Matrix Rank Minimization with Applications

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April 17, 2001

Outline

1. **Problem statement and motivation**

2. Existing approaches

3. New approaches and results:

   3.1. Trace and log-det heuristics for PSD matrices
   3.2. Semidefinite embedding
   3.3. Trace and log-det heuristics for general matrices

4. Applications

Problem statement

Rank Minimization Problem (RMP):

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, X \\ \text{subject to} & X \in \mathcal{C}, \end{array}$$

where $X \in \mathbf{R}^{m \times n}$ is the variable, $\mathcal{C}$ is a convex set

- a difficult **non-convex** problem (NP-hard)
- arises in many application areas
- in practice means finding **simpler** systems, models with fewer parameters, **lower order** or **less complexity** ... (Occam's razor)

this talk: new methods to (approximately) solve RMP

Example 1: constrained factor analysis

goal: find lowest rank covariance matrix close to measured (noisy) $\hat{\Sigma}$, satisfying prior info

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\,\Sigma \\ \text{subject to} & \|\Sigma - \hat{\Sigma}\| \leq \epsilon \\ & \Sigma \succeq 0 \\ & \Sigma \in \mathcal{C}, \end{array}$$

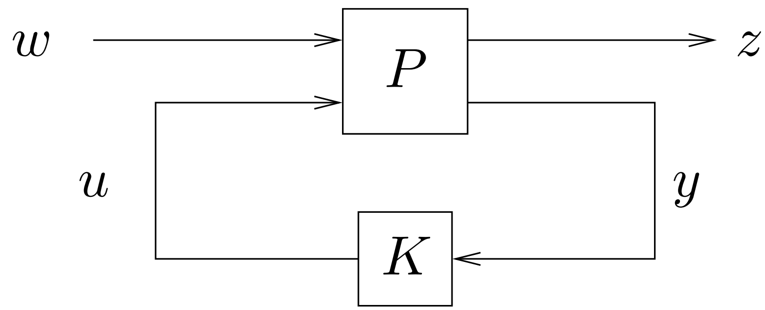
where $\Sigma \in \mathbf{R}^{n \times n}$, $\hat{\Sigma}$ is measured covariance, ϵ is tolerance, $\mathcal{C}$ denotes assumptions or prior information on Σ

$\mathbf{Rank}\,\Sigma$ corresponds to the number of factors that explain Σ

without last constraint, can solve by eigenvalue decomposition (EV); but in general, a difficult problem

Example 2: reduced order controller design

goal: given system P, design stabilizing controller K with lowest possible order to achieve specified performance



in practical implementation, lower order controller means savings in computation time and memory usage

problem can be expressed as RMP (details later)

Special case: vector sparsity problems

suppose $X = \mathbf{diag}(x) = \begin{bmatrix} x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_n \end{bmatrix}, \quad x \in \mathbf{R}^n$,

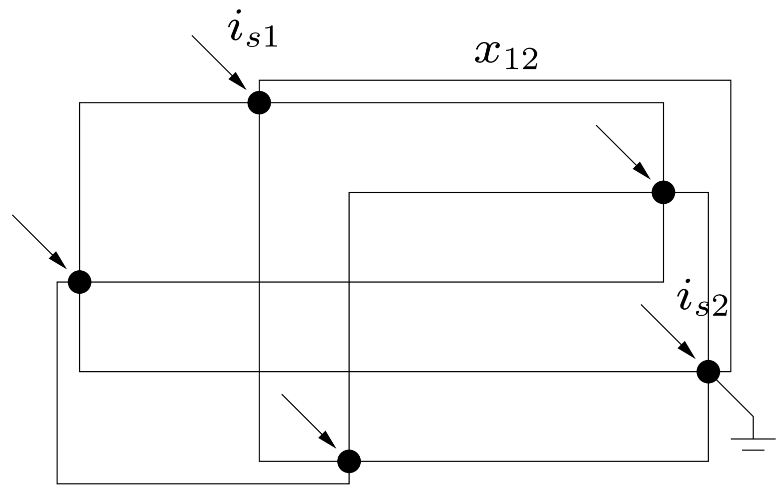
then $\mathbf{Rank}\, X =$ number of non-zero entries in x,

RMP reduces to finding the **sparsest** vector in $\mathcal{C}$:

$$\begin{array}{ll} \text{minimize} & (\#\text{ of non-zero } x_i\text{'s}) \\ \text{subject to} & x \in \mathcal{C}. \end{array}$$

in practice means finding simple models, efficient designs, sparse signal representations; detecting sparse signals in noise, ... (same theme as before)

Example: sparse power-ground mesh design



- n nodes, some grounded, connected by full graph of wires,
  i_{sk}= current injected to node k

- choose wire widths x_{ij} between nodes i & j subject to convex constraint on x ($e.g.$, $0 \le x_{ij} \le W_{\max}$, low power dissipation)

$x_{ij} = 0$ means no wire between i & j, thus **sparse** mesh

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Exact methods

- special cases with analytical solutions ($e.g.$, via SVD, EV)

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, X \\ \text{subject to} & \|X - A\|_{2,F} \leq b \end{array}$$

  - solution: sum of r largest terms in SVD of A, r smallest number s.t. $\sigma_{r+1} \leq b$
  - applications: factor analysis, MDS, array signal processing, sub-space based methods, ...

