On a Dynamic Extension of the Theory of Graphs

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Abstract

We consider a graph of a given order whose topology is dictated by an underlying controlled dynamics. This dynamic graph structure provides a convenient framework for modeling distributed dynamic systems where the *topology* of the interaction among its elements evolves in time. The static theory of graphs has been an active area of research in non-enumerative combinatorics. In this paper, we propose a framework for studying dynamic graphs as an area that lies at the intersection of dynamical systems and combinatorics. We also highlight the directions which we find to be particularly promising in this venue.

Keywords: Dynamic graphs; distributed systems; matrix evolutions

1 Introduction

We embark on a dynamic extension of the theory of graphs. Graph theory has traditionally provided a convenient framework for studying the intrinsic properties of the collective interaction of a finite or an infinite set of elements. This theory has been particularly useful in proposing arrangements of objects that satisfy a given set of pairwise relationships and have extremal properties. Graph theory has been successful in a wide array of disciplines including, social sciences, economics, geography, architecture, transportation, electrical networks, production planning and control, traffic network design, and genetics, among many others [8]. At the same time, much of the traditional theory of graphs is concerned with the *static* arrangements of, and assignments among, the objects in a set. Exceptions to this statement include the areas of graph transformations, random graphs [2], and a proposal for studying graphs with time varying topologies which was motivated by their potential applications in the computer sciences (e.g., languages, databases, compilers) [10].

By a *static* graph we mean the familiar structure of vertices and edges- with possibly weights associated with them- that is independent of time. From a dynamic point of view, however, a static graph captures the topology of the interaction between the elements of a set at a frozen time instance. Motivated by the introduction of distributed architectures for many of the future scientific, military, and civilian space missions [14, 15] (see Figures 1, and 2), we initiate the development of a theoretical framework for studying *dynamic graphs*. This attempt is substantiated by our earlier observation [12] that the interaction between the various elements of a distributed space system is, generally, dynamic: it topology is strongly time and state dependent. In fact, we expect that a successful theory of dynamic graphs will shed light on the intricate and dynamic dependency of the distributed spacecraft control architecture and algorithms on one hand, and the spacecraft communication and sensing topology, on the other. Concurrently, we aim at developing the theoretical and algorithmic tools needed for the control system design in situations where the elements of the system are distributed, interact over a network, and the topology of their interaction evolves over time [13].

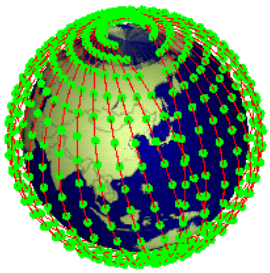

Figure 1: The original Teledesic design [15]

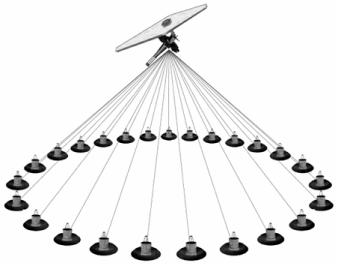

Figure 2: A conceptual configuration for the Terrestrial Planet Imager [14]

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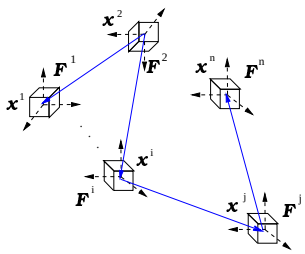

Figure 3: Distributed space control system hierarchical graph [12]

2 Motivation

The role and significance of a dynamic theory of graphs for the distributed space systems design was pointed out in [12] and more recently in [13]. In [12], graphs are initially employed to represent the hierarchical structure of the multiple spacecraft control system (Figure 3). The control hierarchy is then allowed to advance in time in order to stay compatible with the possibly time varying properties of the communication and sensing infrastructure. This infrastructure is represented by, and referred to as, the formation information graph. In [12] it is then argued that the space system's control architecture strongly affects, and is effected by, the topology of the corresponding sensing and communication graphs. The stability and performance characteristics of the resulting interconnected dynamic system, evolving through a strong interaction between the sensing, communication, and control architectures, are by no means traditional or well-understood. This system essentially involves a feedback mechanism through a graph structure- leading to a new class of dynamical systems referred to as graph-driven hybrid systems [13]. The control system design therefore can only proceed by an understanding of the intrinsic properties of the evolution of the control hierarchy and the underlying communication and sensing infrastructure.

