Quadratically Constrained Attitude Control via Semidefinite Programming

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***Abstract*— We consider the attitude control problem as augmented with an arbitrary number of nonconvex quadratic constraints. By utilizing the implicit magnitude constraint of the state vector (i.e., quaternion), we are able to reduce this problem to a convex quadratically constrained quadratic program or a semi-definite program. This result leads to an efficient solution strategy for general quadratically constrained attitude control problems. An example demonstrates the proposed methodology.**

I. Introduction

Consider a science space mission with the task of collecting images of distant stars and other heavenly bodies. In such a mission, the spacecraft is often required to change its orientation (attitude) in some optimal manner while ensuring that its sensitive optical instruments are not exposed to very bright objects in the sky, such as the sun. A control problem that is of great practical importance is thus to determine the necessary control torques such that the cones emanating from these sensitive optical devices exclude the bright object(s) during the re-orientation maneuver. This problem, which we refer to as the Quadratically Constrained Attitude Control (Q-CAC), can in principle be formulated as an optimal nonlinear control problem subject to nonconvex quadratic constraints.

The Q-CAC problem has arisen in the context of many space science missions [1]. Efficient solution strategies for the Q-CAC have not been available so far due to its perceived computational difficulty and possible membership in the class of NP-hard problems.[1] In fact, the two sources of problem complexity- nonlinearity of the attitude dynamics and nonconvexity of the associated quadratic constraints- have often been considered separately. In this venue, the goal has been to solve either the nonconvex quadratically constrained control problem for linear plants [7], or the linearly constrained control problems for the nonlinear plants [4], [6], [14], [16], or even the more promising linearly constrained control problems for linear plants [5], [10]. We particularly note the work of Yakubovich [17] on the quadratically constrained optimal control problems and employing S-procedure and the multiplier method as the core of its solution strategy. Meanwhile, motivated by the new advances in the computational methods for solving optimization problems defined over matrix spaces, a wide range of control problems have been approached via the theory of linear matrix inequalities (LMIs) and semidefinite programming (SDP) [3], [15]. Specifically, we note the successful LMI approaches to resolve optimization problems with a limited number of quadratic constraints, as reported for example in [2], [13].

In the present paper we employ an SDP approach for the general Q-CAC problem, by including, and in fact exploiting, both the nonlinear and nonconvex features in the problem. Specifically, our main contribution is an SDP-based solution strategy for Q-CACs augmented with an *arbitrary* number of nonconvex quadratic constraints. In retrospect, we believe that the underlying ideas of our approach will contribute to the development of new solution methods for a much wider class of constrained control problems beyond that of the attitude control.

[1]The researchers and engineers who *need* to solve this class of problems have nevertheless proposed creative solution strategies for their solution [1].

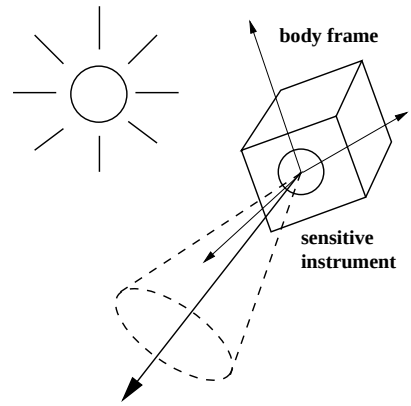

Fig. 1. Sun-avoidance constraint

The outline of the paper is as follows. In §2 we formalize the Q-CAC problem; §3 is devoted to the main result, where we present the general convexification procedure for handling nonconvex quadratic attitude constraints. We then proceed to compute the required control torques via an SDP-based procedure for the discretized Q-CAC problem. A numerical example in §4 concludes our presentation.

First, a few words on the notation. The space of real numbers, real n dimensional vectors, $m \times n$ real matrices, $n \times n$ symmetric matrices, and $n \times n$ positive semi-definite matrices, will be denoted by $\mathbf{R}$, $\mathbf{R}^n$, $\mathbf{R}^{m\times n}$, $\mathcal{S}^n$, and $\mathcal{S}^n_+$, respectively. I_n and $0_{m\times n}$ will denote the $n \times n$ identity and the $m \times n$ zero matrices; **Diag** (x) will designate the square diagonal matrix of appropriate dimensions with the vector x on its diagonal. The notation $\|q\|$ designates the 2-norm of vector $q \in \mathbf{R}^n$; $\langle A, B\rangle$ is the inner product for square symmetric matrices A and B, defined as the trace of their matrix product.

