AA 578

Optimization and System Sciences

Aeronautics & Astronautics
University of Washington
Winter 2005

Mehran Mesbahi

AA 578
Optimization and Systems Sciences
Mehran Mesbahi
Office: Gug. 315C, Tel: 206-543-7937
Email: *mesbahi@aa.washington.edu*
Class Time: T/Th: 10:30-11:50 am, Room: LOEW 222
Class website: *www.aa.washington.edu/faculty/mesbahi/courses/optimization/*

Introduction

This course is on finite dimensional- mostly convex- optimization with applications in engineering- mostly control, system theory, and estimation. Optimization theory is a rich subject, with many deep theoretical results and a wide array of applications. The theory contains areas that are quite distinct: combinatorial optimization, integer programming, linear and nonlinear programming (mathematical programming), calculus of variations, and optimal control. The main mathematical tools used in combinatorial optimization are enumerative and non-enumerative combinatorics (and thus the terminology). In the case of calculus of variation and optimal control, the tools are mostly functional analytic (along with topology, non-smooth analysis, etc.). In the case of mathematical programming, we rely heavily on analysis, geometry, and linear algebra. Linear programming is distinguished in this classification- it seems to be right on the boundary of combinatorial and continuous optimization.

The textbook for the course is *Convex Optimization* by Boyd and Vandenberghe, subsequently refereed to as [BV]. We will have reading assignments from [BV]; most of the assigned homework will also be selected from it. Bi-weekly homework assignments are an integral part of the course: it will count for 65% of the final grade. We will also have a class project which will contribute 35% to your final grade. At this point, I will reserve the option of giving you a midterm as well- in that case- the grade break down will look like: homework (40% of the final grade), midterm (30% of the final grade) and project (30% of the final grade). The date of the exam (if there is one) will be determined later.

Topics

I will cover the following topics:

1. introduction and background: examples, analysis, linear algebra, algorithms
2. convex sets, separation theorems, theorem of alternatives, convex functions, convex optimization
3. duality and its applications
4. linear and semi-definite programming; linear matrix inequalities
5. optimization algorithms and software
6. applications in systems, control, estimation, combinatorics

Depending on the available time, we might also be able to touch upon one or more of the following topics: (1) sums-of-squares, (2) model-predictive control, (3) non-smooth optimization, (4) fixed point theory.

Some suggested references on classical analysis and linear algebra are as follows: (1) *Finite Dimensional Vector Spaces* by P. Halmos (2) *Matrix Analysis* by R. A. Horn and C. Johnson, (3) *Principles of Mathematical Analysis* by W. Rudin.

Lecture Notes

Lecture notes for the week are posted on the class website on Fridays following the lectures. The purpose of the notes is to summarize, for the ease of studying or reference, the main points of the lecture. They are not meant to substitute for the reading assignments, nor are they guaranteed to be without errors; however, I will try to make them as error-free as possible.

Notation

See the Appendix A of [BV]; there might be few differences between the notation used in the class and [BV]. I will point them out as the occasion arise. To summarize, I will use $\mathbf{R}$: real number; $\mathbf{R}_+$: nonnegative real numbers; $\mathbf{R}_-$: non-positive real numbers; $\mathbf{C}$: complex numbers; $\mathbf{R}^{p\times q}$: $p \times q$ matrices with real entries; $\mathcal{S}^n$: $n \times n$ symmetric matrices; $\mathcal{S}^p_+$: their positive semi-definite subset; $\mathcal{S}^p_{++}$: their positive definite subset; $\alpha, \beta, \gamma, \ldots$: scalars; $a, b, c, d, \ldots, x, y, z, w, \ldots$: vectors; $\mathbf{1}$ is the vector of all ones; $A, B, C, \ldots$: matrices or sets; $X, Y, , \ldots$ matrices (usually symmetric); $A \backslash B$: the set difference between A and B- all the elements in A that are not in B.

A GLIMPSE OF THEORY

1 Week One

Main ideas: Examples, vector spaces, norms, inner product, analysis review, an existence result

1.1 Examples and Motivation

1. Some quotes. Euler: "... nothing at all takes place in the universe in which some rule of maximum or minimum does not appear." Chebyshev: "Most practical questions can be reduced to problems of largest and smallest magnitudes ... it is only by solving these problems that we can satisfy the requirements of practice which always seeks the best, the most convenient." Kepler: "Near the maximum the decrements on both sides are in the beginning only imperceptible." Laplace: " All the effects of nature are only the mathematical consequence of a small number of immutable laws." Lagrange: "I do not know."

2. Heron's problem: Given two points x, y on the same side of a given line, find a point z on the line such that the sum of distances from x to z and z to y is minimum (symmetry, triangle inequality)- reflection of light and properties of mirrors.

3. Discrete (or discrete version of a continuous) optimal control problems; least squares.

4. Minimum fuel force and thrust generation for spacecraft (from [BV] lecture notes): we are given a rigid planar body with the center of mass located at the origin. We need to find the magnitude of the required forces exerted by the thrusters located at locations $p_i = (p_{ix}, p_{iy})$, each directed at angle θ_i, such that desired total force and torque is generated, while minimizing fuel usage.

$$\min_u \mathbf{1}^T u \quad \text{s.t.} \quad \underbrace{\begin{bmatrix} \cos\theta_1 & \dots & \cos\theta_n \\ \sin\theta_1 & \dots & \sin\theta_n \\ p_{1y}\cos\theta_1 - p_{1x}\sin\theta_1 & \dots & p_{ny}\cos\theta_n - p_{nx}\sin\theta_n \end{bmatrix}}_{F} u = \underbrace{\begin{bmatrix} f_x^d \\ f_y^d \\ \tau^d \end{bmatrix}}_{f^d}, \ 0 \le u \le \mathbf{1}.$$

5. How about having thruster that can fire in both directions:

$$\min_u \sum |u_i| \quad \text{s.t.} \quad Fu = f^d, \ |u_i| \le 1.$$

6. How about minimizing the number of thrusters needed to generate the required force and torque:

$$\min \text{ (number of thrusters on)} \quad \text{s.t.} \quad Fu = f^d, \ 0 \le u \le \mathbf{1}.$$

7. Combinatorial optimization; Max-Cut.

8. Some terminology: feasible, infeasible, unbounded, active and inactive constraints, local and global, ϵ-suboptimal, min versus max, slack variables, functional or black-box description of objective/constraints, convex and non-convex optimization, NP-hard, polynomial-time solvable, integer programming.

