}{L}} = \frac{V_0}{r}\frac{1}{L^2} e^{\frac{\pm r}{L}} = \frac{1}{L^2} V(r)$

Thus $(\nabla^2 - \frac{1}{L^2})V = 0$ and Laplace's (Poisson's with $\rho = 0$) equation is solved.

(b) (4) Experimental searches for a non-infinite value of L use a geometry first employed by Cavendish in which two concentric conducting shells (radii $r_1 < r_2$) are maintained at a common potential V_0 by an infinitely thin conducting wire. If $L = \infty$, the usual electrostatics is applicable and the charge on the inner sphere must vanish. Explain why.

There is spherical symmetry. Any electric field has a radial direction. The potential difference between the two spheres is 0, so the integral of $\mathbf{E} \cdot \mathbf{r}$ along a radius from the inner sphere to the outer sphere vanishes. Thus $E = 0$. Then the integral $\oint d\mathbf{A} \cdot \mathbf{E}$ over a Gaussian sphere just outside the inner sphere vanishes, by Gauss law the enclosed charge vanishes.

(c) (5) Obtain the general form of $V(r)$ in the (charge-free) region between the two spheres.

$V(r) = \frac{1}{r}(Ae^{\frac{-r}{L}} + Be^{\frac{r}{L}})$

(d) (5) Given your expression for $V(r)$ from part(c) find the equation(s) that $V(r)$ must satisfy to satisfy the boundary conditions.

$V(r_1) = \frac{1}{r_1}(Ae^{\frac{-r_1}{L}} + Be^{\frac{r_1}{L}})$

$V(r_2) = \frac{1}{r_2}(Ae^{\frac{-r_2}{L}} + Be^{\frac{r_2}{L}})$

(e) The $V(r)$ of the previous problem leads to an electric field $E(r) \neq 0$, which you may take as given. Would $E(r)$ be consistent with Gauss's law? Explain why or why not.

Gauss's law is not satisfied: $\int_{\mathcal{V}} d^3r \boldsymbol{\nabla} \cdot \mathbf{E} = \int_{\mathcal{S}} d\mathbf{A} \cdot \mathbf{E}$ and using the new Poisson's equation and $\mathbf{E} = -\boldsymbol{\nabla} V$ to get

$\int_{\mathcal{S}} d\mathbf{A} \cdot \mathbf{E} = \int_V d^3r(-\nabla^2 V) = \int d^3r(\frac{\rho}{\epsilon_0} - \frac{V}{L^2}) = Q_{\text{enclosed}} - \frac{1}{L^2}\int d^3r V(r)$. There is an extra term if L is finite and Gauss's law is not satisfied.