A Rank Minimization Heuristic with Application to Minimum Order System Approximation

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Abstract

Several problems arising in control system analysis and design, such as reduced order controller synthesis, involve minimizing the rank of a matrix variable subject to linear matrix inequality (LMI) constraints. Except in some special cases, solving this rank minimization problem (globally) is very difficult. One simple and surprisingly effective heuristic, applicable when the matrix variable is symmetric and positive semidefinite, is to minimize its trace in place of its rank. This results in a semidefinite program (SDP) which can be efficiently solved.

In this paper we describe a generalization of the trace heuristic that applies to general nonsymmetric, even non-square, matrices, and reduces to the trace heuristic when the matrix is positive semidefinite. The heuristic is to replace the (nonconvex) rank objective with the sum of the singular values of the matrix, which is the dual of the spectral norm. We show that this problem can be reduced to an SDP, hence efficiently solved. To motivate the heuristic, we show that the dual spectral norm is the convex envelope of the rank on the set of matrices with norm less than one.

We demonstrate the method on the problem of minimum order system approximation.

1 Introduction

In recent years there has been a growing interest in problems that involve minimizing the rank of a matrix over a convex set. Applications arise in diverse areas such as minimum order controller design [Mes99], factor analysis in statistics [Sha82], and Euclidean distance matrix problems [TT93], among others. The general matrix rank minimization problem can be expressed as

$$\begin{array}{ll} \text{minimize} & \mathbf{Rank}\, X \\ \text{subject to} & X \in \mathcal{C}, \end{array} \qquad (1)$$

where $X \in \mathbf{R}^{m\times n}$ is the optimization variable and $\mathcal{C}$ is a convex set, e.g., described by LMIs. It is well known that in general this problem is hard to solve [VB96, §7.3]. Various heuristics have been developed to handle problems of this type; see, e.g., [BG96, SIG98, Dav94]. One simple and surprisingly effective heuristic, applicable when the matrix is symmetric positive semidefinite, is to minimize its trace in place of its rank. This is used in [Par00, Mes99] to design reduced order controllers.

This heuristic obviously does not apply to problems in which the matrix is non-symmetric, or non-square, since the trace is not even defined, let alone a good convex surrogate for the rank. In this paper, we present a generalization of the trace heuristic that can be applied to *any* matrix. The heuristic is to solve the problem

$$\begin{array}{ll} \text{minimize} & \|X\|_* \\ \text{subject to} & X \in \mathcal{C}, \end{array} \qquad (2)$$

in place of (1), where

$$\|X\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i(X),$$

where $\sigma_i(X) = \sqrt{\lambda_i(X^TX)}$ denote the singular values of X. The norm $\|\cdot\|_*$ is sometimes called the *nuclear norm* or *Ky-Fan n-norm* (see [HJ91]), and is the dual of the spectral (or maximum singular value) norm of a matrix, i.e.,

$$\|X\|_* = \sup\{\mathbf{Tr}\, Y^TX \mid \|Y\| \le 1\},$$

where $\|\cdot\|$ denotes the maximum singular value or spectral norm. Note that while the original problem (1) is in general a difficult optimization problem, the dual spectral norm minimization problem (2) is a convex optimization problem, and therefore (at least in principle) easily solved.

If the matrix variable X is symmetric and positive semidefinite, then its singular values are the same as its eigenvalues, and therefore the dual spectral norm $\|X\|_*$ reduces to $\mathbf{Tr}\, X$. In this case, the heuristic (2) reduces to the trace minimization heuristic.

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Another special case occurs when the matrix X is diagonal, say, $X = \mathbf{diag}(x)$, where $x \in \mathbf{R}^n$. In this case, $\mathbf{Rank}\, X$ is the same as the number of nonzero entries of the vector x, i.e., its *cardinality* $\mathbf{Card}\, x$. Problem (1) then reduces to the problem of finding the *sparsest* (i.e., minimum cardinality) vector x in a convex set. For this special case, the heuristic dual spectral norm minimization problem (2) becomes the problem of minimizing the ℓ_1 norm of a vector x over a convex set C. Minimizing the ℓ_1 norm is a well-known heuristic for minimizing the cardinality; see for example [HHB99, CD94].