Motivated by its potential applications in the design of distributed space systems, we consider a rather specific class of dynamic graphs; the time evolution of these graphs are dictated by the underlying spacecraft dynamics, and as such, can be referred to as state-driven dynamic graphs. Specifically, we will study the *invariance* and the *reachability* properties of these structures. Moreover, we propose formalisms that can lead to the solution methods for the dynamic graph problems. In this paper, we take "connectedness" as a pro-typical graph property that is required to be invariant as the structure evolves in time. We should point out however that other graph properties could be considered as well; the edge connectivity, the genus of the graph, the girth, and the chromatic number [2], are potential candidates for our future work.

3 Notation and background

A few words on the terminology and the notation. Our treatment of graph theoretic concepts is basic. A (simple, finite) graph G is defined by the ordered triplet (V, E, ϕ), where V (vertices) and E (edges) are arbitrary finite sets and ϕ is an one-to-one incidence mapping (no multiple edges) that assigns to each element $e \in E$ a nonempty set $\phi(e) \subseteq V$ of cardinality exactly two (no loops). It is often more convenient to denote the graph as the pair $G = (V, E)$, and simply represent ϕ by listing the elements of E. The triplet $(\widetilde{V}, \widetilde{E}, \phi)$, for which $\widetilde{V} \subseteq V$ and $\widetilde{E} \subseteq E$ is called a subgraph of G. The order of graph G, $|G|$, is the cardinality of its vertex set; $e(G)$, the cardinality of the edge set, is called its size. An edge $e \in E$ is said to be incident on $u, v \in V$, if $\phi(e) = \{u, v\}$. In this case, given that multiple edges between two vertices in the graph are not allowed, we identify the edge e by uv. The cardinality of the set of edges incident on the vertex v is called its degree and will be denoted by d_v; the degree sequence for a graph of order n is $\{d_v\}_{v \in V(G)}$, arranged in a nonincreasing order. One can arbitrary label the vertex set and the edge set of the graph of order n and size m: $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$. Two graphs G and H are called isomorphic, $G \simeq H$, if there is an adjacency preserving one-to-one correspondence between their vertex sets. For a given labeling of the graph G, a walk is defined as an alternating sequence of vertices and edges, $v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ $(k \geq 1)$, for which e_i is incident on v_i and v_{i+1} for all $i = 1, \ldots, k$. A graph is connected if each pair of its vertices is joined by a walk. A component of a graph is any of its connected subgraphs. A graph is complete if for every distinct pair of elements $u, v \in V$, there is an edge $e \in E$ for which $e = uv$; K_n will denote the complete graph of order n. Given $G = (V, E)$, the graph $G - e$ (respectively $G + e$) is obtained by deleting (respectively adding) the edge e to G. The set of all labeled graphs of order n is denoted by $\mathcal{G}^n$.

We also recall an enumeration formula: the number of combinations of n distinct objects, taken m at a time will be denoted by $C(n, m) := n!/(m!\,(n-m)!)$. Additionally, we use the following notations: E_{ij} will denote a square matrix of appropriate dimensions with all of its entries set to zero except the ij-th entry, which is set to one. For two symmetric matrices A, B of the same size, $A \bullet B$ denotes the trace of their product.

4 Graph of graphs

Consider a formation of n elements labeled as $1, 2, \ldots, n$. We view each element i, as a discrete controlled dynamical system, with state x_i, whose time evolution is described through a difference equation of

the form

$$x_i(k+1) = f_i(x_i(k), u_i(k)), \quad i = 1, \ldots, n; \qquad (4.1)$$

we assume that the functional forms of the f_i's are the same (the formation is dynamically homogeneous), and for $k = 0, 1, \ldots$, one has $x_i(k) \in X$ and $u_i(k) \in U$. Furthermore, our working assumption will be the complete controllability of the dynamical systems (4.1): given any initial state, there exists a control input that steers the system to a desired target state (possibly, set-valued) in a finite time. In the case of a formation of multiple (rigid) spacecraft, each x_i can represent the inertial position and velocity, as well as the orientation and angular velocity of the space vehicles.