II. Problem Statement

The formal Q-CAC problem considered in this paper is as follows:[2] find the control torques $u_i(t) \in \mathbf{R}^3$ $(i = 1, 2, 3)$, over the time interval $t \in [t_0, t_f]$, subject to the initial and the terminal conditions on the angular velocity vector $\omega(t)$ and the attitude quaternion $q(t)$,

$$(\omega(t_0), q(t_0)) \quad \text{and} \quad (\omega(t_f), q(t_f)),$$

with

$$\begin{aligned} \omega(t) &= [\,\omega_1(t), \omega_2(t), \omega_3(t)\,]^T, \\ q(t) &= [\,q_1(t), q_2(t), q_3(t), q_4(t)\,]^T, \end{aligned}$$

the rigid body equations of motion,

$$J_1\,\dot\omega_1(t) - (J_2 - J_3)\,\omega_2(t)\,\omega_3(t) = u_1(t), \tag{2.1}$$

$$J_2\,\dot\omega_2(t) - (J_3 - J_1)\,\omega_3(t)\,\omega_1(t) = u_2(t), \tag{2.2}$$

$$J_3\,\dot\omega_3(t) - (J_1 - J_2)\,\omega_1(t)\,\omega_2(t) = u_3(t), \tag{2.3}$$

[2]As stated, Q-CAC is a constrained, rather than an optimal, control problem.

(J_i being the body inertia along axis i), boundedness of the angular velocities and control torques,

$$|\omega_i(t)| \leq \gamma_1 \quad \text{and} \quad |u_i(t)| \leq \gamma_2, \quad i = 1, 2, 3, \ \forall t$$

(given the constants $\gamma_1, \gamma_2 > 0$), norm preserving kinematic constraint

$$\dot{q}(t) = \frac{1}{2}\,\Omega(t) q(t), \tag{2.4}$$

with

$$\Omega(t) = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) & \omega_1(t) \\ -\omega_3(t) & 0 & \omega_1(t) & \omega_2(t) \\ \omega_2(t) & -\omega_1(t) & 0 & \omega_3(t) \\ -\omega_1(t) & -\omega_2(t) & -\omega_3(t) & 0 \end{bmatrix},$$

guaranteeing that

$$\|q(t)\| = 1 \quad \text{for} \quad t \in [t_0, t_f],$$

and finally, the attitude constraints

$$q(t)^T\, \widetilde{A}_i(x, y, \theta)\, q(t) \leq 0, \quad \text{for} \quad i = 1, \ldots, m, \tag{2.5}$$

with

$$\widetilde{A}_i(x, y, \theta) = \widetilde{A}_i = \begin{bmatrix} A_i & b_i \\ b_i^T & d_i \end{bmatrix} \in \mathbf{R}^{4\times 4}, \tag{2.6}$$

and

$$\begin{aligned} A_i &:= x_i y_i^T + y_i x_i^T - (x_i^T y_i + \cos\theta)\, I_3, \\ b_i &:= y_i \times x_i, \quad d_i := x_i^T y_i - \cos\theta. \end{aligned}$$

To see how constraints of the form (2.5) specify the exclusion zones in the attitude space- cones emanating from the spacecraft's sensitive instruments that need to exclude the bright objects in the sky during the maneuver - one proceeds as follows [1]. First consider the unit celestial vector x (specified in the inertial coordinates) and the unit body vector y (specified in the body coordinates).[3] We would like the time evolution of the cone with a half-angle θ around the inertially represented vector y, y_I, to exclude x at all times. Enforcing the inequality

$$x^T y_I \leq \cos\theta, \quad \theta \in (0, \pi), \tag{2.7}$$

guarantees that in fact a minimum angular separation θ is maintained between the two vectors x and y_I (see Fig. 2).[4] Inequality (2.7) is now translated in terms of the constraints on the orientation of the spacecraft. Since y_I represents y in the inertial coordinate frame in (2.7) it satisfies

$$y_I = y - 2(q_0^T q_0) y + 2(q_0^T y) q_0 - 2q_4 (y \times q_0), \tag{2.8}$$

where the spacecraft attitude quaternion is represented by

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ -- \\ q_4 \end{bmatrix} = \begin{bmatrix} q_0 \\ -- \\ q_4 \end{bmatrix};$$

see for example [8], [11]. We now expand the quadratic form (2.5),

$$\begin{aligned} 2(q_0^T y)(q_0^T x) - (q_0^T q_0)(x^T y) + q_4^2 (x^T y) + 2q_4 q_0^T (y \times x) \\ \leq (q_0^T q_0 + q_4^2) \cos\theta, \end{aligned}$$

[3]For notational brevity, we will drop the subscript i.