The rest of the paper is organized as follows. In §2 we motivate the heuristic by showing that the dual spectral norm is the *convex envelope* of the rank function on the set of matrices with norm less than one, which allows us to interpret the heuristic as a type of relaxation of the original rank minimization problem. In §3 we show how the dual spectral norm can be represented by an LMI, so when the feasible set $\mathcal{C}$ is described by LMIs, the dual spectral norm problem (2) can be formulated as a semidefinite program (SDP), and so can be solved using standard, existing software [WB96, AHN+97, Stu98, FK95, GN93, EG95]. In §4, we demonstrate the effectiveness of the heuristic by applying it to the problem of minimum order system approximation.

2 Convex envelope of rank

Let $f : \mathcal{C} \to \mathbf{R}$, where $\mathcal{C} \subseteq \mathbf{R}^n$. The *convex envelope* of f (on $\mathcal{C}$) is defined as the largest convex function g such that $g(x) \leq f(x)$ for all $x \in \mathcal{C}$ (see, e.g., [HUL93]).

Theorem 1 *The convex envelope of the function* $\phi(X) = \mathbf{Rank}(X)$*, on* $\mathcal{C} = \{X \in \mathbf{R}^{m\times n} \mid \|X\| \leq 1\}$*, is* $\phi_{\text{env}}(X) = \|X\|_*$.

This theorem has the following implications for problem (1) and the heuristic (2). Suppose the feasible set is bounded by M, i.e., for all $X \in \mathcal{C}$, we have $\|X\| \leq M$. The convex envelope of $\mathbf{Rank}\, X$ on $\{X \mid \|X\| \leq M\}$ is given by $\frac{1}{M}\|X\|_*$. In particular, for all $X \in \mathcal{C}$, we have $\mathbf{Rank}\, X \geq \frac{1}{M}\|X\|_*$. It follows that if p_{rank} denotes the optimal value of the rank minimization problem (1) and p_* denotes the optimal value of the dual spectral norm minimization problem (2), we have

$$p_{\text{rank}} \geq \frac{1}{M} p_*.$$

In other words, by solving the heuristic problem, we obtain a lower bound on the optimal value of the original problem (provided we can identify a bound M on the feasible set).

See the appendix for the proof of theorem 1.

3 Dual spectral norm minimization via SDP

The heuristic problem (2) is a convex problem and can be handled using a variety of convex optimization algorithms. In this section, we show how to express the problem as an SDP, when the constraints are given by LMIs. The advantage of such a formulation is that we can use widely available SDP solvers to readily solve the problem.

We will use the following result:

Lemma 1 *For* $X \in \mathbf{R}^{m\times n}$ *and* $t \in \mathbf{R}$*, we have* $\|X\|_* \leq t$ *if and only if there exist matrices* $Y \in \mathbf{R}^{m\times m}$ *and* $Z \in \mathbf{R}^{n\times n}$ *such that*

$$\begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \geq 0, \qquad \mathbf{Tr}\, Y + \mathbf{Tr}\, Z \leq 2t. \tag{3}$$

In other words, the condition $\|X\|_* \leq t$ can be represented as an LMI. This observation is made also in [VB96, §3.1].

This lemma can be used to express the dual spectral norm minimization problem (2) as an SDP. We first write problem (2) as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|X\|_* \leq t \\ & X \in \mathcal{C}, \end{array}$$

with variables X and t. Then, using the lemma above, we express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, Y + \mathbf{Tr}\, Z \\ \text{subject to} & \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \geq 0 \\ & X \in \mathcal{C}, \end{array} \tag{4}$$

where $Y = Y^T$ and $Z = Z^T$ are new variables. The problem (4) is an SDP, provided the constraint set $\mathcal{C}$ is expressed as an LMI. See appendix B for the proof of the lemma.

4 Minimum order system approximation

In this section we apply the rank minimization heuristic to the minimum order system approximation problem. Such problems arise, for example, in model reduction problems that come from overparametrization in subspace system identification [Jac94, McK95, OM96], and $\mathcal{H}_\infty$ model reduction [HJN92].