1.2 Background

Finite dimensional optimization heavily uses tools from three branches of mathematics: analysis, geometry, and linear algebra. We will start with some linear algebra, add geometry, and then review the key analytic notions that are subsequently used in the course.

It seems that in engineering, we can not really escape from an algebraic structure called a vector space.

1.2.1 Algebra and Linear Algebra

1. For a set **F** to be a field, it has to be closed under two binary operations (addition and multiplication), both operations must be associative, commutative, and have distinct identity elements in **F** ; additive inverses exist and multiplicative inverses exist except for the additive identity, and multiplication operation must be distributive over the addition. We will mainly work with the field of real numbers **R** , and occasionally with the field of complex numbers **C** .

2. A vector space over **F** (scalars) is a set of objects, called vectors, which is closed under a binary operation (addition) that is associative, commutative, and has an identity element; moreover, for all $\alpha, \beta \in$ **F** and all $x, y \in$ **V**, $\alpha(x+y) = \alpha x + \alpha y, (\alpha+\beta)x = \alpha x + \beta x, \alpha(\beta x) = (\alpha\beta)x$ and $ex = x$, where $e \in$ **F** is the multiplicative identity.

3. Let $S \subseteq$ **V**. The span of S is the set all possible linear combinations of the vectors in S; S is called a linear independent set (or a set with are linearly independent vectors) if none of its elements can be expressed as a linear combination of the others (otherwise it is called linearly dependent). A basis for **V** is a linearly independent set whose span is **V** . If **V** admits a finite basis, it is called finite dimensional.

1.2.2 Geometry

1. Let **V** be a vector space over **R**. A function

$$\langle ., . \rangle : \mathbf{V} \times \mathbf{V} \to \mathbf{R}$$

is an inner product if it is symmetric, self-positive ($\langle x, x \rangle = 0$ if and only if $x = 0$), and it is additive (individually, in each of its arguments) and homogeneous (with respect to the scalar multiplication). One has

$$| \langle x, y \rangle |^2 \leq \langle x, x \rangle \ \langle y, y \rangle \,, \tag{1.1}$$

for all $x, y \in$ **V** (Cauchy-Schwarz).

In this course, we will mainly work with **finite dimensional vector spaces over R with an inner product defined on them**- Euclidean spaces. An arbitrary Euclidean space will be denoted by **E**.

2. Let **V** be a vector space over **R**. A function

$$\|.\| : \mathbf{V} \to \mathbf{R}$$

is a norm if it is positive (except when its argument is the zero vector), positive homogeneous (with respect to the scalars),

$$\|\alpha\, x\| = |\alpha|\, \|x\|, \quad \alpha \in \mathbf{R},$$

and satisfies the triangular inequality

$$\|x + y\| \leq \|x\| + \|y\|.$$

A norm is really a topological notion, but we can induce a topology with the aid of geometry ...

3. The inner product induces a (canonical) norm:

$$\|x\| := \langle x, x \rangle^{1/2} \,.$$

4. $\mathbf{E} = \mathbf{R}^n$; $\langle x, y\rangle := x^T y$. $\mathbf{E} = \mathcal{S}^n$; $\langle X, Y\rangle :=$ **Trace** $XY = \sum_i \sum_j X_{ij} Y_{ij} =$ **Trace** YX. Note that the inner product induces a norm on the respective Euclidean space.

5. The unit ball is the set $B = \{x \in \mathbf{E} \,|\, \|x\| \leq 1\}$. The ball of radius r centered at x_c is the set $B(x_c, r) := \{x \in \mathbf{E} \,|\, \|x - x_c\| \leq r\}$.

   We find a good use for some basic set theoretic operations ...

6. For $\Lambda \subseteq \mathbf{R}$ and $C \subseteq \mathbf{E}$
$$\Lambda C := \{\lambda x \,|\, \lambda \in \Lambda,\ x \in C\}.$$

7. The sum of the two sets $C, D \subseteq \mathbf{E}$ is defined by
$$C + D := \{x + y \,|\, x \in C,\, y \in D\}; \quad C - D := C + (-D).$$

8. A set $C \subseteq \mathbf{E}$ is convex if for all $x, y \in C$
$$\lambda x + (1 - \lambda)\, y = y + \lambda(x - y) \in C, \quad \text{whenever} \quad \lambda \in [0, 1].$$

9. A set $C \subseteq \mathbf{E}$ is a cone, if $\mathbf{R}_+ C = C$.

1.2.3 Analysis

From analysis, we need to understand openness, closed-ness, compactness, and lacking or having an interior, etc.:

1. A point x is in the interior of the set $D \subseteq \mathbf{E}$ (**int** D) if there is a real $\delta > 0$ such that $x + \delta B \subseteq D$.

2. x is the limit of a sequence of points $x^1, x^2, \ldots, x^n$ in $\mathbf{E}$, written as
$$x^j \to x \quad \text{as} \quad j \to \infty, \quad \text{if} \quad \|x^j - x\| \to 0 \quad \text{as} \quad j \to \infty.$$

3. The closure of D , cl D, is the set of limits of sequences of points in D. The boundary of D, bd D is cl $D \backslash$**int** D.

4. $D \subseteq \mathbf{E}$ is open if $D =$ **int** D and closed if $D =$ cl D.