- special cases that can be reduced to convex problems ($e.g.$, [Mesbahi '97])

- general case: global optimization (branch and bound)
  - impractical for problem sizes of interest

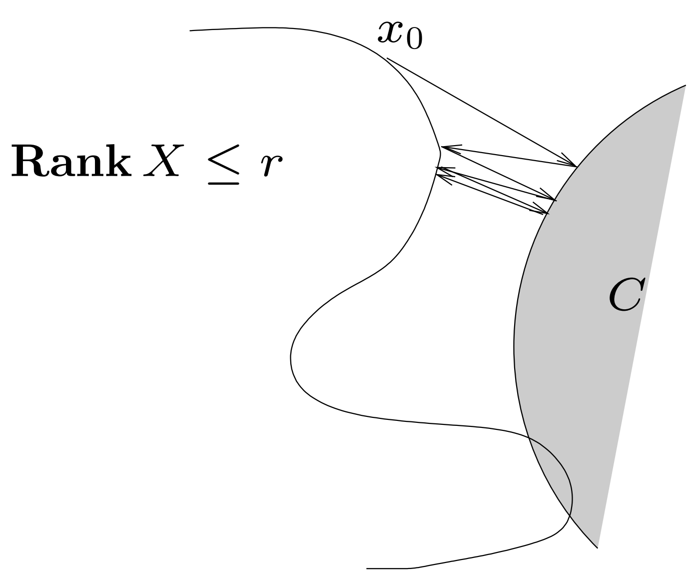
Heuristic methods

- Factorization methods ($e.g.$, [Iwasaki '99])

  idea: $\mathbf{Rank}\, X \le r$ iff $\exists\, F, G$ s.t. $X = FG$, where $F \in \mathbf{R}^{m\times r}$, $G \in \mathbf{R}^{r\times n}$

- Alternating projections [Grigoriadis & Beran '00]

  idea: project alternately onto set of matrices of fixed rank, and constraints set



- Analytic anti-centering/potential reduction [David '94]

  low rank solutions are expected to lie on the boundary of feasible set. Newton algorithm (in interior point methods) follows a path from the boundary to the analytic center.

  idea: follow the Newton path backwards from analytic center to the boundary

Comments on methods

- factorization methods:
  - require user-supplied initial point—non-trivial
  - in our experience, perform worse than other methods

- alternating projections:
  - require user-supplied initial point—non-trivial
  - typically require a large number of iterations (100s– sometimes 1000s), thus computational cost very high if iterations cannot be done analytically

- analytic anti-centering/potential reduction:
  - requires user-supplied initial point (& and is highly sensitive to it)
  - applicable only to positive semidefinite (PSD) matrices
  - complicated implementation

Our goal

Provide methods to (approximately) solve RMP that

- are **general:** handle any matrix, any convex constraint
- do not require user to supply an initial point
- are numerically **efficient** ($e.g.$, based on solving convex problems)
- have supporting **theory**
- are **effective** in practice

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Trace heuristic for PSD matrices

observation: for $X = X^T \geq 0$, minimizing trace tends to give low-rank solutions in practice [Mesbahi '97, Pare '00]

suggests the following:

RMP:	Trace heuristic:
$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, X \\ \text{subject to} & X \in \mathcal{C} \end{array}$	$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, X \\ \text{subject to} & X \in \mathcal{C} \end{array}$

- simple yet **effective** in practice
- **convex** problem, hence efficiently solved, no initial point needed
- if feasible set bounded, provides **lower bound** on objective
- variation: weighted trace minimization

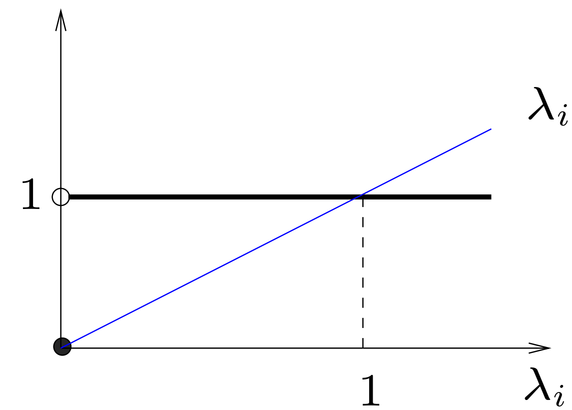
$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, W X \\ \text{subject to} & X \in \mathcal{C}, \end{array}$$

where $W = W^T > 0$.

intuition: replace $\mathbf{Rank}\, X$ with convex function $\mathbf{Tr}\, X$

- note $\mathbf{Rank}\, X =$ (# of non-zero λ_i's), $\quad \mathbf{Tr}\, X = \sum_i \lambda_i$

  where λ_is are eigenvalues of X.