Let $p_1, \ldots, p_N \in \mathbf{C}$ be a set of complex numbers with conjugate symmetry, i.e., whenever p_i is complex, there is some j such that $p_j = \bar{p}_i$. We consider the family of proper rational matrices given by

$$H(s) = R_0 + \sum_{i=1}^{N} \frac{R_i}{s - p_i}, \tag{5}$$

where $R_i \in \mathbf{C}^{m \times n}$ satisfy conjugate symmetry: whenever $p_i = \bar{p_j}$, we have $R_i = \bar{R}_j$. We consider p_i, the poles of the rational matrix H, as fixed; the residues R_i are the variables that we will use for approximation (subject to the conjugate symmetry constraint). The McMillan degree, i.e., the order of a minimal state space realization, of the rational matrix H is given by

$$\deg(H) = \sum_{i=1}^{N} \mathbf{Rank}(R_i).$$

Our goal is to determine values of the residue matrices R_i that minimize the MacMillan degree, over some set of acceptable approximations.

Let $\omega_1, \ldots, \omega_K \in \mathbf{R}$, and suppose $G_k \in \mathbf{C}^{m \times n}$ are given. We can interpret the ω_k and G_k are sampled frequencies, and measured frequency response matrix, respectively. As a criterion for acceptable fit, we use the simple conditions

$$\|H(j\omega_k) - G_k\| \leq \epsilon, \qquad k = 1, \ldots, K,$$

i.e., that the matrix H, evaluated at the given frequencies, should approximate (in spectral norm), within a tolerance ϵ, the given data.

The problem of finding the minimum order approximation is then given by

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \mathbf{Rank}(R_i) \\ \text{subject to} & \|H(j\omega_k) - G_k\| \leq \epsilon, \quad k = 1, \ldots, K \\ & R_j = \bar{R}_i \text{ for } p_j = \bar{p}_i \end{array} \tag{6}$$

where the optimization variables are the $R_i \in \mathbf{C}^{m \times n}$. Note that $H(j\omega_k)$ is a linear function of the variables R_i. The objective can also be expressed as the rank of the block diagonal matrix with blocks $R_1, \ldots, R_N$, so this problem has the minimum rank form (1) (with complex matrices, however, instead of real matrices).

For a discussion on optimization over an affine family of linear systems, see [BGFB94, §10.1].

4.1 Dual spectral norm heuristic

The heuristic dual spectral norm method, and the results concerning convex envelope and the LMI representation, are all readily extended to the complex case, with the Hermitian conjugate substituted in place of the transpose.

We now form the heuristic problem (2) associated with the minimum order approximation problem (6). We obtain

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \|R_i\|_* \\ \text{subject to} & \|H(j\omega_k) - G_k\| \leq \epsilon, \quad k = 1, \ldots, K \\ & R_j = \bar{R}_i \text{ for } p_j = \bar{p}_i. \end{array} \tag{7}$$

This is a convex optimization problem in the variables $R_0, \ldots, R_N$.

4.2 SDP representation

We can express the problem (7) as an SDP as follows. We introduce variables t_i, and express problem (7) as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} t_i \\ \text{subject to} & \|R_i\|_* \leq t_i \quad i = 1, \ldots, N \\ & \|H(j\omega_k) - G_k\| \leq \epsilon \quad k = 1, \ldots, K \\ & R_j = \bar{R}_i \text{ for } p_j = \bar{p}_i. \end{array}$$

Using lemma 1, we can replace the first constraint by its LMI equivalent; the approximation constraints can also be expressed via LMIs using Schur complements. Thus we obtain the following SDP:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \mathbf{Tr}\, Y_i + \mathbf{Tr}\, Z_i \\ \text{subject to} & \left[\begin{array}{cc} Y_i & R_i \\ R_i^* & Z_i \end{array}\right] \geq 0 \quad i = 1, \ldots, N \\ & \left[\begin{array}{cc} \epsilon I & (H(j\omega_k - G_k) \\ (H(j\omega_k) - G_k)^* & \epsilon I \end{array}\right] \geq 0, \\ & \quad k = 1, \ldots, K \\ & R_j = \bar{R}_i \quad \text{for } p_j = \bar{p}_i, \end{array} \tag{8}$$

where $R_i \in \mathbf{C}^{m \times n}$, $Y = Y^* \in \mathbf{C}^{m \times m}$, and $Z = Z^* \in \mathbf{C}^{n \times n}$ are the variables. (Note that since Y_i and Z_i are Hermitian, their traces are real, so the objective is real.) This is a (complex) SDP.