5. $D \subseteq \mathbf{E}$ is bounded if there is a real k such that
$$D \subseteq kB.$$

6. The set $D \subseteq \mathbf{E}$ is compact if it is closed and bounded.

7. One of the most important results in classical analysis is the following:

   Theorem 1.1 (Bolzano-Weierstrass) *Bounded sequences in* $\mathbf{E}$ *have convergent subsequences.*

8. Let $D \subseteq \mathbf{E}$ and $f : D \to \mathbf{R}$. The function f is continuous (on D) if
$$f(x^j) \to f(x) \quad \text{when} \quad x^j \to x.$$

9. The set of real numbers is ordered.

1.2.4 Optimization

Finite dimensional optimization theory is concerned with the problem of minimization or maximization of a real-valued function over a subset of a finite dimensional inner product space.

1. Given a set $\Gamma \subseteq \mathbf{R}$, the infimum of Γ ($\inf \Gamma$) is the greatest lower bound on Γ; the least upper bound on Γ is denoted by $\sup \Gamma$ (supremum).

2. To make sure that $\inf$ and $\sup$ always exists (what is the $\inf$ of $\mathbf{R}$?) we append $-\infty$ and $+\infty$ to $\mathbf{R}$; we write $\mathbf{R} \cup \{+\infty\}$ if necessary. One has, by convention, $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

3. Let $D \subseteq \mathbf{E}$ and $f : D \to \mathbf{R}$. Note that $f(D) \subseteq \mathbf{R}$. The global minimizer of f in D is a point $\bar{x}$ where f attains its infimum, i.e., $\bar{x} \in D$ and

$$\inf_D f = \inf\{f(x) \,|\, x \in D\} = f(\bar{x}).$$

   In this case $\bar{x}$ is *optimal solution* of the optimization problem $\inf_D f$.

4. Recall that the level sets of a function $f : D \to \mathbf{R}$ are, for each $\alpha \in \mathbf{R}$,

$$L_f(\alpha) := \{x \in D \,|\, f(x) \leq \alpha\}.$$

   Another important result in analysis, well, it is really in *optimization*, is the following existence result ...

5. **Theorem 1.2 (Weierstrass)** *Suppose that the set $D \subseteq \mathbf{E}$ is nonempty and closed and that all the level sets of the continuous function $f : D \to \mathbf{R}$ are bounded. Then f has a global minimizer (in D).*

   Proof: Since D is nonempty, $\inf_D f < +\infty$. Consider a decreasing sequence $\alpha^i \to \inf_D f$ ($i = 1, 2, \ldots$). Now construct a sequence of vectors $x^i \in \mathbf{E}$ such that $x^i \in L_f(\alpha^i)$; note that

$$f(x^i) \to \inf_D f.$$

   This sequence is bounded: for all $i \geq 1$, $x^i \in L_f(\alpha^i) \subseteq L_f(\alpha^1)$. Thus it has a convergent subsequence (Theorem 1.1), i.e., there exists a increasing sequence $j_1, j_2, \ldots,$ and x^* such that $x^{j_1}, x^{j_2}, \ldots \to x^*$. The set D is closed thus $x^* \in D$. Moreover,

$$f(x^{j_1}), f(x^{j_2}), \ldots \to f(x^*).$$

   Thereby $\inf_D f = f(x^*)$. $\square$

   Note that the proof does not give us a clue on actually finding this global minimizer.

Reading: [BV]: B.1 and B.2, 2.1-2.3, 2.5, 3.1-3.2
Homework # 1: (due 1/23) [BV]: 2.1, 2.3, 2.4, 2.12, 2.20, 2.21, 2.22

2 Week Two

Main ideas: subspaces, convex sets, examples, separation

2.1 Subspaces, affine sets, convex sets, examples

1. Subspaces: $S \subseteq \mathbf{E}$ such that for $x, y \in S$, $\lambda_1 x + \lambda_2 y \in S$ for all $\lambda_1, \lambda_2 \in \mathbf{R}$. The dimension of S is the cardinality of the set of linearly independent vectors in S that span S.

2. Affine sets: $C \subseteq \mathbf{E}$ such that for $x, y \in C$, $\lambda x + (1-\lambda) y \in C$ for all $\lambda \in \mathbf{R}$.

3. If $C \in \mathbf{E}$ is affine and $x_0 \in C$, then $C - x_0 := \{x - x_0 \,|\, x \in C\}$ is a subspace; the dimension of C is the dimension of $C - x_0$ for any $x_0 \in C$.

4. Suppose that you define a property for $C \subseteq \mathbf{E}$. Then the hull of C with respect to this property is the smallest set that contains C that has the property. Examples of *property* is convexity, being affine, etc.; in this way we get co S, **aff** S, ...

5. The affine dimension of a set is the dimension of its affine hull.

6. The relative interior of $C \subseteq \mathbf{E}$ is its interior relative to its affine hull:

$$\textbf{rel int}\, C := \{x \in C \,|\, B(x, \epsilon) \cap \textbf{aff}\, C \subseteq C, \text{ for some } \epsilon > 0\}.$$

7. Intersections of (possibly infinitely many) convex sets is convex: if x and y belong to the intersection, so does $\lambda x + (1-\lambda) y$ for any $\lambda \in [0, 1]$.

8. Halfspaces in $\mathbf{E}$ are parametrized by $0 \neq a \in \mathbf{E}$ and $\beta \in \mathbf{R}$: they assume the form

$$H_{a,\beta} := \{x \in \mathbf{E} \,|\, \langle a, x \rangle \leq \beta\} = \{x \in \mathbf{E} \,|\, L_{a,\beta}(x) \leq 0\}, \tag{2.2}$$

   where $L_{a,\beta}(.) := \langle a, . \rangle - \beta$ defines an affine (linear plus a constant) functional on $\mathbf{E}$.

9. Some examples of convex sets and cones unit balls, second order cone, subspaces, hyperplanes, halfspaces, and ellipsoids.