We can juxtapose the state of the elements in terms of (formation) state x, whose time evolution can similarly be described by

$$x(k+1) = f(x(k), u(k)); \qquad (4.2)$$

if the formation control sequence (or control law) has already been specified, then we will also refer to the autonomous analog of (4.2),

$$x(k+1) = f(x(k)). \qquad (4.3)$$

We note that for $k = 0, 1, \ldots$,

$$x(k) \in \mathcal{X} := \underbrace{X \times X \ldots \times X}_{n \text{ times}}.$$

As it was argued in [12], architecting the distributed spacecraft control system is strongly influenced by the formation sensing and communication infrastructure. We will not, however, delve into the multiple spacecraft crosslink issues (e.g., data rates, coding) and focus instead on the topology of the corresponding information channels (communication and sensing). We abstractly represent this topology through the graph G, and as hinted in §2, we let G to be a function of the formation state x- we write $G(x(t))$ to highlight this dependency. We can thus formally define (state-driven) dynamic graphs in terms of mappings from the formation configuration space $\mathcal{X}$ to the set of all labeled graphs of order n, $\mathcal{G}^n$. In this venue, we assume that the vertex set of G, $V(G)$, having cardinality n, is fixed. The edge set of G, $E(G)$, on the other hand, will be a function of the formation state. This establishes a correspondence between the formation state and labeled graphs on n vertices. Since the configuration state of the formation, $\mathcal{X}$, is generally uncountable and the number of labeled graphs on n vertices is finite [9],

$$N := 2^{C(n,2)}, \qquad (4.4)$$

the graph $G(x(t))$ does not uniquely determine the formation state $x(t)$ at time t. Note that the number N (4.4) is the set of all subgraphs of the complete graph

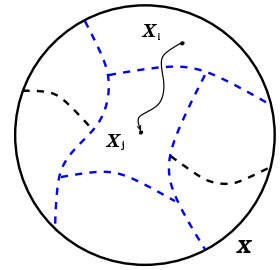

Figure 4: The equivalence class of the neighboring graphs

K_n. We observe that the set of labeled information graphs induces an equivalence class on $\mathcal{X}$: for $x, y \in \mathcal{X}$ we can write

$$x \simeq y \quad \text{if} \quad G(x) \simeq G(y); \qquad (4.5)$$

the notation '$\simeq$' between the graphs $G(x)$ and $G(y)$ in (4.5) can be interpreted in terms of an isomorphism. We now label the subgraphs of the complete graph as

$$G_1, G_2, \ldots, G_N, \quad \text{where } N \text{ is as defined in (4.4)} \qquad (4.6)$$

Next, we associate with each of these indexed graphs, the equivalence class $\mathcal{X}_i \subseteq \mathcal{X}$. We say that G_r is a neighboring graph of G_s with respect to the controlled dynamical system (4.2), if for $x(k_0) \in \mathcal{X}_r$, there is a control sequence $u(k_0), u(k_0+1), \ldots, u(k_0+p-1)$, such that $x(k_0+p) \in \mathcal{X}_s$, and that if $x(k) \in \mathcal{X}_q$ for some $k_0 \leq k \leq k_0+p$, then $q = r$ or $q = s$ (see Figure 4). For now, it is natural to let "neighboring" be a reflexive relation on the set of labeled graphs of order n. Note that in the case of a controlled dynamical system, the notion of a neighboring graph is dependent on the class of admissible control laws. As the formation state evolves in its configuration space during the indexed time interval $k = 0, 1, \ldots, k_f$, either controlled or autonomously, the formation graph undergoes the transformation

$$G(0) \to G(f).$$

Let us proceed to represent such a transformation by introducing the graph of graphs. Let $\mathcal{G}$ be the graph whose vertex set $V(\mathcal{G})$ consists of all the subgraphs of K_n. The edge set of $\mathcal{G}$ is now defined with respect to the system (4.2): there is an edge between vertices G_i and G_j of $\mathcal{G}$ if they are neighboring graphs with respect to (4.2) and the class of admissible control sequences. (see Figure 5). As pointed out previously, in this paper we proceed to view $\mathcal{G}$ as an undirected graph.

