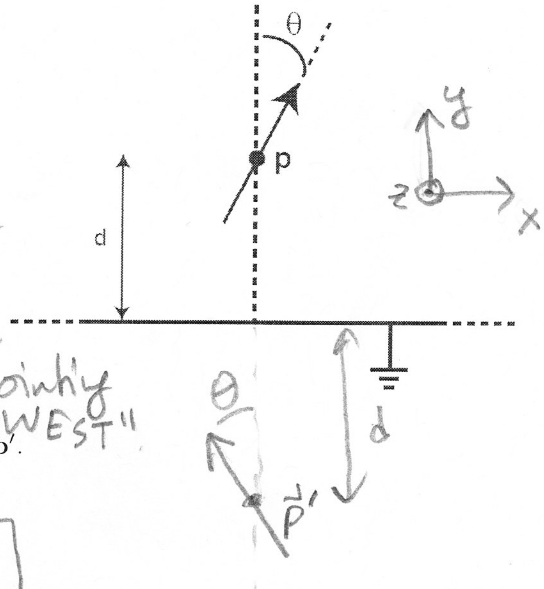
NAME: **Student ID:** **Score:**

1. (25 pts total) *Real and Image Dipoles*
A perfect dipole $\mathbf{p}$ is situated a distance d above an infinite grounded conducting plane (see figure). The dipole makes an angle θ with the perpendicular to the plane.



(a) (7 pts) Describe in words the magnitude, location and angular orientation of the corresponding image dipole $\mathbf{p}'$. (Hint: Consider the images of the positive and negative charges that constitute $\mathbf{p}$)

Image dipole $\vec{p}\,'$ has magnitude $|\vec{p}\,'| = |\vec{p}|$. Located distance d below the plane along perpendicular from $\vec{p}$ to the plane. $\vec{p}\,'$ lies along the plane of the paper oriented at an angle θ to the vertical dashed line pointing "NORTHWEST".

(b) (7 pts) Determine the electric field experienced by $\mathbf{p}$ due to $\mathbf{p}'$.

Field due to image dipole at $\vec{p}$

$$\vec{E} = \frac{|\vec{p}\,'|}{4\pi\epsilon_0 (2d)^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

written in spherical coordinates with image dipole at origin. Using Cartesian coordinates as shown in figure $\hat{r} \to \hat{y}$ & $\hat{\theta} \to \hat{x}$, and $\vec{E} = \frac{p}{4\pi\epsilon_0 (2d)^3}\left[2\cos\theta\,\hat{y} + \sin\theta\,\hat{x}\right]$ where $p = |\vec{p}| = |\vec{p}\,'|$.

(c) (7 pts) Find the torque on $\mathbf{p}$ due to this field.

Torque on $\vec{p}$ due to field: $\vec{N} = \vec{p} \times \vec{E}$

$\vec{p} = p\sin\theta\,\hat{x} + p\cos\theta\,\hat{y}$ and $\vec{N} = \frac{p^2}{4\pi\epsilon_0 (2d)^3}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta & \cos\theta & 0 \\ \sin\theta & 2\cos\theta & 0 \end{vmatrix}$

$$\Rightarrow \vec{N} = \frac{p^2}{4\pi\epsilon_0 (2d)^3}\left[2\sin\theta\cos\theta - \sin\theta\cos\theta\right]\hat{z}$$

$$\vec{N} = \frac{p^2}{8\pi\epsilon_0 (2d)^3}\sin 2\theta\,\hat{z} \qquad [\text{using } \sin 2\theta = 2\sin\theta\cos\theta]$$

(d) (4 pts) If the dipole is free to rotate (but not translate), in what orientation will it come to rest?