10. Let $A \in \mathbf{R}^{n \times n}$. Consider the differential equation $\dot{x}(t) = Ax(t)$. Then for any initial condition $x_0 \in \mathbf{R}^n$, one has $x(t) \to 0$, if and only if all eigenvalues of A have negative real parts. This condition is equivalent to having

$$\mathcal{S}^n \supseteq \mathcal{L} := \{P \in \mathcal{S}^n_{++} \mid -A^T P - PA \in \mathcal{S}^n_{++}\} \tag{2.3}$$

    be nonempty (Lyapunov theorem). The set L (2.3) is convex.

2.2 Separation theorem- preliminaries

1. A separating functional for the sets $C, D \in \mathbf{E}$, is a function $f : \mathbf{E} \to \mathbf{R}$, such that $f(x) \leq 0$ on C and $f(x) \geq 0$ on D; if one of these inequalities is strict, then we call f a strictly separating functional.

2. Let $C \in \mathbf{E}$. A strictly separating functional for the set C and a point $y \notin C$ is a function $f : C \to \mathbf{R}$, such that $f(x) \leq 0$ on C but not at y. A strictly separating functional is a one step membership dis-qualifier for a point not in C.

3. Our main objective this week is to prove Theorem 2.2: for a closed convex set $C \in \mathbf{E}$ and any point $y \notin C$, there is a halfspace that contains C but not y.

2.3 Directional derivatives

1. For $\delta > 0$ and $g : (0, \delta) \to \mathbf{R}$, define
$$\liminf_{t\downarrow 0} g(t) := \lim_{t\downarrow 0} \inf_{(0,t)} g(t) \quad \text{and} \quad \limsup_{t\downarrow 0} g(t) := \lim_{t\downarrow 0} \sup_{(0,t)} g(t).$$
The limit $\lim_{t\downarrow 0} g(t)$ exists if and only if the above two expressions are equal.

2. The directional derivative of a real-valued function f at $\bar{x}$, in a direction $d \in \mathbf{E}$, is
$$f'(\bar{x}; d) := \lim_{t\downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t},$$
when this limit exists.

3. When $f'(\bar{x}; d) = \langle a, d\rangle$ for all $d \in \mathbf{E}$ and a vector $a \in \mathbf{E}$, then f is said to be (Gâteaux) differentiable at $\bar{x}$; in this case $\nabla f(\bar{x})$ denotes this vector $a \in \mathbf{E}$.

2.4 Normal cone

1. The normal cone to a convex set C at point $\bar{x}$ is defined as
$$N_C(\bar{x}) := \{d \in \mathbf{E} \mid \langle d, x - \bar{x}\rangle \le 0, \text{ for all } x \in C\}.$$

2. The normal cone to a subset of $\mathbf{E}$ at any one of its points is a closed convex cone.

2.5 First order necessary condition and separation theorem

1. Let $C \subseteq \mathbf{E}$. The vector $\bar{x} \in C$ is a global minimizer of $f : C \to \mathbf{R}$ on C if $f(\bar{x}) \le f(x)$ for all $x \in C$.

2. Let $C \subseteq \mathbf{E}$. The vector $\bar{x} \in C$ is a local minimizer of $f : C \to \mathbf{R}$ on C if $f(\bar{x}) \le f(x)$ for all $x \in C$ close to $\bar{x}$.

3. **Proposition 2.1** *(***First order necessary condition***) Suppose that $C \subseteq \mathbf{E}$ is convex and $\bar{x}$ is a local minimizer of $f : C \to \mathbf{R}$. Then for any point $x \in C$, the directional derivative, if it exists, satisfies*
$$f'(\bar{x}; x - \bar{x}) \ge 0.$$
In particular, if f is differentiable at $\bar{x}$, then one has
$$-\nabla f(\bar{x}) \in N_C(\bar{x}).$$
Proof: If for some $x \in C$, $f'(\bar{x}; x - \bar{x}) < 0$, then $f(\bar{x} + t(x - \bar{x}) < f(x)$ for sufficiently small $t > 0$; since C is assumed to be convex, $\bar{x} + t(x - \bar{x}) \in C$- a contradiction. □

4. **Theorem 2.2** *(***Separation***) Let $C \subseteq \mathbf{E}$ be closed and convex and $y \notin C$. Then there exists a pair $(\beta, a) \in \mathbf{R} \times \mathbf{E}$, $a \neq 0$, such that*
$$\langle a, y\rangle > \beta \ge \langle a, x\rangle \quad \textit{for all} \quad x \in C.$$
Proof: Let $f := \|x - y\|^2/2$. By Theorem 1.2, f has a minimizer on C; $\bar{x}$. By Proposition 2.1 $-\nabla f(\bar{x}) = y - \bar{x} \in N_C(\bar{x})$. Thus $\langle y - \bar{x}, x - \bar{x}\rangle \le 0$ for all $x \in C$. Now let $a := y - \bar{x}$ and $\beta := \langle y - \bar{x}, \bar{x}\rangle$. □

Theorem 2.3 *(***Supporting hyperplane***) Let $C \subseteq \mathbf{E}$ be convex,* **int** *$C \neq \emptyset$ and $\bar{x} \in$ bd C. Then C admits a supporting hyperplane at $\bar{x}$: there exists $a \in \mathbf{E}$ such that $\langle a, x\rangle \ge \langle a, \bar{x}\rangle$ for all $x \in C$.*

3 Week 3

Main ideas: More on separation, first order necessary condition, and their consequences, convex functions

3.1 More on separation and first order necessary condition

1. Let $C \in \mathbf{E}$ be closed and convex. Then C is the intersection of all halfspaces that contain it.

2. In finding the halfspace $H_{a,\beta}$ (2.2) containing a closed convex cone C but not $y \notin C$,

$$\langle a, x\rangle \leq \beta < \langle a, y\rangle, \quad \text{for all } x \in C,$$

we can always set $\beta = 0$: $\beta \geq 0$ since the vector zero belongs to the cone and if $\langle a, x\rangle > 0$ for some $x \in C$, then $\langle a, \alpha\, x\rangle \geq \langle a, y\rangle$ for a sufficiently large $\alpha \geq 0$.

