NAME: SOLUTIONS **Student ID:** **Score:**

Physics 321 **Autumn 2017**

EXAM # 1
9 AM - 1020 AM, Tuesday October 24

Write your name and ID number at the top of this page and on pages 2-5.

Clearly show all your reasoning.

You are not allowed to use calculators, computers or other programmable devices during the exam.

You are not allowed to use your phone during the exam.

This is a closed-book exam. Textbooks, class notes and other class material are not allowed.

Relevant formulae and equations are provided within this exam (the last 4 pages).

Clearly note all constants and assumptions you use.

Show all your work and your final answers in the spaces provided. If you need to use the back of a page to complete your answer, clearly indicate this. Scratch work will not be graded.

Extra paper is available at the front of the classroom.

If you have a question during the exam, raise your hand.

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1. (25 pts total) *Parallel Plates and Sphere*
Two infinitely large non-conducting parallel plates are placed parallel to the xy-plane, one at $z = +d$, the other at $z = -d$. The plate at $z = +d$ is positively charged with uniform surface charge density σ and the one at $z = -d$ has uniform surface charge density $-\sigma$. Consider the point O located at $(x, y, z) = (0, 0, 0)$ and the point P at $(d, 0, 0)$.
(a) (5 pts) What is the electric field at O and at P?

Electric field at O and P are the same by the symmetry of the parallel plate arrangement. The field from each individual plate points in the $(-\hat{z})$ direction and has magnitude $\frac{\sigma}{2\epsilon_0}$ making the total field $-\frac{\sigma}{\epsilon_0}\hat{z}$ at both O and P.

A non-conducting sphere with radius $d/2$, with uniform volume charge distribution and total charge Q is now positioned centered at O.
(b) (6 pts) What is the new electric field at O and at P?

We can use the superposition principle to add the contributions from the parallel plates and the sphere. Considering *only* the sphere, we can use Gauss' law and spherical symmetry to obtain that the electric field inside the sphere varies linearly with distance from the origin (and is therefore zero at the origin), and the field outside resembles the field of a point charge Q positioned at the origin. Thus, at point O, we have $\mathbf{E}_O = -\frac{\sigma}{\epsilon_0}\hat{z}$. At point P which is outside the sphere, we have, $\mathbf{E}_P = -\frac{\sigma}{\epsilon_0}\hat{z} + \frac{Q}{4\pi\epsilon_0 d^2}\hat{x}$.

Now suppose the sphere was a conductor with the same total charge Q.
(c) (5 pts) What is the new electric field at O? Explain.

If the sphere was a conductor, the charges in the conductor would redistribute themselves such that the electric field inside the conductor was zero. Since the point O lies inside the conductor, the field there would be zero.

(d) (5 pts) Describe in one or two sentences the distribution of charge on the conducting sphere.

The charge inside the conducting sphere is zero and the excess charge Q lies on the outer surface, distributed in such a way as to cancel the electric field along the $-\hat{z}$ from the parallel plates. This also means that the charge distribution is not spherically symmetric.

(e) (4 pts) Is the new electric field at P different from your answer to (b)? Explain your reasoning.

Yes, it is different. We can again use the superposition principle and add the field from parallel plates and the conducting sphere. The charge on the conducting sphere is distributed in such a way as to cancel the electric field inside it from the parallel plates, and is therefore *not* spherically symmetric. Thus the field from the conducting sphere in the region outside it is not that of a point charge Q at the origin, and $\mathbf{E}_P$ is different now than it was in (b). It still points in the $-\hat{z}$ direction but has smaller magnitude.

2. (25 pts total) *Charges on a Plane*
Two positive charges each of magnitude q are fixed on the xy-plane at $(0, 0)$ and $(4, 0)$. For the following questions, do not include "at infinity" as your answer. If there are no such locations, state so explicitly.
(a) (6 pts) Where on the plane can you place a test charge Q such that it feels no force?

We can see that for any point where $y \neq 0$, there will be a non-zero force on Q in the $\hat{y}$ direction and so total force cannot be zero. Thus, any solution has to lie on the $x-$axis. By inspection it must be at the midpoint of the two fixed charges i.e. at $(2, 0)$. To work it out generally, we first note that for $x < 0$ and for $x > 4$, the electric field from both fixed charges point in the same direction, so the force cannot be zero in either of these regions. For $0 < x < 4$, let's assume Q feels no force at the point $(x, 0)$. There is no y component of force from the fixed charges. Setting the x component to zero, we have $\frac{1}{x^2} = \frac{1}{(4-x)^2}$. Simplifying, we have $16 - 8x + x^2 = x^2$, which solves for $x = 2$. Thus test charge Q feels no force at $(2, 0)$.