Suppose that we are given a graph property $\mathcal{P}$ (e.g., connectedness) that is to remain invariant under the formation controlled dynamics. First, let us remove the vertices of $\mathcal{G}$ that do not satisfy the property $\mathcal{P}$ (and all the edges incident on them). Denote this modified graph by $\mathcal{G}_{\mathcal{P}}$. We say that the graph G_f is $\mathcal{P}$-reachable

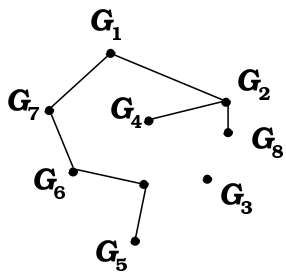

Figure 5: A graph of graphs, $\mathcal{G}$, for three spacecraft

from G_0 if there is a walk from G_0 to G_f on $\mathcal{G}_{\mathcal{P}}$ (Note once more that $\mathcal{P}$ in this paper will be equivalent to "connectedness").

Proposition 4.1 *Suppose that the initial and the final formation states, $x(t_0), x(t_f) \in \mathcal{X}$ (or $x(t_f) \subseteq \mathcal{X}$) are given. Furthermore, suppose that the corresponding formation graphs, G_0 and G_f, satisfy the property $\mathcal{P}$. A necessary condition for the existence of $T \geq 0$ and a control sequence on the indexed time interval $k = 0, 1, \ldots, T$, steering x_0 to x_f, and keeping the corresponding (state-driven) dynamic graph $\mathcal{P}$-invariant is that G_f is $\mathcal{P}$-reachable from G_0 on $\mathcal{G}$.*

Corollary 4.2 *A necessary condition for the existence of an admissible control sequence, steering an arbitrary initial formation state to an arbitrary final one, while preserving the connectedness of the underlying formation graph is that the corresponding graphs belong to the same connected component of $\mathcal{G}_{\mathcal{P}}$.*

We conclude this section by considering two candidate graph transformations that in principle can represent the neighboring graphs. Let G_i be the initial graph of order n and size m. The graph G_{i+1} is called a (single step, edge rotated) transform of G_i, if for some distinct vertices $u, v, w \in V(G_i)$, for which $uv \in E(G_i)$, $uw \notin E(G_i)$, $G_{i+1} \equiv G_i - uv + uw$; the edge uv is said to have been rotated to uw [4] (see Figure 6). In this case we write,

$$G_i \xrightarrow{\text{e}} G_{i+1}$$

Moreover, the graph G_f is said to be reachable from G_0 under edge-rotations,

$$G_0 \overset{\text{e}}{\rightsquigarrow} G_n,$$

if there is a sequence of single step edge rotations of the form

$$G_0 \xrightarrow{\text{e}} G_1 \xrightarrow{\text{e}} \ldots \xrightarrow{\text{e}} G_n.$$

We note that an initial graph G_0 can be transformed to some other graph G_n of the same size (and order) via an edge rotation, if and only if G_0 is neither empty nor complete.

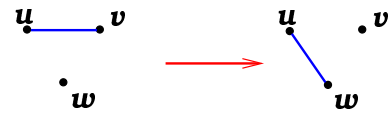

Figure 6: Rotating the edge uv

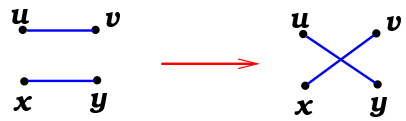

Figure 7: 2-switch

Proposition 4.3 *Let G_0 and G_f be connected graphs having the same order and same size, corresponding to the initial and final formation information graphs. Then G_f is $\mathcal{P}$-reachable from G_0 under edge rotations.*

Proof: The proof essentially builds on the result of [4]. ■

Another graph transformation which might represent the neighboring dynamics is the 2-switch. Let $G_i = (V(G_i), E(G_i))$. For vertices $u, v, x, y \in V(G)$ such that $uv, xy \in E(G_i)$, but $uy, xv \notin E(G_i)$, we say that G_{i+1} is a 2-switch of G_i and write

$$G_i \xrightarrow{\text{s}} G_{i+1},$$

if $G_{i+1} = G_i - uv - xy + uy + xv$ (see Figure 7). Moreover, the graph G_f is said to be 2-switch reachable from G_0,

$$G_0 \overset{\text{s}}{\rightsquigarrow} G_n,$$

if there is a sequence of 2-switches of the form

$$G_0 \xrightarrow{\text{s}} G_1 \xrightarrow{\text{s}} \ldots \xrightarrow{\text{s}} G_n.$$