3. Separation theorem Theorem 2.2 can be extended to two convex sets: the following result is known as the Eidelheit Separation Theorem (see also §1.5 of [BV]): let $C, D \subseteq \mathbf{E}$ be convex with C having interior points and D containing no interior point of D. Then there exists a hyperplane that separates C and D.

4. Theorem 2.2 says that a closed convex set admits an affine strictly separating functional corresponding to any point that does not belong to it.

5. Supporting hyperplane:

6. Some examples of normal cones: (a) let $C = [a, b] \subseteq \mathbf{R}$. Then

$$N_C(a) = \mathbf{R}_-,\ N_C(b) = \mathbf{R}_+, \quad \text{and} \quad N_C(\bar{x}) = \{0\} \quad \text{for} \quad 0 < \bar{x} < b;$$

(b) let $C = B$ (the unit ball in $\mathbf{E}$). Then $N_C(\bar{x}) = \mathbf{R}_+\bar{x}$; this follows from the Cauchy-Schwarz inequality (1.1); (c) let C be a subspace, i.e., for $x, y \in C$, $\alpha x + \beta y \in C$ for all $\alpha, \beta \in \mathbf{R}$. Then

$$N_C(\bar{x}) = C^\perp := \{y \in \mathbf{E} \mid \langle x, y\rangle = 0 \text{ for all } x \in C\}.$$

To see this, without loss of generality, let $\langle d, \bar{x}\rangle \geq \langle d, x\rangle > 0$ for some $x \in C$. But for some large enough α, $\langle d, \alpha x\rangle = \alpha\,\langle d, x\rangle > \langle d, \bar{x}\rangle \geq \langle d, \alpha x\rangle$, a contradiction.

7. Rayleigh quotient: Let $f : \mathbf{R}^n \backslash \{0\}$ be a continuous function such that $f(\lambda x) = f(x)$ for all $\lambda > 0$ and $x \neq 0$. Then f has a minimizer. Let $f(x) := x^T Ax/\|x\|^2$ for $A \in \mathcal{S}^n$. What does the necessary conditions, Prop. 2.1, suggest?

3.2 Convex functions

1. Let $C \subseteq \mathbf{E}$ be convex. The function $f : C \to \mathbf{R}$ is convex if for all $x, y \in C$ and all $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)\, y) \leq \lambda f(x) + (1 - \lambda)\, f(y). \tag{3.4}$$

If whenever x, y are distinct and $0 < \lambda < 1$ one has strict inequality in (3.4) f is said to be strictly convex. The function f is called (strictly) concave when $-f$ is (strictly) convex.

2. We will allow f to take on the value $+\infty$, but define its domain as the set of points that it does not use this privilege- dom $f := \{x \in \mathbf{E} \mid f(x) < +\infty\}$- if dom$f \neq \emptyset$ then f is said to be proper. An example of a proper extended-valued function is the indicator function for a nonempty set $C \subseteq \mathbf{E}$: $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. Note that $\inf f_D = \inf f + \delta_D$.

3. Let $f_i : \mathbf{E} \to \mathbf{R}$ be convex. Then $\sup_i f$ is convex.

4. Let $C \subseteq \mathbf{R}^n$ be convex and $f : C \to \mathbf{R}$ be convex relative to C. Then the set of optimal solutions of $\inf_C f$ is convex.

5. Convexity of sets and functions are closely related: for example the level sets of a convex function is convex. However, even if all level sets of the function is convex, the function itself does not have to be convex- we call such functions quasi-convex. A set which completely characterizes the convexity of f is its epigraph:
$$\mathbf{E} \times \mathbf{R} \supseteq \text{epi } f := \{(x, \alpha) \mid f(x) \leq \alpha\};$$
f is convex if and only if epi f is convex.

6. Some operations on convex functions preserve their convexity: nonnegative weighted sums and compositions with an affine function. We already saw that the $\sup$ of a (possibly infinitely) many convex functions is convex. Thus follows the convexity of
$$\begin{aligned} f : \mathbf{R}^n \to \mathbf{R}, &\qquad f(x) := \max_i \{x_1, \ldots, x_n\}, \\ f : \mathbf{R}^n \to \mathbf{R}, &\qquad f(x) := \sup_{y \in C} \|x - y\|, \quad C \subseteq \mathbf{R}^{\mathbf{n}}, \\ f : \mathcal{S}^n \to \mathbf{R}, &\qquad f(X) := \max \text{eigenvalue of } X. \end{aligned}$$
Note that convexity is not preserved under general convex compositions, e.g., if $h(x) = -x$ and $g(x) = x^2$, then $h(g(x)) = -x^2$. However, if g is convex and h is nondecreasing, then $h \circ g$ is convex.

7. The image of a convex set under a linear map is convex.

8. The set $C \subseteq \mathbf{E}$ is convex if its intersection with every line is convex. The function f is convex if and only if it is convex relative to every line. For this reason we start by letting $\mathbf{E} = \mathbf{R}$.

9. Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ be convex. Then $p(u) := \inf_x f(x, u)$ is convex (on $\mathbf{R}^m$), as well as the set $P(u) := \arg\min_x f(x, u)$ for each u. The proof goes as follows: consider the projection of $E := \{(x, u, \alpha) \mid f(x, u) < \alpha < \infty\}$ on its second and third coordinates.

10. Let $C \subseteq \mathbf{R}$ be an interval, $f : C \to \mathbf{R}$ be convex on C, and $x < y < z$ (all in C). Then f is convex if and only if
$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y},$$
i.e., if for each x we define the slope function
$$s_x(y) := \frac{f(y) - f(x)}{y - x}, \quad x \neq y \in C,$$
then s is nondecreasing. In particular, if f is differentiable and C is an open interval, then f is convex if and only if $f'(x)$ is nondecreasing.