(b) (7 pts) Now suppose that the fixed charge at $(4, 0)$ is changed to a negative charge $-q$. Where can you place a test charge Q such that it feels no force?

We can see that for any point where $y \neq 0$, there will be a non-zero force on Q in the $\hat{x}$ direction and so total force cannot be zero. Thus any solution has to lie on the $x-$axis at $(x, 0)$. We first note that for $0 < x < 4$, the electric field from both fixed charges point in the same direction, so the force cannot be zero in this region. For $x < 0$, we have $\frac{1}{x^2} = \frac{1}{(4-x)^2}$, which solves for $x = 2$ (from (a)), but we must reject this solution because it is does not satisfy $x < 0$. Similarly for $x > 4$, we have $\frac{1}{x^2} = \frac{1}{(x-4)^2}$, again giving $x = 2$, which again we must reject. Thus, there is no location at which Q feels no force.

(c) (6 pts) Now suppose that the fixed charge at $(4, 0)$ is changed to $-2q$. Where can you place a test charge Q such that it feels no force?

We can use an approach similar to (b). Again we reject the region $0 < x < 4$. For $x < 0$, we have $\frac{1}{x^2} = \frac{2}{(4-x)^2}$ which simplifies to the quadratic equation $16 - 8x + x^2 = 2x^2 \implies x^2 + 8x - 16 = 0$ which gives the two solutions (using quadratic formula) $x = 4(\sqrt{2} - 1)$ and $x = -4(\sqrt{2} + 1)$. Since $x < 0$, we reject the first solution. Thus Q feels no force at the location $(-4(\sqrt{2} + 1), 0)$. For $x > 4$, we have $\frac{1}{x^2} = \frac{2}{(x-4)^2}$ which gives the same solutions $x = 4(\sqrt{2} - 1)$ and $x = -4(\sqrt{2} + 1)$ both of which have to be rejected since neither satisfies $x > 4$. So for this situation, Q feels no force only at the location $(-4(\sqrt{2} + 1), 0)$.

(d) (6 pts) Now suppose that the fixed charge at $(4, 0)$ is changed to $+2q$. Where can you place a test charge Q such that it feels no force?

As in (a), we can see that for any point where $y \neq 0$, there will be a non-zero force on Q in the $\hat{y}$ direction and so total force cannot be zero. Assume the total force is zero at $(x, 0)$. Again we note that for $x < 0$ and for $x > 4$, the electric field from both fixed charges point in the same direction, so the force cannot be zero in either of these regions. For $0 < x < 4$, we have $16 - 8x + x^2 = 2x^2 \implies x^2 + 8x - 16 = 0$ which gives the two solutions (using quadratic formula) $x = 4(\sqrt{2} - 1)$ and $x = -4(\sqrt{2} + 1)$. Since $0 < x < 4$, we reject the second solution. Thus Q feels no force at the location $(4(\sqrt{2} - 1), 0)$.

3. (25 pts total) *Electrostatic Potential*
A grounded spherical conducting shell (inner radius R_1, outer radius R_2) contains an insulating (non-conducting) solid sphere with radius R_1 and uniform volume charge density ρ.
(a) (8 pts) Find the electric fields in the three regions $r < R_1$, $R_1 < r < R_2$, $r > R_2$.

We first note that the problem has spherical symmetry and Gauss' law may be usefully applied to Gaussian spheres concentric with the system. The field everywhere will then have the form $\mathbf{E}(\mathbf{r}) = E(r)\hat{r}$. The field in the conducting material (between R_1 and R_2) is zero. Since there is a uniform volume charge density for $r < R_1$, there miust be charge $-\rho(\frac{4}{3}\pi R_1^3)$ on the inner surface of the conducting shell making the charge enclosed by a Gaussian sphere with radius between R_1 and R_2 equal to zero. Since the conductor is grounded, there is no charge on the outer surface. Thus for a Gaussian sphere with radius larger than R_2, the charge enclosed is again zero and there is no electric field for $r > R_2$. For $r < R_1$, Gauss' Law gives $E(r)4\pi r^2 = \rho(\frac{4}{3}\pi r^3)$ which solves for $E(r) = \frac{\rho r}{3\epsilon_0}$. Putting it all together, we get

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \frac{\rho r}{3\epsilon_0}\hat{r}, & \text{if } r < R_1 \\ 0, & \text{if } R_1 < r < R_2 \\ 0, & \text{if } r > R_2 \end{cases}$$

(b) (10 pts) Find the electrostatic potential at the center. Take the reference point to be at infinity.