4.3 Complex semidefinite programs

The complex SDP (8) can in turn be expressed as a real SDP, using the fact that for any Hermitian matrix $X \in \mathbf{C}^{n \times n}$, the matrix inequality $X \geq 0$ is equivalent to

$$\left[\begin{array}{cc} \Re X & -\Im X \\ \Im X & \Re X \end{array}\right] \geq 0,$$

which is an ordinary (real) LMI in the (real) matrix variables $\Re X$ and $\Im X$.

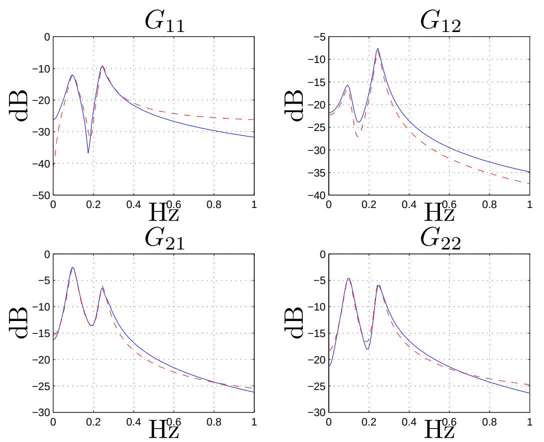

Figure 1: Original 8th order data (solid), and 6th order approximation (dashed).

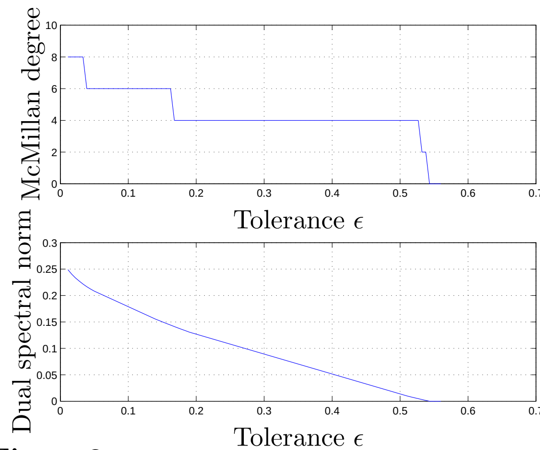

Figure 2: Tradeoff curves. The horizontal axis gives the approximation tolerance ϵ. The top plot shows the MacMillan degree obtained by the dual spectral norm heuristic. The bottom plot shows the minimum dual spectral norm.

4.4 Numerical example

In this section, we demonstrate the techniques above on numerical data, generated from a generic system model.

The problem data was generated as follows. We used an 8th order, 2-input 2-output transfer matrix F, which was normalized so that $\|F\|_\infty = \sup_\omega \|F(j\omega)\| = 1$. The frequencies ω_k, $k = 1, \ldots, K = 128$ were chosen as linearly spaced from 0Hz to 1Hz, and G_k was taken as the value of the 8th order model at ω_k: $G_k = F(j\omega_k)$. For the poles $p_1, \ldots, p_8$, we took the poles of F, which appear in four complex conjugate pairs. Two pairs are clustered at $\pm$0.10Hz, the other two are around $\pm$0.24Hz.

The system approximation problem then becomes a model reduction problem: we keep the poles of the original system, and modify the residue matrices; the goal is to reduce the order, while respecting a model reduction transfer matrix error. We used SDPSOL [WB96] to solve the resulting SDPs.

As an example, (8) is solved with $\epsilon = 0.05$ (-26dB). The result is a 6th order approximation. Figure 1 shows the magnitude plot of the original system (F) and the approximation result (i.e., H).

By solving the dual spectral norm problem (8) for a range of values of the tolerance ϵ from very small to 0.55, the tradeoff curve in figure 2 is obtained. The staircase curve is the actual rank objective from (6), evaluated for the optimizer of (8). This provides an upper bound on the optimal rank objective in (6). The curve below it is the dual spectral norm objective value of (8).