- if $\lambda_i \leq 1$, $\mathbf{Tr}\, X \leq \mathbf{Rank}\, X$.

method applicable only when $X = X^T \geq 0$

Log-det heuristic for PSD matrices

suggested heuristic for $X = X^T \geq 0$: [Fazel, Hindi, Boyd '01]
($0 < \delta \ll 1$ small constant for regularization)

RMP:	Log-det heuristic:
minimize $\mathbf{Rank}\, X$	minimize $\log\det(X + \delta I)$
subject to $X \in \mathcal{C}$	subject to $X \in \mathcal{C}$

- objective is **non-convex** (in fact, concave)
- can use any local optimization method to find a local minimum; in particular, can iteratively linearize and solve linearized (convex) problem at each step.

yields iterative method:

$$X_{k+1} = \underset{X \in \mathcal{C}}{\operatorname{argmin}}\, \mathbf{Tr}(X_k + \delta I)^{-1} X$$

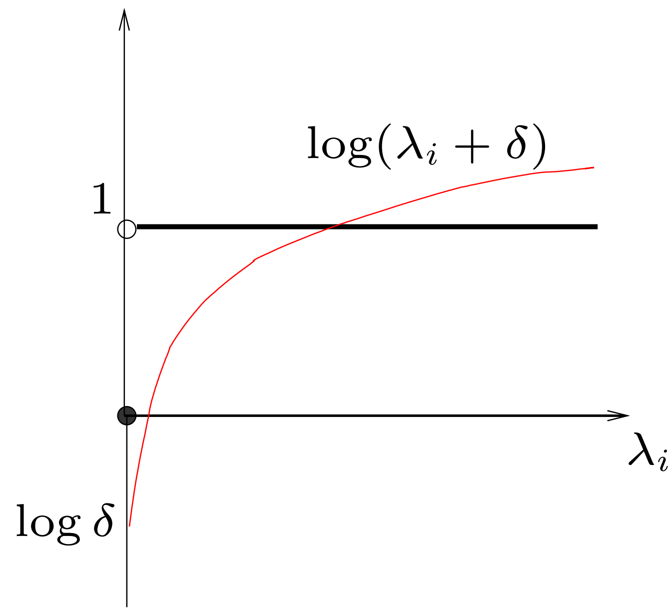
$i.e.$, iterative **weighted trace** minimization:

$$\begin{aligned} X_{k+1} &= \operatorname*{argmin}_{X \in \mathcal{C}} \mathbf{Tr}\, W_k X \\ W_k &= (X_k + \delta I)^{-1}. \end{aligned}$$

- each iteration a **convex** problem, hence efficiently solved
- can show iterations converge to a local minimum of $\log\det(X + \delta I)$ (using concavity of $\log\det(X + \delta I)$ in X)
- if $X_0 = I$, first iteration same as trace heuristic (thus, iterations refine the result of trace heuristic)
- in practice, **very few** iterations needed (about 5 or 6)

intuition: note $\mathbf{Rank}\, X =$ (# of non-zero λ_i's),

$$\log\det(X + \delta I) = \log \prod_i (\lambda_i + \delta) = \sum_i \log(\lambda_i + \delta)$$



again, method applicable only when $X = X^T \geq 0$

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Semidefinite embedding

question: can we extend last two heuristics to general matrices?

Lemma 1. [Fazel, Hindi, Boyd '01] *Let* $X \in \mathbf{R}^{m\times n}$*. Then* $\mathbf{Rank}\, X \leq r$ *iff* $\exists\ Y = Y^T \in \mathbf{R}^{m\times m}$ *and* $Z = Z^T \in \mathbf{R}^{n\times n}$ *s.t.*

$$\mathbf{Rank} \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \leq 2r, \quad \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \succeq 0.$$

conclusion: can relate the rank of a general matrix to that of a PSD (symmetric) matrix

thus, can **embed** general rank problem in a larger one involving only PSD matrices

Proof: ($\Rightarrow$) suppose $\mathbf{Rank}\, X = r_0 \leq r$. then $X = L\, R$, with $L \in \mathbf{R}^{m \times r_0}$, $R \in \mathbf{R}^{r_0 \times n}$, and $\mathbf{Rank}\, L = \mathbf{Rank}\, R = r_0$.

setting $Y = L\, L^T$ and $Z = R^T\, R$ satisfies conditions in lemma, since

$$\begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} = \begin{bmatrix} L \\ R^T \end{bmatrix} \begin{bmatrix} L^T & R \end{bmatrix} \geq 0.$$

($\Leftarrow$) w.l.o.g, suppose $\mathbf{Rank}\, Y \leq \mathbf{Rank}\, Z$. by Schur complements

$$\begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} \text{(i)} & Y \geq 0 \\ \text{(ii)} & X^T(I - YY^\dagger) = 0 \\ \text{(iii)} & Z - X^T Y^\dagger X \geq 0 \end{cases}.$$

from (ii), and using $\mathbf{Rank}\, X = n - \dim \mathcal{N}(X) = m - \dim \mathcal{N}(X^T)$ follows

$$\begin{aligned} \mathcal{N}(X^T) \supseteq \mathcal{N}(Y) \quad & \Rightarrow \quad \dim \mathcal{N}(X^T) \geq \dim \mathcal{N}(Y), \\ & \Rightarrow \quad \mathbf{Rank}\, Y \geq \mathbf{Rank}\, X^T = \mathbf{Rank}\, X, \end{aligned}$$

thus, $2\,\mathbf{Rank}\, X \leq \mathbf{Rank}\, Y + \mathbf{Rank}\, Z$, or $\mathbf{Rank}\, X \leq r$. $\square$

Equivalent PSD form of RMP

recall RMP

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, X \\ \text{subject to} & X \in \mathcal{C}, \end{array}$$

via embedding lemma, is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank} \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \geq 0 \\ & X \in \mathcal{C}, \end{array}$$

with variables $X \in \mathbf{R}^{m \times n}$, $Y \in \mathbf{R}^{m \times m}$ and $Z \in \mathbf{R}^{n \times n}$.