[4]Note that there is no need to consider the case of $\theta \in [\pi, 2\pi]$.

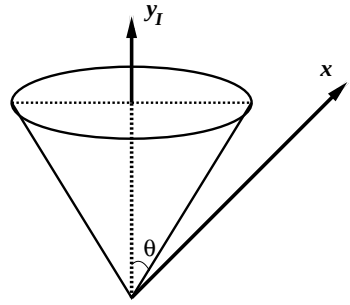

Fig. 2. The exclusion zone

which can be written as

$$\begin{aligned} x^T y - 2(q_0^T q_0)(x^T y) + 2(q_0^T y)(q_0^T x) - 2q_4 (x \times y)^T q_0 \\ \leq \cos\theta; \end{aligned}$$

in view of (2.8), this last expression is equivalent to (2.7).

III. Main Result

In this section, we will consider finding the required control torques in the Q-CAC problem of §2. In this venue, we will first state and prove Proposition 3.1. We then proceed to discretize the kinematic and dynamic constraints (2.1)-(2.4) before employing the proposition in the context of constrained attitude control problem.

Let

$$\mathcal{B}_\eta := \{x \in \mathbf{R}^n \mid \|x\| = \eta\}, \tag{3.9}$$

where η is a nonnegative real number.

Proposition 3.1: Given the matrices $W_i \in \mathcal{S}^n$, $b_i \in \mathbf{R}$, $i = 1, 2, \ldots, m$, and $\eta > 0$, let

$$\mathcal{W} := \{x \in \mathcal{B}_\eta \mid x^T W_i x \geq b_i, \ i = 1, \ldots, m\}, \tag{3.10}$$

be nonempty. Then a feasible element of $\mathcal{W}$ can be found by solving a magnitude constrained SDP.

Proof: Our strategy is to convexify the quadratic constraints

$$x^T W_i x \geq b_i, \quad \text{for} \quad i = 1, \ldots, m.$$

In view of (3.9), determining an element $x \in \mathcal{W}$ (3.10) is equivalent to finding a rank one feasible matrix, $X = xx^T$, for the inequalities

$$\langle W_i, X \rangle \geq b_i, \quad i = 1, \ldots, m, \tag{3.11}$$

$$\langle I_n, X \rangle = \eta^2. \tag{3.12}$$

For $i \in \{1, \ldots, m\}$, let

$$W_i = \widehat{W}_i + \lambda_1 I_n, \tag{3.13}$$

where λ_1 is greater than the largest eigenvalue of W_i. Note that $-\widehat{W}_i \in \mathcal{S}_+^n$ (transformations similar to (3.13) have appeared in the design of optimization algorithms, specifically, in the context of solving the trust region subproblem [12]).[5]

Thus,

$$\begin{aligned} \langle W_i, X \rangle &= \left\langle \widehat{W}_i + \lambda_1 I_n, X \right\rangle \\ &= \left\langle \widehat{W}_i, X \right\rangle + \lambda_1 \langle I_n, X \rangle \\ &= \left\langle \widehat{W}_i, X \right\rangle + \lambda_1 \eta^2 \\ &\geq b_i. \end{aligned}$$

[5]This similarity was pointed out to us by V. Balakrishnan.

Hence (3.11)-(3.12) is feasible if and only if the set defined by inequalities

$$\left\langle -\widehat{W}_i, X \right\rangle \quad \leq \quad \lambda_1 \eta^2 - b_i, \; i = 1, \ldots, m, \tag{3.14}$$

$$\langle I_n, X \rangle \quad = \quad \eta^2, \tag{3.15}$$

is feasible. Since $X = xx^T$, the inequality (3.14) is equivalent to having

$$\mu_1 (p_1^T x)^2 + \ldots + \mu_n (p_n^T x)^2 \leq \lambda_1 \eta^2 - b_i, \tag{3.16}$$

where $-\widehat{W}_i = PDP^T$,

$$P = [\, p_1, \; \ldots, \; p_n \,]^T \quad (p_i \in \mathbf{R}^n),$$

and

$$D = \mathbf{Diag}\, ([\, \mu_1, \; \ldots, \; \mu_n \,]^T)$$

($\mu_j \geq 0, j = 1, \ldots, n$). The quadratic inequality (3.16) however is convex, assuming an equivalent LMI representation

$$\begin{bmatrix} I_n & l(x) \\ l(x)^T & \lambda_1 \eta^2 - b_i \end{bmatrix} \geq 0, \tag{3.17}$$

with

$$l(x) := [\, \sqrt{\mu_1}\, p_1^T x, \; \ldots \; , \sqrt{\mu_n}\, p_n^T x \,]^T.$$