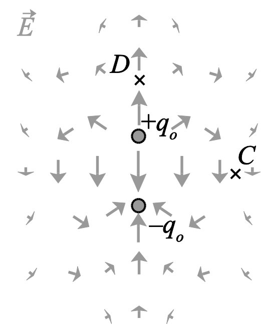
For $r < R_1$, we have the potential $V(r) = -\int_\infty^r \mathbf{E} \cdot \hat{r} dr = -\int_{R_1}^r \frac{\rho r}{3\epsilon_0} dr = \frac{\rho(R_1^2 - r^2)}{6\epsilon_0}$. The potential at the center ($r = 0$) is then $V(0) = \frac{\rho R_1^2}{6\epsilon_0}$

(c) (7 pts) Find the electrostatic energy of the system.

By the work-energy theorem, the electrostatic energy is the work done to assemble the system starting from all charges at infinity. This is given by $\frac{1}{2}\int_{\text{allspace}} \rho V d\tau$. The integrand is zero for $r > R_1$ since V is zero there. Using $V(r)$ from (b), the electrostatic energy is $\frac{1}{2}\int_0^{R_1} \rho \frac{\rho(R_1^2 - r^2)}{6\epsilon_0} 4\pi r^2 dr = \frac{\pi\rho^2}{3\epsilon_0}\{\int_0^{R_1} R_1^2 r^2 dr - \int_0^{R_1} r^4 dr\} = \frac{\pi\rho^2}{3\epsilon_0}\{R_1^5/3 - R_1^5/5\} = \frac{2\pi\rho^2 R_1^5}{45\epsilon_0}$.

IV. [25 points total] This page contains two independent parts, A and B.

A. The electric field of a dipole is shown in the vector field diagram at right. The dipole consists of a single positive point charge above and a single negative point charge below.



Vector field diagram
(not a field line diagram)

i. [5 pts] At point *D,* is the divergence of the electric field *positive, negative,* or *zero?* Explain your reasoning.

Zero.

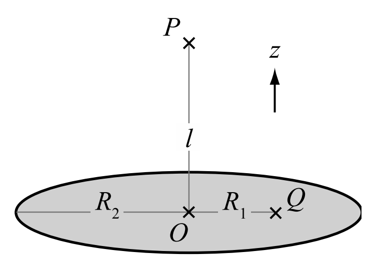
The divergence of the electric field is proportional to the charge density by $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}$*. As there is no charge at point D, there is no divergence of the electric field.*

ii. [5 pts] At point *C,* in what direction is the curl of the electric field? If the curl of the electric field is zero, state so explicitly. Explain your reasoning.

Zero.

A property of electrostatic fields is that the curl of the electrostatic field is always zero.

B. A circular disk of radius R_2 centered at the origin and oriented in the *xy*-plane is charged with a surface charge density of $\sigma(s,\phi,z) = a_o s$. Point *P* is located a distance *l* above the origin along the *z*-axis, at coordinates (0, 0, *l*). Point *Q* is located on the disk a distance R_1 from the origin, at coordinates (R_1, 0, 0).



$\sigma\,(s,\ \varphi,\ z) = a_o s$
Perspective view

i. Consider a small region of the disk around point *Q*.

[8 pts] How much charge is located in that small region? Use an appropriate coordinate system and variables provided above. Show your work and/or explain your reasoning.

$$dq = a_o s^2 ds d\phi$$

To find a charge from a surface density, we need to multiply by an area, as $\sigma = \frac{dq}{dA}$*. The infinitesimal area for a part of a disk is* $s ds d\phi$ *(as one side is* $s d\phi$ *and another side is* ds*).*

[3 pts] What is the distance from the coordinate (R_1, 0, 0) to point *P?* Show your work.

The distance can be seen by making the right triangle POQ. The distance can be found by Pythagorean theorem, so $d = \sqrt{R_1^2 + l^2}$.

The electric potential at a point relative to infinity is $V = kq/d$ for a point charge, where *d* is the distance between the point charge and where the electric potential is measured, and *k* is a constant.

ii. [4 pts] Based on your answers above, write an integral expression for the electric potential (not electric field) at point *P* due to the charged disk, relative to infinity. You do not need to evaluate the integral.

$$V = \int k\, dq/d = \int_0^{2\pi} \int_0^{R_2} \frac{k a_o s^2 ds d\phi}{\sqrt{s^2 + l^2}}$$