A Proof of theorem 1

To prove the theorem we use *conjugate functions*. Recall that the conjugate f^* of a function $f : \mathcal{C} \to \mathbf{R}$, where $\mathcal{C} \subseteq \mathbf{R}^n$, is defined as

$$f^*(y) = \sup\{y^T x - f(x) \mid x \in C\}.$$

A basic result of convex analysis is that f^{**}, i.e., the conjugate of the conjugate, is the convex envelope of the function f, provided some technical conditions, which are valid here, hold; see theorem 1.3.5 in [HUL93].

Part 1. *Computing* ϕ^*: The conjugate of the rank function ϕ, on the set of matrices with (spectral) norm less than or equal to one, is

$$\phi^*(Y) = \sup_{\|X\| \le 1} (\mathbf{Tr}\, Y^T X - \phi(X)). \qquad (9)$$

Let $q = \min\{m, n\}$, and note that by Von Neumann's trace theorem we have

$$\mathbf{Tr}\, Y^T X \le \sum_{i=1}^{q} \sigma_i(Y)\sigma_i(X), \qquad (10)$$

where $\sigma_i(\cdot)$ denotes the ith largest singular value. Let $X = U_X \Sigma_X V_X^T$ and $Y = U_Y \Sigma_Y V_Y^T$ be the singular value decompositions (SVDs) of X and Y. Since the term $\phi(X)$ in (9) is independent of U_X and V_X, we pick $U_X = U_Y$ and $V_X = V_Y$ to maximize the first term in (9). It follows that

$$\phi^*(Y) = \sup_{\|X\| \le 1} \left(\sum_{i=1}^{q} \sigma_i(Y)\sigma_i(X) - \mathbf{Rank}(X)\right).$$

If $X = 0$, we have $\phi^*(Y) = 0$ for all Y, and if $\mathbf{Rank}(X) = r$, $1 \le r \le q$, then $\phi^*(Y) = \sum_{i=1}^r \sigma_i(Y) - r$. So $\phi^*(Y)$ can be expressed as:

$$\begin{aligned}\phi^*(Y) \quad = \quad & \max\{0,\ \sigma_1(Y)-1, \ldots,\ \sum_{i=1}^r \sigma_i(Y) - r, \\ & \ldots,\ \sum_{i=1}^q \sigma_i(Y) - q\},\end{aligned}$$

The largest term in this set is the one that sums all *positive* $(\sigma_i(Y) - 1)$ terms. We conclude that

$$\phi^*(Y) \quad = \quad \textstyle\sum_{i=1}^q (\sigma_i(Y)-1)_+, \tag{11}$$

where a_+ denotes the positive part of a, i.e., $a_+ = \max\{0, a\}$.

Part 2. *Computing* ϕ^{**}: We will now find the conjugate of ϕ^*, defined as

$$\phi^{**}(Z) = \sup_Y (\mathbf{Tr}\, Z^T Y - \phi^*(Y)),$$

for all $Z \in \mathcal{C}^{m\times n}$. As before, we choose U_Y and V_Y such that $U_Z^T U_Y = I$ and $V_Y^T V_Z = I$ to get

$$\phi^{**}(Z) = \sup_Y (\sum_{i=1}^q \sigma_i(Z)\sigma_i(Y) - \phi^*(Y)).$$

We will consider two cases, $\|Z\| > 1$ and $\|Z\| \le 1$:

If $\|Z\| > 1$, we can choose $\sigma_1(Y)$ large enough so that $\phi^{**}(Z) \to \infty$. To see this, note that in

$$\phi^{**}(Z) = \sup_Y (\sum_{i=1}^q \sigma_i(Z)\sigma_i(Y) - (\sum_{i=1}^r \sigma_i(Y) - r)),$$

the coefficient of $\sigma_1(Y)$ is $(\sigma_1(Z) - 1)$ which is positive.