- X is **general**; Y and Z are symmetric PSD
- any method for minimizing rank of PSD matrices applies
- can exploit structure of X ($e.g.$, block-diagonal, symmetric) to reduce number of variables

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Trace heuristic for general matrices

applying trace heuristic to PSD form of RMP yields

$$
\begin{array}{ll}
\text{minimize} & \mathbf{Tr}\begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} \\
\text{subject to} & \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \geq 0 \\
& X \in \mathcal{C},
\end{array}
$$

can show is equivalent to

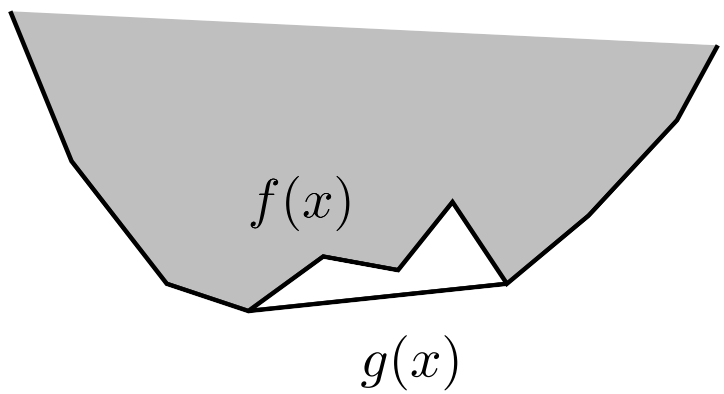
$$
\begin{array}{ll}
\text{minimize} & \|X\|_* \\
\text{subject to} & X \in \mathcal{C},
\end{array}
$$

where $\|X\|_* = \sum_i \sigma_i(X)$, called **nuclear norm** of X, is dual of the spectral (or maximum singular value) norm.

question: how is $\|X\|_*$ related to $\mathbf{Rank}\, X$?

Convex envelope

definition: convex envelope of $f : \mathcal{C} \to \mathbf{R}$, $\mathcal{C} \subseteq \mathbf{R}^n$, is largest convex function g s.t. $g(x) \le f(x)$ for all $x \in \mathcal{C}$.



- 'best' convex lower approximation
- epigraph of g is convex hull of epigraph of f

Convex envelope of rank

Lemma 2. [Fazel, Hindi, Boyd '01] $\|X\|_*$ *is the convex envelope of* $\mathbf{Rank}\, X$ *on the set* $\{X \in \mathbf{R}^{m \times n} \mid \|X\| \leq 1\}$.

conclusions:

- trace heuristic minimizes the **convex envelope** of rank ($i.e.$, the *best* convex approximation to rank) over a bounded set
- if feasible set bounded, heuristic provides lower bound on objective (can be incorporated in a branch and bound method)

lemma provides theoretical support for use of trace heuristic

Special case: ℓ_1 heuristic for vectors

suppose $X = \mathbf{diag}(x)$, $x \in \mathbf{R}^n$, applying trace heuristic yields

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & x \in \mathcal{C}. \end{array}$$

- well-known ℓ_1 heuristic for finding sparse solutions
- used in LASSO methods in statistics, signal decomposition by basis pursuit, . . .
- $\|x\|_1$ is convex envelope of '# of non-zero entries of x' over $\{x \mid \|x\|_\infty \leq 1\}$
- thus, trace heuristic can be considered an extension of ℓ_1 heuristic to matrix case

Log-det heuristic for general matrices

applying log-det heuristic to PSD form of RMP yields

$$\begin{array}{ll}\text{minimize} & \log\det\left(\begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix} + \delta I\right) \\ \text{subject to} & \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \geq 0 \\ & X \in \mathcal{C}. \end{array}$$

can show is equivalent to

$$\begin{array}{ll}\text{minimize} & \sum_i \log(\sigma_i(X) + \delta) \\ \text{subject to} & X \in \mathcal{C}, \end{array}$$

can iteratively linearize as before, obtain iterations in X, Y, Z.