If $\vec{p}$ is free to rotate, then it will respond to the torque and rotate in the plane of the paper. For initial $\theta < \pi/2$ as shown, $\vec{N}$ will tend to rotate $\vec{p}$ in a way as to reduce θ. Assuming there is some damping, the dipole will come to rest when $\theta = 0$ (and $\vec{N} = 0$)

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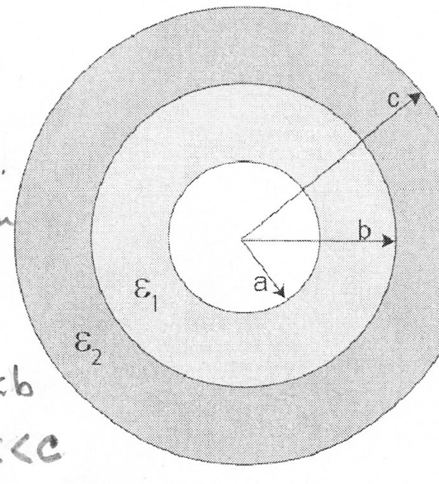
2. (25 pts total) *Coaxial Cable*

A long coaxial cable with cylindrical symmetry consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius c (cross-section shown in figure). The space between is filled from a to b with material of dielectric constant ϵ_1, and from b to c with material of dielectric constant ϵ_2, as shown. (Copper is a conductor.)

(a) (10 pts) What is the capacitance per unit length of this cable?



Put charge per unit length $+\lambda$ on inner conductor and $-\lambda$ on outer conductor.

By symmetry $\vec{D}(\vec{r}) = D(s)\hat{s}$ everywhere.

For $a<s<c$, $\oint \vec{D}\cdot d\vec{A} = Q_{f,enc}$, for Gaussian cylinder length l. Then, $D 2\pi s l = \lambda l$

$\Rightarrow D = \frac{\lambda}{2\pi s}$. By symmetry $\vec{E} = E(s)\hat{s}$ everywhere and $\vec{D} = \epsilon \vec{E} \Rightarrow \vec{E} = \begin{cases} \frac{\lambda \hat{s}}{2\pi\epsilon_0\epsilon_1 s} & a<s<b \\ \frac{\lambda \hat{s}}{2\pi\epsilon_0\epsilon_2 s} & b<s<c \end{cases}$

$\Rightarrow \Delta V = -\int_c^a \vec{E}\cdot d\vec{l} = \frac{\lambda}{2\pi\epsilon_0\epsilon_1}\ln\left(\frac{b}{a}\right) + \frac{\lambda}{2\pi\epsilon_0\epsilon_2}\ln\left(\frac{c}{b}\right)$

$\Rightarrow \frac{C}{l} = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0}{\frac{1}{\epsilon_1}\ln(b/a) + \frac{1}{\epsilon_2}\ln(c/b)}$ = Capacitance per unit length.

(b) (8 pts) Suppose the inner and outer conductors are charged with charge per unit length λ and $-\lambda$ respectively. Determine the polarization in the region between the conductors.

$\vec{P} = \epsilon_0 \chi_e \vec{E} = \epsilon_0(\epsilon_r - 1)\vec{E}$.

For $a<s<b$, $\vec{P} = \frac{\lambda}{2\pi}\left(\frac{\epsilon_1 - 1}{\epsilon_1}\right)\frac{\hat{s}}{s}$

For $b<s<c$, $\vec{P} = \frac{\lambda}{2\pi}\left(\frac{\epsilon_2 - 1}{\epsilon_2}\right)\frac{\hat{s}}{s}$

(c) (7 pts) For the situation described in (b), determine the bound surface charges per unit length at radii a, b, c.

$\sigma_b = \vec{P}\cdot\hat{n}$ Bound surface charge density.

At radius a, $\sigma_b = \frac{\lambda}{2\pi}\left(\frac{\epsilon_1 - 1}{\epsilon_1}\right)\frac{1}{a}\hat{s}\cdot(-\hat{s}) = -\frac{\lambda}{2\pi}\left(\frac{\epsilon_1 - 1}{\epsilon_1}\right)\frac{1}{a}$

Bound surface charge per unit length $= \sigma_b(2\pi a) = -\lambda\left(\frac{\epsilon_1 - 1}{\epsilon_1}\right)$

At radius c, $\sigma_b = \frac{\lambda}{2\pi}\left(\frac{\epsilon_2 - 1}{\epsilon_2}\right)\frac{1}{c}\hat{s}\cdot\hat{s} = \frac{\lambda}{2\pi}\left(\frac{\epsilon_2 - 1}{\epsilon_2}\right)\frac{1}{c}$