Since each of the inequality constraints defining the set $\mathcal{W}$ can be reformulated as convex LMI constraints in the variable x in a similar manner, the statement of the proposition now follows. ■

We note that the proof of Proposition 3.1 also implies that if the set $\mathcal{B}_\eta$ consists of the elements satisfying

$$\|x\| \geq \eta$$

for some real number η, then (3.17) will be a sufficient, but not necessary, condition for guaranteeing the inequality

$$x^T W_i x \geq b_i.$$

Proposition 3.1 states that once the magnitude constraint of the unknown variable is satisfied automatically, any set of nonconvex quadratic inequalities in this variable can be transformed to a set of LMIs. The metamorphosis does not completely eliminate nonconvexity, however, as the relation that restrict the magnitude of a constrained vector is in fact, nonconvex. Nevertheless, as we show shortly, for the *attitude* constrained control problem, this possible complication does not arise; the constrained state (the unit quaternion) is always guaranteed to belong to set the $\mathcal{B}_1$.

A. Discretization

In this section, we discretize the original continuous-time Q-CAC problem via Euler's first method. The original Q-CAC problem is subsequently transformed to finding the control torques $u_i(k)$ ($i = 1, 2, 3$) for $k = 0, \ldots, N-1$, such that given the initial and terminal conditions

$$(\omega(0), \, q(0)) \quad \text{and} \quad (\omega(N), \, q(N)),$$

with

$$\begin{aligned} \omega(k) &= [\, \omega_1(k), \omega_2(k), \omega_3(k)]^T, \\ q(k) &= [\, q_1(k), q_2(k), q_3(k), q_4(k) \,]^T, \end{aligned}$$

the objective

$$\|\omega(k) - \omega(N)\|^2 + \|q(k) \otimes q(N)^{-1}\|^2 \tag{3.18}$$

is minimized. The constraints of this optimization problem are as follows: the kinematic and dynamic constraints,[6]

$$\begin{aligned} F(k)x(k) &= y(k), \\ |G_1 x(k)| &\leq \gamma_1 \, [1 \; 1 \; 1]^T, \\ |G_2 x(k)| &\leq \gamma_2 \, [1 \; 1 \; 1]^T, \end{aligned}$$

with

$$F(k) = \left[\begin{array}{c|c|c} -I_3 & J & 0_{4\times 4} \\ \hline 0_{3\times 3} & Q(k) & I_4 \end{array} \right],$$

where

$$J := \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix},$$

$$Q(k) := \frac{\Delta t}{2} \begin{bmatrix} -q_4(k) & q_3(k) & -q_2(k) \\ -q_3(k) & -q_4(k) & q_1(k) \\ q_2(k) & -q_1(k) & -q_4(k) \\ q_1(k) & q_2(k) & q_3(k) \end{bmatrix},$$

$$\begin{aligned} G_1 &:= \begin{bmatrix} I_3 & 0_{3\times 7} \end{bmatrix}, \\ G_2 &:= \begin{bmatrix} 0_{3\times 3} & I_3 & 0_{3\times 4} \end{bmatrix}, \end{aligned}$$

$$x(k) := \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ w(k+1) \\ q(k+1) \end{bmatrix},$$

$$y(k) := \begin{bmatrix} J_1 w_1(k) + \Delta t \, (J_2 - J_3) \, \omega_2(k) \, \omega_3(k) \\ J_2 w_2(k) + \Delta t \, (J_3 - J_1) \, \omega_3(k) \, \omega_1(k) \\ J_3 w_3(k) + \Delta t \, (J_1 - J_2) \, \omega_1(k) \, \omega_2(k) \\ w(k) \\ q(k) \end{bmatrix},$$