Special case: iterative ℓ_1 heuristic for vectors

suppose $X = \mathbf{diag}(x)$, $x \in \mathbf{R}^n$, applying log-det heuristic yields

$$\begin{array}{ll} \text{minimize} & \sum_i \log(|x_i| + \delta) \\ \text{subject to} & x \in \mathcal{C}. \end{array}$$

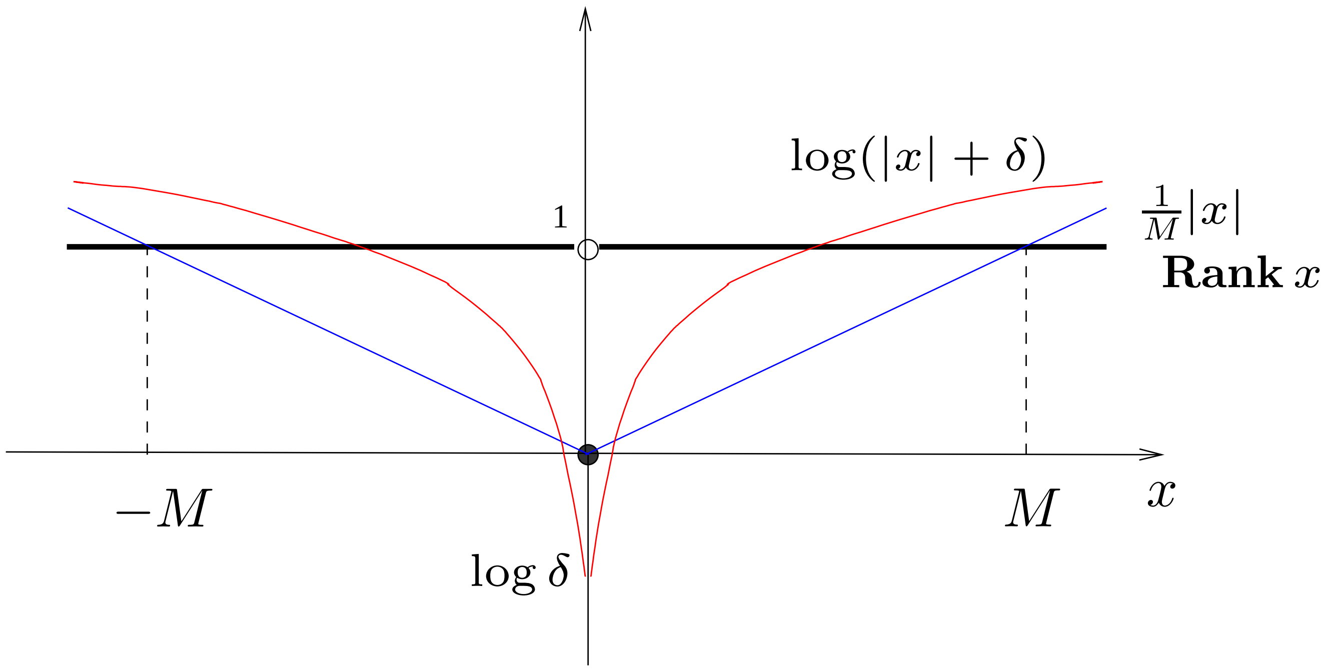
iterative linearization and minimization yields

$$\begin{aligned} x^{(k+1)} &= \operatorname*{argmin}_{x \in \mathcal{C}} \sum_i w_i^{(k)} |x_i| \\ w_i^{(k)} &= \frac{1}{|x_i^{(k)}| + \delta}. \end{aligned}$$

- in each step, minimize **weighted** ℓ_1 **norm** of x
- **interpretation:** if $x_i^{(k)}$ is small, its weight in next minimization step is large, so small entries in x are generally pushed towards zero as far as constraints allow

An interpretation

objectives for the two heuristics plotted for scalar case, $i.e.$, $X = x \in \mathbf{R}$.



- in this case $\sigma(X) = |x|$

- note $\frac{1}{M}|x|$ is convex envelope of $\mathbf{Rank}\, x$ on $\{x \mid |x| \leq M\}$.

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4. **Applications**

   - System realization with time-domain constraints
   - Reduced order controller design
   - Euclidean distance matrix problems
   - Portfolio optimization with fixed transaction costs

System realization with time-domain constraints

goal: find minimum order system whose step response up to nth sample satisfies specs, $e.g.$, certain rise-time, slew-rate, overshoot, settling characteristics and given sample delay ...

