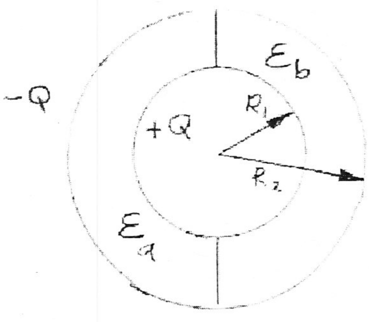
NAME: Student ID: Score:

1. (25 pts total) *Spherical Capacitor with Two Dielectrics*
Consider a spherical capacitor of inner radius R_1 and outer radius R_2 (see figure). The conductors have charge $\pm Q$. The region between R_1 and R_2 is filled with two different linear dielectrics. Half the region has permittivity ϵ_a, the other half has permittivity ϵ_b.

(a) (6 pts) If $\epsilon_a = \epsilon_b$, determine the electric field everywhere.



Gauss' Law for dielectrics & spherical symmetry: $D \cdot 4\pi r^2 = Q$; $D = \epsilon_a E$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \frac{Q}{4\pi\epsilon_a r^2}\hat{r} & R_1 < r < R_2 \\ 0 & r < R_1 \\ 0 & r > R_2 \end{cases}$$

For (b-d), take $\epsilon_a \neq \epsilon_b$.

(b) (8 pts) Assuming that any **E** and **D** fields are purely radial (i.e, no θ or ϕ components), find **E** and **D** in the region between the conductors.

Between conductors, $\vec{E}(\vec{r}) = E(\vec{r})\hat{r}$ (by assumption) $= E(r)\hat{r}$ (since $-\int_{R_2}^{R_1} \vec{E} \cdot d\vec{l}$ is the same, no matter which radial path is taken).

Gauss' Law for dielectrics:

$$D_a 2\pi r^2 + D_b 2\pi r^2 = Q$$

$$\Rightarrow \epsilon_a E 2\pi r^2 + \epsilon_b E 2\pi r^2 = Q$$

$$\Rightarrow E = \frac{Q}{2\pi(\epsilon_a + \epsilon_b) r^2} \quad \text{ie, } \vec{E}(\vec{r}) = \frac{Q}{2\pi(\epsilon_a+\epsilon_b)r^2}\hat{r}$$

on the left: $\vec{D} = \vec{D}_a = \frac{\epsilon_a}{\epsilon_a + \epsilon_b}\frac{Q}{2\pi r^2}\hat{r}$; on right: $\vec{D} = \vec{D}_b = \frac{\epsilon_b}{\epsilon_a+\epsilon_b}\frac{Q}{2\pi r^2}\hat{r}$ $\Big\}$ $R_1 < r < R_2$

(c) (6 pts) What is the polarization in the region between the conductors?

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \epsilon_0 \left[\frac{\epsilon_{a/b}}{\epsilon_0} - 1\right]\vec{E} = (\epsilon_{a/b} - \epsilon_0)\vec{E}$$

On the left: $\vec{P} = \vec{P}_a = \left(\frac{\epsilon_a - \epsilon_0}{\epsilon_a + \epsilon_b}\right)\frac{Q}{2\pi r^2}\hat{r}$

on the right: $\vec{P} = \vec{P}_b = \left(\frac{\epsilon_b - \epsilon_0}{\epsilon_a + \epsilon_b}\right)\frac{Q}{2\pi r^2}\hat{r}$ $\Big\}$ $R_1 < r < R_2$

(d) (5 pts) What are the surface bound charge distributions at R_1 and R_2?