(the sampling time Δt is assumed to be fixed), and the attitude constraints

$$\begin{aligned} q(k+1)^T \, \widetilde{A}_i \, q(k+1) \quad &= \quad x(k)^T H_i x(k) \leq 0, \\ & \qquad i = 1, \ldots, m, \end{aligned} \tag{3.19}$$

where

$$H_i = \begin{bmatrix} 0_{6\times 6} & 0_{6\times 4} \\ 0_{4\times 6} & \widetilde{A}_i \end{bmatrix},$$

and $\widetilde{A}_i$ is as defined in (2.6). The second expression in (3.18)[7] is employed in our performance index for the re-orientation problem as it accurately represents the error between the current attitude $q(k)$ and the desired attitude $q(N)$. In fact,

$$\begin{aligned} q(k) \otimes q(N)^{-1} \quad &= \quad q(k) \otimes \begin{bmatrix} -q_1(N) \\ -q_2(N) \\ -q_3(N) \\ q_4(N) \end{bmatrix} \\ &= \quad Q(N)\, q(k), \end{aligned}$$

where

$$Q(N) := \begin{bmatrix} q_4(N) & q_3(N) & -q_2(N) & -q_1(N) \\ -q_3(N) & q_4(N) & q_1(N) & -q_2(N) \\ q_2(N) & -q_1(N) & q_4(N) & -q_3(N) \\ q_1(N) & q_2(N) & q_3(N) & q_4(N) \end{bmatrix}.$$

[6]The notation $|x|$ refers the vector $[\, |x|_1, \ldots, |x|_n \,]^T$.

[7]The notation is adopted from [11].

B. Control torques

For the index set $k = 0, \ldots, N-1$, we consider the quadratically constrained quadratic program

$$\min_{x(k)} \quad \begin{bmatrix} x(k) \\ 1 \end{bmatrix}^T E(k) \begin{bmatrix} x(k) \\ 1 \end{bmatrix}, \tag{3.20}$$

subject to

$$\begin{aligned} F(k)x(k) &= y(k), & (3.21)\\ |G_1 x(k)| &\leq \gamma_1 [1\ 1\ 1]^T, & (3.22)\\ |G_2 x(k)| &\leq \gamma_2 [1\ 1\ 1]^T, & (3.23)\\ x(k)^T H_i x(k) &\leq 0, \quad i = 1, \ldots, m, & (3.24) \end{aligned}$$

where

$$\begin{aligned} E(k) &= \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1} \\ 0_{3\times3} & I_3 & 0_{3\times4} & \omega(N) \\ 0_{4\times3} & 0_{4\times3} & 0_{4\times4} & 0_{4\times1} \\ 0_{1\times3} & \omega(N)^T & 0_{1\times4} & \omega(N)^T\omega(N) \end{bmatrix} \\ &+ \begin{bmatrix} 0_{6\times6} & 0_{6\times4} & 0_{6\times1} \\ 0_{4\times6} & Q(N)^T Q(N) & 0_{4\times1} \\ 0_{1\times6} & 0_{1\times4} & 0_{1\times1} \end{bmatrix} \in \mathcal{S}_+^{11}, \end{aligned}$$

$F(k) \in \mathbf{R}^{7\times10}$, $G_1, G_2 \in \mathbf{R}^{10\times10}$, and $H_i \in \mathcal{S}^{10}$. As the *linear* constraints (3.21)-(3.23) guarantee that $\|q(k)\| \approx 1$ at each time index k for small values of Δt, Proposition 3.1 now implies that the spacecraft constrained control attitude problem, as augmented with an *arbitrary* number of nonconvex quadratic constraints, can accurately be represented by a (convex) SDP [15]. The convex representation remains valid as long as the errors introduced by linearization are negligible; otherwise, the attitude quaternion is not guaranteed to remain close to a unit vector at each time step. As it pertains to this last remark, we observe that the discretization of (2.4) has the form

$$q(k+1) = \{I_4 + \frac{1}{2}\Delta t\, \Omega(k)\}\, q(k),$$

and thereby

$$\begin{aligned} \|q(k+1)\|^2 &= q(k)^T \{I_4 + \frac{1}{2}\Delta t\, (\Omega(t) + \Omega(t)^T) \\ &+ \frac{1}{4}\Delta t^2 \Omega(t)^T \Omega(t)\}\, q(k) \\ &= \|q(k)\|^2 \\ &+ \frac{1}{4}\Delta t^2 q(k)^T \Omega(t)^T \Omega(t) q(k) \\ &\geq \|q(k)\|^2, \end{aligned} \tag{3.25}$$

since

$$\Omega(t) + \Omega(t)^T = 0.$$

The inequality (3.25), along with the initial condition $\|q(0)\| = 1$, implies that

$$\|q(k)\| \geq 1, \quad \text{for all } k \geq 1. \tag{3.26}$$

By our earlier observation proceeding Proposition 3.1, the SDP approach does still guarantee that the attitude exclusion zone (3.24) is not violated despite errors introduced by the linearization. Nonetheless, as the linearization errors grow, the SDP representation becomes more conservative. The sufficiency of such the approach can be made to be nonrestrictive as long as a proper updating method is used to propagate the attitude [9], or in view of (3.26), by including some positive multiple of

$$\|q(k+1)\|^2$$

in the objective functional (3.20).