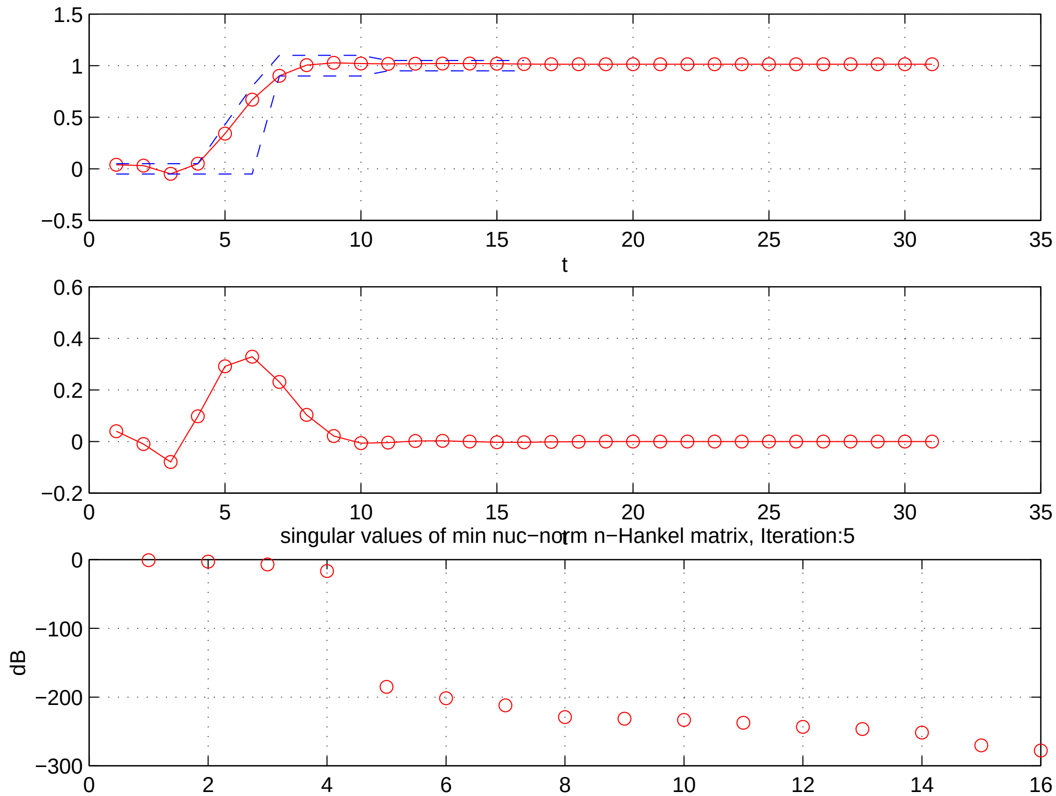
can express as

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, H_n \\ \text{subject to} & l_i \le s_i \le u_i, \quad k = 1, \dots, n \\ & h_{n+1}, \dots, h_{2n-1} \in \mathbf{R}, \end{array}$$

where h_is are impulse response, $s_k = \sum_{i=1}^{k} h_i$ are step response, l_i, u_i are samples of lower and upper time domain specs, and H_n is hankel matrix

$$H_n = \begin{bmatrix} h_1 & h_2 & \dots & h_n \\ h_2 & h_3 & \dots & h_{n+1} \\ \vdots & \vdots & & \vdots \\ h_n & h_{n+1} & \dots & h_{2n-1} \end{bmatrix}.$$

example: $n = 16$, specs denoted by dashed region (blue)

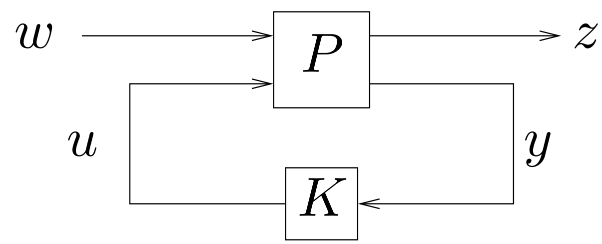


top: step response obtained after 5 iterations of log-det heuristic,
middle: corresponding impulse response,
bottom: singular values of Hankel matrix formed by above impulse response

- conclude that specs can be met by an LTI system of order four

Reduced order controller design

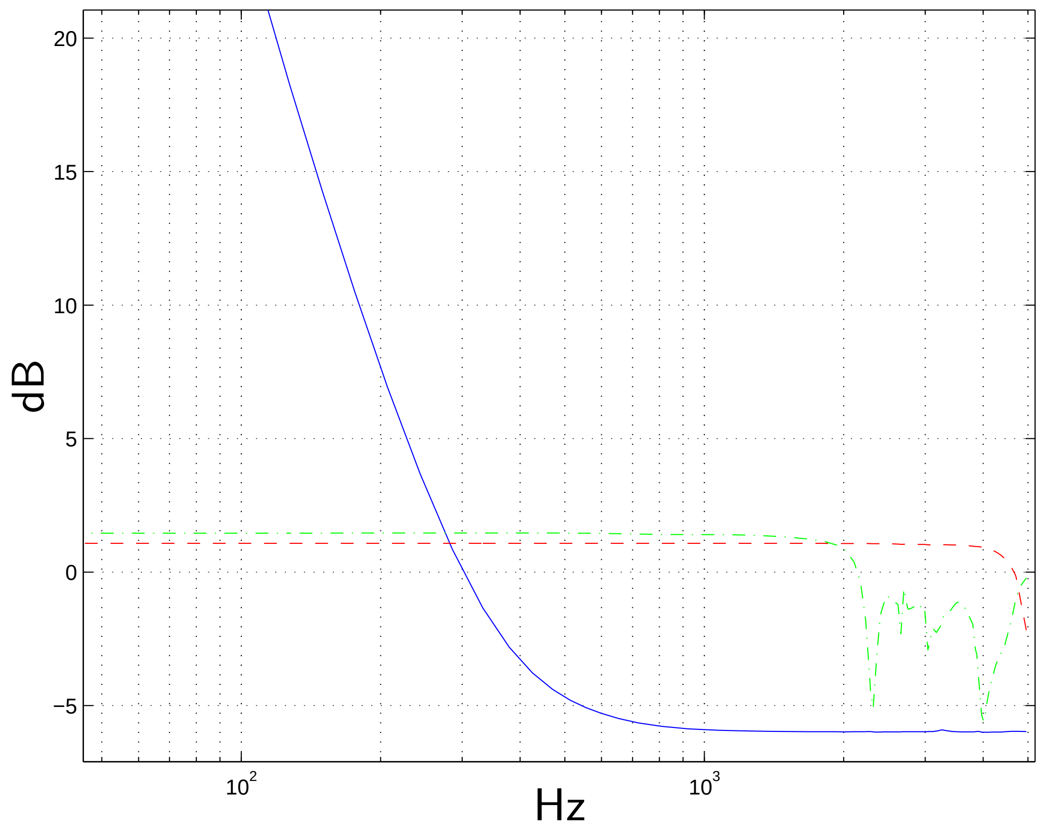
goal: given system P, design stabilizing controller K with lowest possible order to achieve specified performance ($e.g.$, closed loop l_2-gain from w to z less than γ)