Proposition 4.4 *Let G_0 and G_f be graphs having the same degree sequence. Then G_f is 2-switch reachable from G_0.*

5 The Laplacian and graph evolutions

Graphs, as combinatorial structures, can be algebraically represented and characterized. This is exemplifies by the elegant matrix theoretic approach to non-enumerative combinatorics, and in particular to graphs and matroids [3]. For a given labeling of the vertex set, $V(G)$, the adjacency matrix of graph $G = (V, E)$, $A(G) = [a_{ij}]$, is defined by

$$a_{ij} := \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $A(G)$ is always symmetric and the sum of its real eigenvalues is equal to zero. It is convenient to think of the adjacency matrix in terms of the map

$$A : \mathcal{G} \to \{0,1\}^{n\times n},$$

mapping graphs of order n to $n \times n$ symmetric matrices on the boolean field; denote by $\mathcal{A}^n$, the image of all such mappings on the labeled graphs of order n. In other words, there is a one to one correspondence between $\mathcal{A}^n$ and $\mathcal{G}^n$.

We now let the allowable levels of the interaction between node i and other nodes of the formation to depend on the formation state x. Naturally, the resulting dynamic interaction can be captured algebraically through an a *dynamic* adjacency matrix $[a_{ij}]$:

$$a_{ij} := \begin{cases} 0 & \text{if } i = j \\ w_{ij}(x(t)) & \text{otherwise.} \end{cases}$$

for a suitable class of symmetric functionals, $w_{ij} = w_{ji}$ $(i, j = 1, \ldots, n)$. A candidate class of w_{ij}'s includes the characteristic function (of a given set)

$$w_{ij}(x(t)) = \mathbf{1}_{S_{ij}}(x(t)), \quad S_{ij} \subseteq \mathcal{X},$$

for $i \neq j$ $(i, j = 1, \ldots, n)$, where $\mathbf{1}_S(x)$ is 1 if $x \in S$ and zero otherwise.

In particular, we can let

$$w_{ij}(x(t)) = \mathbf{1}_S(x_i(t), x_j(t)), \quad S \subseteq X \times X,$$

indicating that the interaction between spacecraft i and j is only a function of their respective states (and not of the others). An example where this model fails to capture the dynamics of the interaction, consider the situation where a third spacecraft is *in the way* of the direct line-of-sight between the two spacecraft.

At the same time, it is both intuitive and mathematically judicious to let these functions be nonnegative rather than being boolean Two elements however are not allowed to interact negatively. In this way, we arrive at another attractive functional class for w_{ij}'s: the distance function (to a given set) [5],

$$a_{ij}(x(t)) = \mathbf{dist}_{S_{ij}}(x(t)), \quad S_{ij} \subseteq \mathcal{X}$$

(more on this example in the final version of the paper). Yet, another alternative is to interpret the entries $a_{ij}(t)$ probabilistically, assuming values on the closed unit interval.

Thereby, at a given time instance t, through the functionals w_{ij}'s, the formation state x maps to the (state-dependent) adjacency matrix, $A(x(t))$. We note that $A(x(t))$ is defined for all t only after specifying the functionals w_{ij}'s, as well as the dynamics expressed by (4.3).