Bound surface charge per unit length $= \sigma_b(2\pi c) = \lambda\left(\frac{\epsilon_2 - 1}{\epsilon_2}\right)$

At radius b, $\sigma_b = \frac{\lambda}{2\pi}\left(\frac{\epsilon_1 - 1}{\epsilon_1}\right)\frac{1}{b}\hat{s}\cdot\hat{s} + \frac{\lambda}{2\pi}\left(\frac{\epsilon_2 - 1}{\epsilon_2}\right)\frac{1}{b}\hat{s}\cdot(-\hat{s}) = \frac{\lambda}{2\pi b}\left[\frac{\epsilon_1 - 1}{\epsilon_1} - \frac{\epsilon_2 - 1}{\epsilon_2}\right]$

bound surface charge per unit length $= \sigma_b(2\pi b)$
$= \lambda\left[\frac{\epsilon_1 - 1}{\epsilon_1} - \frac{\epsilon_2 - 1}{\epsilon_2}\right]$

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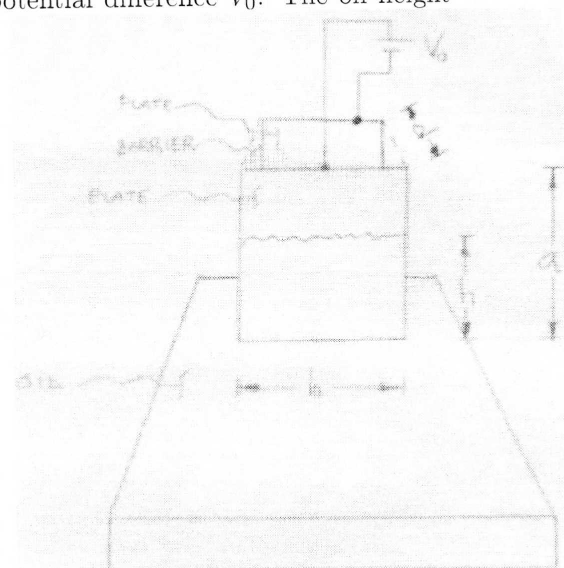
3. (25 pts total) *Dielectrics and Electrostatic Energy*

As shown in the figure, the bottom of an ideal parallel-plate capacitor just contacts the surface of a dielectric oil bath (susceptibility χ_e and mass density ρ). Two sides of the capacitor have an insulating barrier preventing oil from escaping to the sides. The conducting plates have height a, width b, are separated by gap d, and maintained at potential difference V_0. The oil height above the surface is h. Ignore fringing fields.



(a) (7 pts) What is the capacitance as a function of oil height h?

These are two capacitors in parallel configuration.
Total capacitance

$$C = C_1 + C_2$$
$$= \epsilon_0 \frac{(a-h)b}{d} + \frac{\epsilon h b}{d}$$
$$C = \epsilon_0 \frac{(a-h)b}{d} + \frac{\epsilon_0 (1+\chi_e) h b}{d}$$

(b) (5 pts) What is the downward gravitational force on the liquid column as a function of h?

$$F_G = \text{mass} \times g = (\rho h d b) g = \rho h d b g \text{ downwards.}$$

(c) (7 pts) Find the equilibrium height h_{eq} of the oil between the plates.

Electrostatic force upwards $F_E = \frac{1}{2} V_0^2 \frac{dC}{dh}$

$$\frac{dC}{dh} = -\epsilon_0 \frac{b}{d} + \epsilon_0 (1+\chi_e) \frac{b}{d} = \epsilon_0 \chi_e \frac{b}{d} \Rightarrow F_E = \frac{1}{2} V_0^2 \epsilon_0 \chi_e \frac{b}{d}.$$

Equilibrium when upward and downward forces balance.

$$\Rightarrow \rho h_{eq} d b g = \frac{1}{2} V_0^2 \epsilon_0 \chi_e \frac{b}{d} \Rightarrow h_{eq} = \frac{V_0^2 \epsilon_0 \chi_e}{2 \rho g d^2}$$

(d) (6 pts) What are the free surface charge densities on the positively charged plate, above ($\sigma_{f,\text{above}}$) and below ($\sigma_{f,\text{below}}$) the oil?