IV. An Example

We now revisit our motivational example in §1 where a spacecraft is to collect images of several stars without exposing its on-board sensitive instruments to the bright objects in the sky. To simplify our presentation, we assume that only one quadratic constraint-associated with the sun- is present. The physical constants and the initial and terminal conditions used in our example are as follows: spacecraft mass, 1 kg, principle axes of inertia,

$$\begin{aligned} J &= \mathbf{Diag}\,([\,J_1, J_2, J_3\,]^T) \\ &= \mathbf{Diag}\,([\,100, 200, 300\,]^T) \text{ kg m}^2, \end{aligned}$$

initial angular velocity,

$$\omega(t_0) = [\,0, 0, 0\,]^T \text{ rad/sec},$$

final angular velocity,

$$\omega(t_f) = [\,0, 0, 0\,]^T \text{ rad/sec},$$

initial attitude quaternion,

$$q(t_0) = [\,0.5000, 0.5000, 0.5000, 0.5000\,]^T,$$

final attitude quaternion,

$$q(t_f) = [\,0.0258, 0.0258, 0.9990, 0.0258\,]^T,$$

the sun vector in the inertial coordinates,

$$x = [\,0, 0, 1\,]^T,$$

the camera vector in the body coordinates,

$$y = [\,0.750, 0.433, 0.500\,]^T,$$

and finally, the required angular separation, $\theta = 50$ deg. The initial and the desired final attitude quaternions have been chosen to satisfy (3.19), i.e., the sun is outside the constraint cone emanating from the instrument's bore-sight at the initial and terminal spacecraft configurations. For the purpose of our example, it is convenient to introduce the notion of Q-space consisting of vectors of the form

$$\{\, [\,\frac{q_1}{q_4}, \frac{q_2}{q_4}, \frac{q_3}{q_4}\,]^T \,|\, q \text{ a quaternion}\}.$$

Fig. 3 depicts the feasible trajectory of the spacecraft in Q-space (solid line) as found via the SDP solution strategy for our example (the exclusion zone is represented by the meshed surface extended to infinity). The figure depicts the spacecraft attitude trajectory evolving along the lower meshed surface, following the boundary of the constraint surface, and then reaching the final attitude without violating the sun angle constraint. In this example, as a remedy for the errors introduced by linearization (see §3.2), the term $\|q(k+1)\|^2$ has been added to the objective functional at each time step in the corresponding optimization problems.

V. Concluding Remarks

It is generally believed that spacecraft attitude dynamics and control problems are an order of magnitude more challenging than those related to its translational degrees of freedom. In this paper, we have explored this idea in the reverse direction, showing that the attitude control problem when augmented with an arbitrary number of nonconvex quadratic constraints on the attitude variable, can be represented by a convex optimization problem. An analogous statement for the translational degrees of freedom is of course not valid. We note that the approach proposed in this paper is potentially applicable to quadratically state constrained dynamic systems with an implicitly enforced magnitude constraint on the state vector.

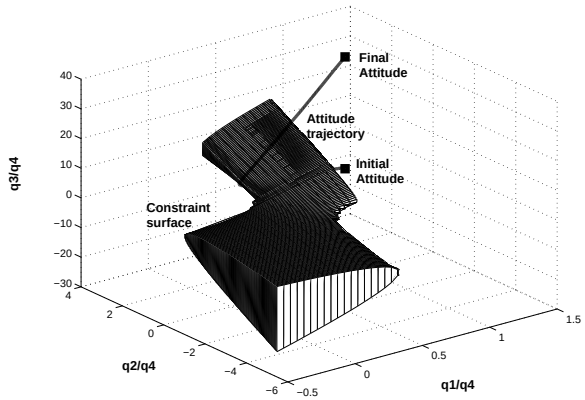

Fig. 3. The attitude trajectory (solid line) in Q-space; the constrained set is outside of the meshed surface

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