Surface bound charge $\sigma_b = \vec{P} \cdot \hat{n}$

At R_1 ($\hat{n}$ points opposite to $\hat{r}$):
- on the left: $\sigma_b = -\left(\frac{\epsilon_a - \epsilon_0}{\epsilon_a + \epsilon_b}\right)\frac{Q}{2\pi R_1^2}$
- on the right: $\sigma_b = -\left(\frac{\epsilon_b - \epsilon_0}{\epsilon_a + \epsilon_b}\right)\frac{Q}{2\pi R_1^2}$

At R_2 ($\hat{n}$ points in same direction as $\hat{r}$):
- on the left: $\sigma_b = \left(\frac{\epsilon_a - \epsilon_0}{\epsilon_a + \epsilon_b}\right)\frac{Q}{2\pi R_2^2}$
- on the right: $\sigma_b = \left(\frac{\epsilon_b - \epsilon_0}{\epsilon_a + \epsilon_b}\right)\frac{Q}{2\pi R_2^2}$

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2. (25 pts total) *Electrostatic Energy*
Consider a conducting sphere in vacuum of radius R concentric with another conducting sphere whose radius approaches infinity.

(a) (6 pts) What is the capacitance of these concentric conducting spheres?

Put charge $+Q$ on inner sphere and $-Q$ on outer sphere.
Potential difference between conductors: $\Delta V = -\int \vec{E}\cdot d\vec{l}$

$$\Delta V = -\int_{\infty}^{R} \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\frac{1}{R}.$$

Note: Gauss' Law gives $\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$ for $r>R$.
$\Delta \vec{E} = 0$ for $r<R$

Capacitance $C = \dfrac{Q}{\Delta V} = \boxed{4\pi\epsilon_0 R}.$

(b) (6 pts) Suppose the inner sphere carries a charge $+Q$. What is the total electrostatic energy in the surrounding space?

Total electrostatic energy = energy stored in capacitor

$$= \frac{1}{2}\frac{Q^2}{C} = \boxed{\frac{Q^2}{8\pi\epsilon_0 R}}$$

Check: $\frac{1}{2}\epsilon_0 \int E^2\,d\tau = \frac{1}{2}\epsilon_0 \int_R^{\infty} \frac{Q^2}{16\pi^2\epsilon_0^2 r^4} \times 4\pi r^2\,dr = \frac{Q^2}{8\pi\epsilon_0 R}$

(c) (7 pts) Consider a concentric spherical region of space with radius R_0 ($R_0 > R$). What is the value of R_0 if two-thirds of the electrostatic energy lies within it?

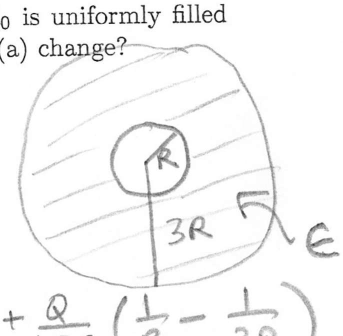
R_0 is defined as:

$$\frac{1}{2}\epsilon_0 \int_R^{R_0} E^2\,d\tau = \frac{2}{3}\times\frac{1}{2}\epsilon_0\int_R^{\infty} E^2\,d\tau.$$

$$\Rightarrow \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{R_0}\right) = \frac{2}{3}\times\frac{Q^2}{8\pi\epsilon_0 R} \Rightarrow \frac{1}{R} - \frac{1}{R_0} = \frac{2}{3}\frac{1}{R} \Rightarrow \frac{1}{R_0} = \frac{1}{3R}$$

$$\Rightarrow \boxed{R_0 = 3R}$$

(d) (6 pts) Now suppose that the spherical-shell region between R and R_0 is uniformly filled with a linear dielectric with permittivity ϵ. How does your answer to part (a) change?