Proof: There is a one-to-one correspondence between $\lambda \in (0, 1)$ and $y \in (x, z)$ through the relation $\lambda = (y - x)/(z - x)$. Thus $f(y) \leq ((z - y)/(z - x))f(x) + ((y - x)/(z - x))f(z)$; now subtract $f(x)$ from both sides of this inequality, and then, subtract $f(z)$ on the second try. $\square$

11. **Theorem 3.1** *Let $C \subseteq \mathbf{R}$ be open interval and $f : C \to \mathbf{R}$ be differentiable. Then f is convex if and only if*
$$f(y) \geq f(x) + f'(x)(y - x), \quad \textit{for all } x, y \in C.$$
If f is twice differentiable, f is convex if and only if $f''(x) \geq 0$ for all $x \in C$.

    Proof: Let $g_y(x) := f(x) - f(y) - f'(y)(x - y)$. Then $g_y'(x) \geq 0$ for all $y < x \in C$ and $g'(x) \leq 0$ for all $y > x \in C$. Thereby $g_y(y) = 0$ is the (global) minimal value of g_y and thus the assertion. Note that if $h_y(x) := f(y) + f'(y)(x - y)$, for each y (a linear function), then $f(x) = \sup_y h_y(x)$ is convex since linear functions are! (for an alternate proof, see §2.1.3 of [BV]). The equivalence of the second order condition with convexity follows from elementary calculus and the fact that f' is nondecreasing. Note that for strict convexity, a necessary condition is NOT the positive definite-ness of $\nabla^2 f$ (e.g., $f(x) = x^4$). □

12. The characterization of convexity in terms of the Hessian is very useful (when f is twice differentiable). Thus follows the convexity of
$$f : \mathbf{R}^n \to \mathbf{R}, \quad f(x) := \frac{1}{2}\langle x, Ax\rangle + \langle a, x\rangle + \gamma, \quad A \in \mathcal{S}^n_+,\ a \in \mathbf{R}^n, \text{ and } \gamma \in \mathbf{R}.$$

13. The convexity of $-\log\det X$, when $X \in \mathcal{S}^n_{++}$, can be established by letting $g(t) := -\log\det(X + tD)$, and evaluating g'' (choosing t such that $X + tD > 0$).

3.3 First order sufficiency condition

1. **Theorem 3.2** *Let $C \subseteq \mathbf{R}^n$ be open and convex and $f : C \to \mathbf{R}$ be differentiable. Then f is convex on C if and only if either*
$$\langle \nabla f(x) - \nabla f(y), x - y\rangle \geq 0, \quad \textit{or} \quad f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle, \quad \textit{for all } x, y \in C. \tag{3.5}$$
Furthermore, if f is twice differentiable, then f is convex if only if $\nabla^2 f$ is positive semi-definite.

    Proof: Let $x \in C$, $d \in \mathbf{R}^n$. Define $g(t) := f(x + td)$ on an open interval for which $y + td \in C$. Then $g'(t) = \langle d, \nabla f(x + td)\rangle$ and $g''(t) = \langle d, \nabla^2 f(x + td)d\rangle$. Let $d = y - x$, $t = 0$ then $t = 1$, and use the fact that $g'(t)$ is a nondecreasing function of t. Note that $g(1) \geq g(0) + g'(0)$. □

2. **Proposition 3.3** *(***First order sufficiency condition***) Let f be convex. Then for any $x, \bar{x} \in C$ the directional derivative exists in $[-\infty, +\infty]$. If $f'(\bar{x}; x - \bar{x}) \geq 0$ for all $x \in C$, or in particular when $-\nabla f(\bar{x}) \in N_C(\bar{x})$ holds, then $\bar{x}$ is a global minimizer.*

    Proof: In the case of the differentiable f this follows from (3.5). □

Reading: [BV]: 3.1-3.2, 3.4-3.5, chapter 4
Homework #2: (due 2/6) [BV]: 3.6, 3.8, 3.11, 3.12, 3.16, 3.40, 4.1, 4.12, 4.16; extra: 4.54, 4.55

4 Week 4

Main ideas: Some extensions of convexity, Lagrange multipliers, duality

4.1 Quasi-convexity and log convexity

1. Let $C \subseteq \mathbf{R}^n$. The function $f : C \to \mathbf{R}$ is quasi-convex if and only if for all $x, y \in C$ and $\lambda \in [0, 1]$,
$$f(\lambda x + (1 - \lambda)\, y) \leq \max\{f(x), f(y)\}.$$
2. Let $f : C \to \mathbf{R}$ be differentiable. Then f is quasi-convex if and only if
$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), y - x\rangle \leq 0.$$
Furthermore, if f is twice differentiable at $\bar{x}$ and $\nabla f(\bar{x}) = 0$ then $\nabla^2 f(\bar{x})$ is positive semi-definite.
3. We will look at the algorithmic implications of quasi-convexity when we study optimization algorithms.
4. The function $f : C \to \mathbf{R}$ is called log-convex if $\log f$ is convex.