Now let $\|Z\| \le 1$. If $\|Y\| \le 1$, then $\phi^*(Y) = 0$ and and the supremum is achieved for $\sigma_i(Y) = 1$, $i = 1, \ldots, q$, yielding

$$\phi^{**}(Z) = \sum_{i=1}^q \sigma_i(Z) = \|Z\|_*.$$

We will now show that if $\|Y\| > 1$, $\phi^{**}(Z)$ is always smaller than the value given above. We have

$$\phi^{**}(Z) = \sup_{\|Y\|>1} \left(\sum_{i=1}^q \sigma_i(Y)\sigma_i(Z) - \sum_{i=1}^r (\sigma_i(Y) - 1) \right).$$

Consider the expression inside the sup. By adding and subtracting the term $\sum_{i=1}^q \sigma_i(Z)$ and rearranging the terms, we get

$$\begin{aligned} &= \textstyle\sum_{i=1}^r (\sigma_i(Y)-1)(\sigma_i(Z)-1) \\ &\quad + \textstyle\sum_{i=r+1}^q (\sigma_i(Y)-1)\sigma_i(Z) + \sum_{i=1}^q \sigma_i(Z) \\ &< \textstyle\sum_{i=1}^q \sigma_i(Z),\end{aligned}$$

where the last inequality holds since the first two sums on the second line always have a negative value.

In summary, we have shown

$$\phi^{**}(Z) = \|Z\|_*,$$

over the set $\{Z \mid \|Z\| \le 1\}$. Thus, over this set, $\|Z\|_*$ is the convex envelope of the function $\mathbf{Rank}(Z)$. □

B Proof of lemma B

Proof: ($\Longleftarrow$) Let Y and Z satisfy the relations (3) above, and let $X = U\Sigma V^T$ be the SVD of X. Here, Σ is of size r, where r is the rank of X. We have

$$\mathbf{Tr} \begin{bmatrix} UU^T & -UV^T \\ -VU^T & VV^T \end{bmatrix} \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} \ge 0,$$

since the trace of the product of two PSD matrices is always non-negative. This yields

$$\mathbf{Tr}\, UU^T Y - \mathbf{Tr}\, UV^T X^T - \mathbf{Tr}\, VU^T X + \mathbf{Tr}\, VV^T Z \ge 0. \tag{12}$$

Since columns of U are orthonormal, we can always add more columns to complete them to a full basis, i.e., there exists U^T such that $[U\ \tilde{U}][U\ \tilde{U}]^T = I$, or $UU^T + \tilde{U}\tilde{U}^* = I$, so $\|UU^T\| \le 1$. So we get $|\mathbf{Tr}\, UU^T Y| \le \sum_i \lambda_i(UU^T)\lambda_i(Y) \le \mathbf{Tr}\, Y$ (using Von Neumann's trace theorem, see [HJ91]) Similarly, for V we have $\mathbf{Tr}\, VV^T Z \le \mathbf{Tr}\, Z$. Also, $\mathbf{Tr}\, VU^T X = \mathbf{Tr}\, V\Sigma V^* = \mathbf{Tr}\, \Sigma$. Using these facts, and (12) above, we get

$$\begin{aligned} &\mathbf{Tr}\, Y + \mathbf{Tr}\, Z - \mathbf{Tr}\, \Sigma \ge 0, \\ &\mathbf{Tr}\, \Sigma \le \tfrac{1}{2}(\mathbf{Tr}\, Y + \mathbf{Tr}\, Z), \\ &\mathbf{Tr}\, \Sigma = \|X\|_* \le t. \end{aligned}$$

($\Longrightarrow$) Suppose $\|X\|_* \le t$. We will show Y and Z can be chosen to satisfy the relations (3). Let $Y = U\Sigma U^T + \gamma I$ and $Z = V\Sigma V^T + \gamma I$, then

$$\mathbf{Tr}\, Y + \mathbf{Tr}\, Z = 2\,\mathbf{Tr}\, \Sigma + \gamma(p+q) = 2\|X\|_* + \gamma(p+q),$$

so if we choose $\gamma = \frac{2(t - \|X\|_*)}{p+q}$, we will have $\mathbf{Tr}\, Y + \mathbf{Tr}\, Z = 2t$.

Also note that

$$\begin{aligned} \begin{bmatrix} Y & X \\ X^T & Z \end{bmatrix} &= \begin{bmatrix} U\Sigma U^T & U\Sigma V^T \\ V\Sigma U^T & V\Sigma V^T \end{bmatrix} + \gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U \\ V \end{bmatrix} \Sigma \begin{bmatrix} U^T & V^T \end{bmatrix} + \gamma I, \end{aligned}$$

which is PSD. □

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