- $x \in \mathbf{R}^n$ state, w and u disturbance and control inputs, z and y performance and measured outputs
- known: there exists stabilizing controller K of order $n_K \leq n$, achieving γ iff there exist PSD matrices $R, S \in \mathbf{R}^{n \times n}$ satisfying certain LMIs.

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank} \begin{bmatrix} R & I \\ I & S \end{bmatrix} \\ \text{subject to} & \begin{bmatrix} R & I \\ I & S \end{bmatrix} \geq 0 \\ & \text{LMIs in } R \text{ and } S. \end{array}$$

example: MIMO flexible positioning mechanism P of order 29, 5 inputs, 3 outputs, with γ within 5% of its minimum achievable value



Maximum singular value plots: open-loop system (blue) closed-loop system with 29th order controller corresponding to γ_{opt} (red), closed loop system with 20th order controller with 5% relaxation of γ_{opt} (green)

- 20th order controller obtained using log-det heuristic, converged in 5 iterations
- controller achieves same performance as full-order one within less than 0.5dB, with 30% reduction in number of states
- leads to savings in computation time and memory usage

Euclidean distance matrix (EDM) with smallest embedding dimension

construct configurations of points from information about interpoint distances

- D is EDM iff $\exists\ x_1, \ldots,\ x_n \in \mathbf{R}^r$ s.t. $D_{ij} = \|x_i - x_j\|^2$
- minimum possible r called **embedding dimension**
- [Schoenberg '35] D is EDM with embedding dimension r iff
  - $D_{ii} = 0$,
  - $VDV \leq 0$, where $V = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$,
  - $\mathbf{Rank}\, VDV \leq r$.

  first two conditions are equivalent to distance properties:
  $D_{ii} = 0, \quad D_{ij} = D_{ji} \geq 0, \quad D_{ik} + D_{ki} \geq D_{ij}$.

goal: given upper & lower bounds on distances D_{ij}, find EDM D with smallest embedding dimension

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, VDV \\ \text{subject to} & D_{ii} = 0 \qquad i = 1, \ldots, n \\ & -VDV \geq 0 \\ & L_{ij} \leq D_{ij} \leq U_{ij} \qquad i, j = 1, \ldots, n. \end{array}$$

applications: statistics/psychometrics (multidimensional scaling), chemistry (molecular conformation)

example: given pairwise distances of 40 points in $\mathbf{R}^6$, perturbed by noise up to 20%, find smallest embedding dimension

EDM with embedding dimension 6 found using trace heuristic ($i.e.$, 1 iteration of log-det)

Portfolio optimization with fixed transaction costs

portfolio with n assets is to be adjusted by performing transactions, after which it will be held over a fixed period

goal: maximize expected final wealth

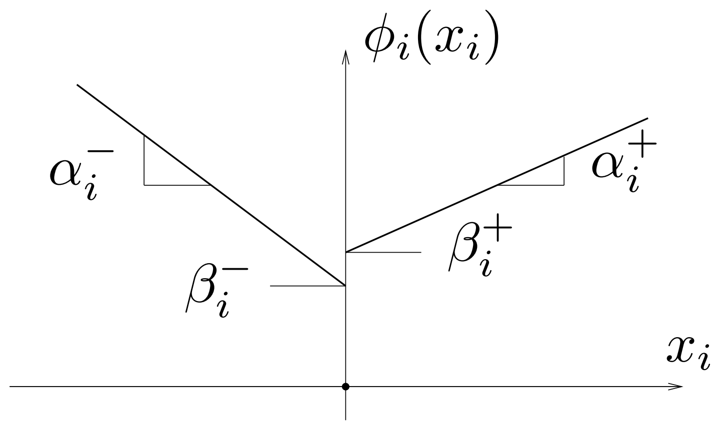
$$\begin{array}{ll} \text{maximize} & \bar{a}^T(w+x) \\ \text{subject to} & \mathbf{1}^T x + \phi(x) \leq 0 \\ & w + x \in \mathcal{S} \end{array}$$

where $w \in \mathbf{R}^n$ denotes current holdings, $x \in \mathbf{R}^n$ amounts transacted, $\bar{a} \in \mathbf{R}^n$ expected return, $\phi : \mathbf{R}^n \to \mathbf{R}_+$ transaction costs, $W = a^T(w+x)$ is end of period wealth