4.2 Lagrange Multipliers and Lagrangian duals

1. The generic optimization problem we consider is
$$\bar{p} := \inf f(x) \quad \text{s.t:} \quad x \in D := \{x \in \mathbf{E} \,|\, g_i(x) \leq 0 \ (i = 1, \ldots, m), h_i(x) = 0 \ (i = 1, \ldots, p)\}, \tag{4.6}$$
where for all i, $g_i, h_i : \mathbf{E} \to \mathbf{R} \cup \{+\infty\}$ are given functionals. The optimal value $\bar{p}$ can assume values in $\bar{\mathbf{R}}$.
2. We assume that dom $f \subseteq (\cap_i \text{dom } g_i) \ \cap \ (\cap_i \text{dom } h_i)$ in (4.6).
3. We adopt the following notation:
$$\mathbf{R}^m \ni g(x) := \begin{bmatrix} g_1(x) & \ldots & g_m(x) \end{bmatrix}^T \quad \text{and} \quad \mathbf{R}^p \ni h(x) := \begin{bmatrix} h_1(x) & \ldots & h_p(x) \end{bmatrix}^T;$$
thus $D = \{x \in \mathbf{E} \,|\, g(x) \leq 0,\ h(x) = 0\}$.
4. The Lagrangian $L : \mathbf{E} \times \mathbf{R}^m_+ \times \mathbf{R}^p \to \mathbf{R} \cup \{+\infty\}$ for (4.6) is defined by
$$L(x, \lambda, \nu) := f(x) + \lambda^T g(x) + \nu^T h(x).$$
5. We say that $(\bar{\lambda}, \bar{\nu}) \in \mathbf{R}^m_+ \times \mathbf{R}^p$ is a Lagrange multiplier pair for a feasible vector $\bar{x}$ in (4.6) if $\bar{x}$ minimizes the function $L(., \bar{\lambda}, \bar{\nu})$ over $\mathbf{E}$, and $\bar{\lambda}_i = 0$ whenever $g_i(\bar{x}) < 0$ (complementary slackness).
6. **Proposition 4.1** *(***Lagrange sufficient optimality condition***) Let $\bar{x}$ be feasible vector for (4.6) that admits a Lagrange multiplier pair. Then $\bar{x}$ is optimal.*

   Proof: For every feasible x one has
$$f(\bar{x}) = f(\bar{x}) + \bar{\lambda}^T g(\bar{x}) + \bar{\nu}^T h(\bar{x}) = L(\bar{x}, \bar{\lambda}, \bar{\nu}) \leq L(x, \bar{\lambda}, \bar{\nu}) = f(x) + \bar{\lambda}^T g(x) + \bar{\nu}^T h(x) \leq f(x).$$
$\square$

7. We note that

$$\sup_{\lambda\in\mathbf{R}^m_+,\nu\in\mathbf{R}^p} L(x,\lambda,\nu) = \begin{cases} f(x) & \text{if} \quad x\in D \\ +\infty & \text{otherwise} \end{cases}$$

and therefore, (4.6) is equivalent to

$$\inf_{x\in\mathbf{E}} \sup_{\lambda\geq 0,\nu} L(x,\lambda,\nu).$$

Note that the argument of the '$\inf$' is unrestricted.

8. In general $\inf_{x\in\mathbf{E}} \sup_{\lambda\geq 0,\nu} L(x,\lambda,\nu) \geq \sup_{\lambda\geq 0,\nu} \inf_{x\in\mathbf{E}} L(x,\lambda,\nu)$.

9. The Lagrange dual function $g : \mathbf{R}^m_+ \times \mathbf{R}^p \to \bar{\mathbf{R}}$ is defined as

$$\phi(\lambda,\nu) := \inf_{x\in\mathbf{E}} L(x,\lambda,\nu).$$

Thus $\bar{p} \geq \sup_{\lambda\geq 0,\nu} \phi(\lambda,\nu)$.

10. $\phi : \mathbf{R}^m_+ \times \mathbf{R}^P \to \bar{\mathbf{R}}$ is a concave function.

11. We define the dual to (4.6) as the optimization problem

$$\bar{d} := \sup \phi(\lambda,\nu) \quad \text{s.t:} \quad \lambda\in\mathbf{R}^m_+,\ \nu\in\mathbf{R}^p$$

12. Example: consider the LP

$$\inf_x c^T x \quad \text{s.t.} \quad x\geq 0,\ Ax-b=0.$$

Then $\phi(\lambda,\nu) = -b^T\nu + \inf(c+A^T\nu-\lambda)^T x$ and thus

$$\phi(\lambda,\nu) = \begin{cases} -b^T\nu & \text{if} \quad A^T\nu-\lambda+c=0 \\ -\infty & \text{otherwise.} \end{cases} \tag{4.7}$$

The dual problem can thus be written as

$$\sup_y b^T y \quad \text{s.t.} \quad A^T y \leq c.$$

4.3 Necessary optimality condition

1. Let $\mathbf{E} = \mathbf{R}^n$; the constraint set D be defined as in the previous lecture. The functionals h_i's are restricted to be affine. In this case the equality constraints can be summarized by $Ax-b=0$, for $A\in\mathbf{R}^{p\times n}$. We assume that the equality constraints are not linearly dependent, i.e., A has full row rank.

2. We say that the constraint set D satisfies the Slater's condition if there exists $\tilde{x}\in$ **int** D (and we assume that $g(\tilde{x})<0$; this holds for example when g's are continuous).

3. **Theorem 4.2** *Let f and g_i's be convex and h_i's be affine (and linearly independent) in (4.6). Furthermore, assume that D satisfies the Slater's condition. Then (a) $\bar{p}=\bar{d}$. (b) If $\bar{p}=f(\bar{x})=\phi(\bar{\lambda},\bar{\nu})=\bar{d}$, for primal feasible $\bar{x}$ and $\bar{\lambda}\geq 0$, then $(\bar{\lambda},\bar{\nu})$ is a Lagrange multiplier pair for $\bar{x}$.*

Proof: If $p = -\infty$ we are done. Otherwise, it suffices to show that $\bar{p} \leq \bar{d}$. Define the set C as

$$\mathbf{R}^{m+p+1} \supseteq C := \{\, [u, v, t]^T \mid \exists x \in D, g(x) \leq u,\ Ax - b = v,\ f(x) \leq t \,\}.$$