Total charge on each capacitor $Q_1 = C_1 V_0$ and $Q_2 = C_2 V_0$.

$$\sigma_{f,above} = \frac{C_1 V_0}{(a-h)b} = \frac{\epsilon_0 \frac{(a-h)b}{d} V_0}{(a-h)b} = \underline{\frac{\epsilon_0 V_0}{d} = \sigma_{f,above}}$$

Similarly, $\sigma_{f,below} = \frac{\epsilon_0 (1+\chi_e) h b}{d} \frac{V_0}{hb} = \frac{\epsilon_0 (1+\chi_e) V_0}{d}$

$= \sigma_{f,below}$.

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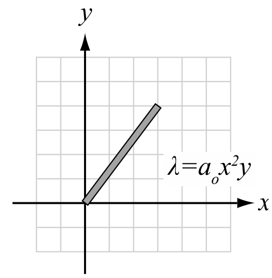
IV. [25 points total] Tutorial question. This page contains two independent parts, A and B.

A. A slanted line charge whose linear charge density is given by $\lambda(x, y) = a_o x^2 y$ is shown at right. The endpoints of the line charge are at the origin and $(x, y) = (3, 4)$.



i. [6 pts] Consider the following *incorrect* student statement about determining the net charge of the system:

"The line charge varies in both x and y, so the monopole moment is $\int_0^3 \int_0^4 a_o x^2 y \, dx \, dy$."

Identify the flaw(s) in the student's reasoning.

The major flaw is that the student has written the integral as $Q_{net} = \iint \lambda dA$ instead of $\int \lambda dl$. The dA integral implies a surface integral, or summing up charges in the rectangle.

ii. [9 pts] Based on your answer above, determine an integral expression for the ***x*-component** of the **dipole moment,** p_x, for the charge distribution above. You do not need to evaluate the integral expression, but it should be in a form that a mathematical program can evaluate.

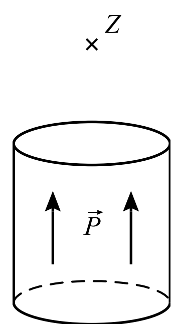
Be sure to explain your reasoning and/or show your work.

The integral for the dipole moment has the form $\vec{p} = \int dq * \vec{r}$, so finding the x-component looks like $p_x = \int dq * x = \int \lambda * dl * x$ which is most easily expressed as

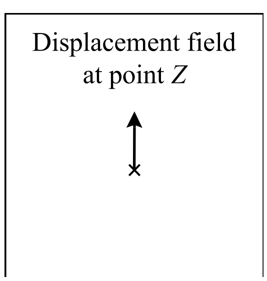
$$\int a_o x^2 y * dl * x = \int_0^3 a_o x^2 (\frac{4}{3}x) \; * (\frac{5}{3}dx) * x = \int_0^3 \frac{20}{9} a_o x^4 dx$$

(Partial credit will be very liberal.)

B. A cylinder with fixed (frozen-in) uniform polarization $\vec{P}$ is shown at right. Point Z is located directly above the cylinder, along the central axis. There are no free charges or an electric field from any other sources.



[10 pts] In what direction is the displacement field at point Z? Sketch an arrow in the box below. Explain your reasoning.



There are two main ways to determine this: through divergence and curl of $\vec{D}$ directly, or through $\vec{E}$.
By looking at the sources of $\vec{D}$, there is no free charge, so there is no divergence. However, there is curl around the side surface of the cylinder (counterclockwise from the top) because $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$. Then by using a right hand rule for curl, you can determine that $\vec{D}$ points up at point Z.

By looking at the sources of $\vec{E}$, the curl is zero in electrostatics. However, the divergence is from the free and bound charges, and there are positive bound charges on the top surface, and negative bound surfaces from the bottom surface, from $\vec{\nabla} \cdot \vec{P} = -\rho_{\text{bound}}$. By superimposing the charges' field at point Z, the net electric field points upward. Now with $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ with $\vec{P}$ being zero, $\vec{D}$ must also point up.

You can also determine $\vec{E}$ from saying that $\vec{P}$ is the dipole density, so the net dipole of the object points upward. From the dipole moment, you can determine the direction of $\vec{E}$ as upward, and the logic follows as above with $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$.