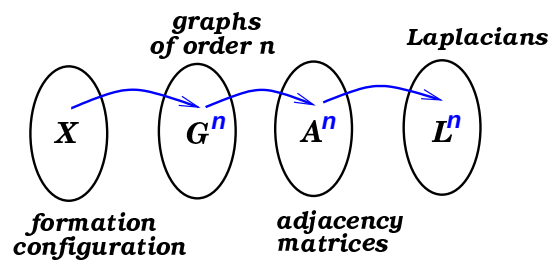

Figure 8: Dynamic graphs in terms of matrix trajectories

Again, we would like to address questions regarding the invariant properties of these dynamic graphs but this time, in the matrix theoretic framework. For this purpose, we found the graph Laplacian to be an ideal algebraic construct. For a graph $G = (V, E)$, let $\Delta(G) = [\delta_{ij}]$ be the degree matrix of graph G- an $n \times n$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of the graph $L(G)$ is defined as the difference between the degree matrix and the adjacency of the graph, i.e.,

$$L(G) := \Delta(G) - A(G).$$

The Laplacian $L(G)$ is always rank deficient and positive semi-definite. In fact, the rank of $L(G)$ is $n - 1$ if and only if G is connected [6]. Similar statements hold when even we consider, instead, the weighted Laplacian of a graph- when the formation information matrix can assume nonnegative entries. The (static) weighted Laplacian for the graph G, $L(G) = [l_{ij}]$, is defined by letting

$$l_{ij} := \begin{cases} \sum_j w_{ij} & \text{if } i = j \\ -w_{ij} & \text{otherwise.} \end{cases}$$

Now given the dynamical system (4.3) and the nonnegative functionals w_{ij}'s, we consider $L(x(t)) = [l_{ij}]$, where

$$l_{ij} := \begin{cases} \sum_j w_{ij}(x(t)) & \text{if } i = j \\ -w_{ij}(x(t)) & \text{otherwise.} \end{cases}$$

As the formation state evolves through (4.2), one obtains a sequence of positive semi-definite matrices $L(x(t))$; in fact, we obtain a matrix trajectory in $\mathcal{L}^n$, the space of $n \times n$ graph Laplacians. The new definition of a dynamic graph can now proceed in terms of mappings from the formation configuration space to the subset of positive semi-definite matrices (see Figure 8). It is instructive at this point to recall the analogy of these graph matrix trajectories with those defined by Riccati difference equations.

Consider first the connectedness of the graph defined through the autonomous dynamical system (4.3). In view of §4 , the following is immediate.

Proposition 5.1 *The dynamic graph is connected if and only if f (4.3) leaves the set*

$$\mathcal{X} \supseteq \mathcal{C} := \{x : \textbf{rank } L(x) = n-1\} \tag{5.7}$$

invariant.

Let us call the set of control sequences that take the initial formation state x_0 to a desired state x_f, and leave all the intermediate information graphs connected, information-friendly. For a controlled formation with a time evolution described by (4.2), we thereby arrive at the following observation.

Corollary 5.2 *The set of information-friendly control sequences for the formation (4.2) are exactly those that leave $\mathcal{C}$ (5.7) invariant.*

Let us elaborate on a direct implication of Corollary 5.2. For this purpose, consider again (4.2)

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots, T.$$

The characterization of all control sequences that take a given initial state x_0 to a desired final state x_T, can be achieved through the nonlinear equation of the form

$$x_T = f(\ldots f(f(x_0, u_0), u_1)) = h(x_0, u_0, \ldots, u_{T-1})$$

for some appropriately defined function h. Of course, we can append to this nonlinear equation, constraints on the state as well as controls, leading to a general nonlinear program. More to the point however, is the addition of the constraints of the form

$$\textbf{rank } L(f(x_k, u_k)) = n-1,$$

to the nonlinear program to ensure the connectedness the formation information graph during the corresponding formation maneuver. The one-step version of this problem, characterizing all the control actions which leave the formation connected at the next time index, can in fact be written in terms of an optimization problem over the space of symmetric matrices.

Proposition 5.3 *Suppose that the formation dynamics described through the discrete control dynamic system (4.2). Define the set*

$$\mathcal{Z} := \{Z \,|\, E_{ii} \bullet Z - \sum_j z_{ij} = 0, z_{ij} - w_{ij}(f(x_k, u)) = 0,$$
$$\frac{1}{2}(E_{ij} + E_{ji}) \bullet Z - z_{ij} = 0 \ (i \neq j), \ Z - Z^T = 0\}.$$

Suppose that the formation information graph is connected at index k. Then the control input u_k leads to a connected graph at $k+1$ if and only if the maximal rank of the matrices in $\mathcal{Z}$ is $n-1$.

Acknowledgements

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