- $\mathcal{S}$ denotes other convex constraints on portfolio, $e.g.$,

  bound on risk $\mathbf{E}\,(W - \mathbf{E}W)^2 = (w+x)^T\Sigma\,(w+x) \leq \sigma_{\max}^2$

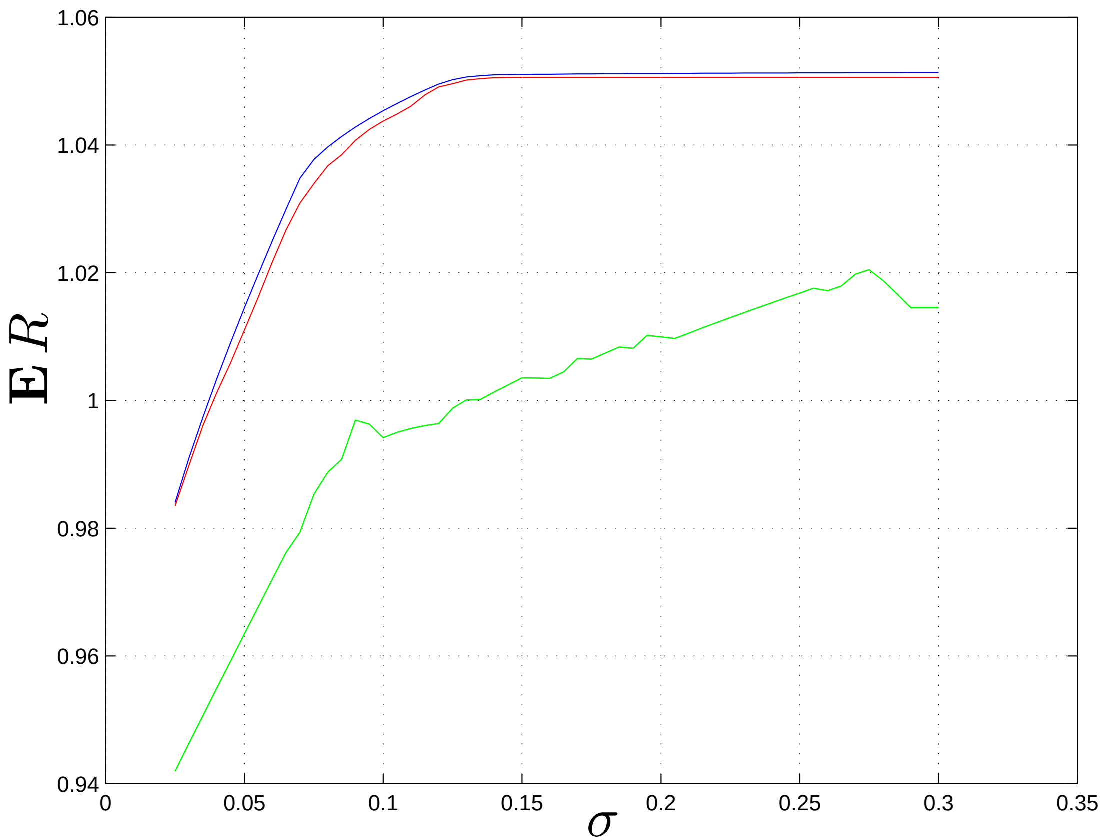
- transaction costs: $\phi(x) = \sum_{i=1}^n \phi_i(x_i)$, with



  (fixed plus linear cost model)

- seek **sparse** vector of transactions due to fixed transaction costs

example: 100 stocks plus a riskless asset with constraints on shorting



return vs risk tradeoff curve: global upper bound from ℓ_1 heuristic (blue), lower bound from solution obtained by iterative ℓ_1 (red), and solution computed without regard for fixed costs (green)

Summary and Contributions

- RMP: difficult non-convex problem, many applications

  Contributions:

  - provided theoretical support for trace heuristic
  - developed log-det heuristic
    - iterative weighted trace minimization implementation
  - proved semidefinite embedding result: "general RMP can be reduced to a PSD one"
  - extended both trace and log-det heuristics to general matrices
  - showed connections to vector sparsity problems
  - developed iterative ℓ_1 heuristic for vector sparsity
  - cataloged applications of RMP in various areas

Publications

Related papers:

- Trace and log-det heuristics for matrix rank minimization (working paper) [Fazel, Hindi, Boyd '01]
- A rank minimization heuristic with application to minimum order system approximation (American Control Conference '01) [Fazel, Hindi, Boyd '00]
- Portfolio optimization with linear and fixed transaction costs and bounds on risk (accepted for publication in *Operations Research*) [Lobo, Fazel, Boyd '00]

Other work:

- Approximations for partially coherent optical imaging systems [Fazel, Goodman, Boyd '98]
- Electron-beam proximity correction using convex optimization

Acknowledgements

- **advisor:** Prof. Boyd
- **committee:** Profs Donoho, Saunders, and Tomlin
- **group:** Haitham, Arash, Miguel, Cesar, Mikael, Lin, Bob, Sunghee, Yirong and Denise
- Prof. Kailath and his (former) group: Babak, Yao-Ting, Bijit
- **former teachers:** Mr. Jafari, Dr. Hashemi
- **friends:** at ISL, at PSA, . . .
- **family**