Field inside dielectric is reduced by a factor of ϵ/ϵ_0.
Thus, potential difference between inner and outer conductor (charged $\pm Q$) is:

$$\Delta V = -\int_{\infty}^{3R} \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_{3R}^{R} \frac{Q}{4\pi\epsilon r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\frac{1}{3R} + \frac{Q}{4\pi\epsilon}\left(\frac{1}{R} - \frac{1}{3R}\right)$$

$$\Delta V = \frac{Q}{4\pi R}\left(\frac{1}{3\epsilon_0} + \frac{2}{3\epsilon}\right).$$

$$\Rightarrow \text{Capacitance} = \frac{Q}{\Delta V} = \boxed{\frac{4\pi R}{1/3\epsilon_0 + 2/3\epsilon}}$$

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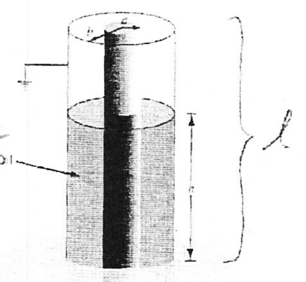
3. (25 pts total) *Dielectrics and Electrostatic Energy*
Two long coaxial conducting cylinders (inner radius a, outer radius b, total height l) stand vertically in a tank of dielectric oil (susceptibility χ_e and mass density ρ). The inner cylinder is maintained at potential V, the outer cylinder is grounded (see figure). The liquid oil column rises between the cylinders to a height h. Assume no fringing fields.



(a) (8 pts) What is the capacitance as a function of h?

Above oil, using cylindrical symmetry:

$E \cdot 2\pi r l = \frac{\lambda l}{\epsilon_0}$ [Gauss' Law; λ = linear charge density]

$\Rightarrow E = \frac{\lambda}{2\pi r \epsilon_0}$ $\Rightarrow$ ΔV between conductors

$= -\frac{\lambda}{2\pi\epsilon_0}\int_b^a \frac{1}{r}\,dr = \frac{\lambda}{2\pi\epsilon_0}\ln(b/a)$

Capacitance above oil $C_1 = \frac{\lambda(l-h)}{\Delta V} = \frac{2\pi\epsilon_0}{\ln(b/a)}(l-h)$.

Similarly, in the region with oil, $C_2 = \frac{2\pi\epsilon_0(1+\chi_e)}{\ln(b/a)}h$.

$\boxed{\text{Capacitance (Q in parallel)} = C_1 + C_2 = \frac{2\pi\epsilon_0}{\ln(b/a)}\{l + \chi_e h\}.}$

(b) (4 pts) What is the downward gravitational force on the liquid column as a function of h?

Downward gravitational force = mass of oil column × g

$= \rho\, \underbrace{\pi(b^2-a^2)h}_{\text{volume}}\, g$

(c) (7 pts) Find the equilibrium height h_{eq} of the oil between the cylinders.

Downward gravitational force balances the upward electrostatic force.

Electrostatic force $= \frac{d}{dh}\left(\frac{1}{2}CV^2\right) = \frac{1}{2}V^2\frac{dC}{dh} = \frac{1}{2}V^2\frac{2\pi\epsilon_0}{\ln(b/a)}\chi_e$

Equilibrium height h_{eq} balances electrostatic force with gravitational one.

$\rho\pi(b^2-a^2)g h_{eq} = \frac{1}{2}V^2\frac{2\pi\epsilon_0}{\ln(b/a)}\chi_e \Rightarrow \boxed{h_{eq} = \frac{V^2\epsilon_0\chi_e}{\{\ln(b/a)\}\rho g(b^2-a^2)}}$

(d) (6 pts) What are the free surface charge densities (σ_f) on the inner cylinder in the regions above and below the oil?

Total charge on inner cylinder $Q = CV = \frac{2\pi\epsilon_0}{\ln(b/a)}(l+\chi_e h)V$

Charge above oil $= Q_1 = C_1V = \frac{2\pi\epsilon_0}{\ln(b/a)}(l-h)V$

σ_f above oil $= \frac{Q_1}{2\pi a(l-h)} = \boxed{\frac{\epsilon_0 V}{a\ln(b/a)}}$

Charge below oil surface $= Q_2 = C_2V = \frac{2\pi\epsilon_0(1+\chi_e)hV}{\ln(b/a)}$

σ_f below oil surface $= \frac{Q_2}{2\pi a h} = \boxed{\frac{\epsilon_0(1+\chi_e)V}{a\ln(b/a)}}$