Let $y_\epsilon := [0, 0, \bar{p} - \epsilon]^T \in \mathbf{R}^{m+p+1}$, for all $\epsilon > 0$. The set C is closed and convex and $y_\epsilon \notin C$ for an arbitrary $\epsilon > 0$. Thus for an arbitrary $\epsilon > 0$, there exists $[\bar{\lambda}, \bar{\nu}, \mu]^T \neq 0$ and $\beta \in \mathbf{R}$, such that

$$\bar{\lambda}^T u + \bar{\nu}^T v + \mu t \geq \beta > \mu y_\epsilon, \quad \text{and thereby} \quad \bar{\lambda}^T g(x) + \bar{\nu}^T (Ax - b) + \mu f(x) \geq \mu \bar{p}, \tag{4.8}$$

for all $x \in \mathbf{E}$. We note that $\mu \geq 0$ and $\bar{\lambda} \geq 0$. By Slater's condition and the rank assumption, μ cannot be equal to zero. If it did, $\bar{\lambda} = 0$, $\bar{\nu} \neq 0$, and $A^T \bar{\nu} \neq 0$. Let $y := -A^T \bar{\nu}$. Since $(A^T \bar{\nu})^T \tilde{x} - \bar{\nu}^T b = 0$, $(A^T \bar{\nu})^T (\tilde{x} + ty) - \bar{\nu}^T b < 0$ for small $t > 0$ for which $\tilde{x} + ty \in D$. But that contradicts (4.8) with $\mu = 0$. Thus $\mu > 0$, and (4.8) implies that for all $x \in \mathbf{E}$, $L(x, (\bar{\lambda}/\mu), (\bar{\nu}/\mu)) \geq \bar{p}$, i.e., $\bar{d} \geq \bar{p}$.

We observe that

$$f(\bar{x}) = \inf_x \{\, f(x) + \bar{\lambda}^T g(x) + \bar{\nu}^T h(x) \,\} \leq f(\bar{x}) + \bar{\lambda}^T g(\bar{x}) + \bar{\nu}^T h(\bar{x}),$$

and therefore $\bar{\lambda}^T g(\bar{x}) = 0$. By definition $\bar{x}$ minimizes $\phi(\bar{\lambda}, \bar{\nu})$. □

4. Let f and g_i's be differentiable. Then a necessary condition for $(\bar{\lambda}, \bar{\nu})$ to be a Lagrange multiplier pair for $\bar{x}$ is that

$$\nabla f(\bar{x}) + \sum_i \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_i \bar{\nu}_i \nabla h_i(\bar{x}) = 0. \tag{4.9}$$

5. A feasible point $\bar{x}$ is said to satisfy the Karush-Kuhn-Tucker (KKT) condition, or is a KKT point, if there is a pair $(\bar{\lambda}, \bar{\nu})$, $\bar{\lambda} \geq 0$, such that (4.9) and complementary slackness condition holds.

6. Thus, the convexity of f and g's, along with the affinity of h's and the Slater's condition, implies that the optimal of (4.6) is a KKT point.

7. If f and g's are convex, a KKT point is optimal for (4.6) since it admits a Lagrange multiplier pair.

8. Let $\bar{x}$ be a KKT point with respect to the Lagrange multiplier pair $(\bar{\lambda}, \bar{\nu})$. If f, g_i's are convex and h_i's are affine, then $\bar{x}$ is the optimal for (4.6).

9. If $= f(\bar{x}) = \phi(\bar{\lambda}, \bar{\nu}) = \bar{d}$, for every primal and dual feasible points one has

$$f(x) \geq f(\bar{x}) = \phi(\bar{\lambda}, \bar{\nu}) \geq \phi(\lambda, \nu);$$

thus the bound $f(x) - \phi(\lambda, \nu)$ can be used as a stopping criteria in primal-dual algorithms.

10. KKT implies that $-\nabla f(\bar{x}) \in N_D(\bar{x})$. Let D be defined by inequality constraints $g_i(x) \leq 0$ $(i = 1, \ldots, m)$. For every feasible y, one has

$$\langle -\nabla f(\bar{x}), y - \bar{x} \rangle = \sum_i \bar{\lambda}_i \langle \nabla g_i(\bar{x}), y - \bar{x} \rangle = \sum_i \bar{\lambda}_i (g_i(\bar{x}) + \langle \nabla g_i(\bar{x}), y - \bar{x} \rangle) \leq \sum_i \bar{\lambda}_i g_i(y) \leq 0.$$

A GLIMPSE OF APPLICATIONS

5 Week 5

Main ideas: Linear Programming, matrix games, data fitting, approximation, diet problem

5.1 Linear programming

1. Linear programming (LP) is the problem of minimizing a linear functional over a set defined by the intersection of hyperplanes and/or halfspaces. Given $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $c \in \mathbf{R}^n$,

$$(P): \quad \min c^T x \quad \text{subject to} \quad Ax \geq b,\ x \geq 0. \tag{5.10}$$

2. Consider $\inf_{x\in D} f(x)$. How can one verify the optimality of a candidate optimal solution? Note that the definition, find $\bar{x}$ such that $f(\bar{x}) \leq f(x)$ for all $x \in D$ (if the $\inf$ is achieved), is NOT descriptive enough! (setting $\nabla f(x) = 0$ for example, partially does the job for local optimality when f is differentiable and D is an open set).

3. The dual of (5.10) is another LP

$$(D): \quad \max b^T y \quad \text{subject to} \quad A^T y \leq c,\ y \geq 0. \tag{5.11}$$

   We thus end up calling (5.10) primal; hence the "(P)" to the left of the equation.

4. If x is primal feasible and y is dual feasible, then $b^T y \leq c^T x$.

5. **Theorem 5.1** *If (P) and (D) are feasible then there exist a primal feasible and dual feasible vectors $\bar{x}$ and $\bar{y}$, such that $c^T\bar{x} = b^T\bar{y}$.*

6. **Theorem 5.2** *If (P) has an optimal solution $\bar{x}$ then the dual has an optimal solution $\bar{y}$, and $c^T\bar{x} = b^T\bar{y}$.*

7. Suppose that $\bar{x}$ and $\bar{y}$ are primal and dual feasible. Then $\bar{x}$ and $\bar{y}$ are optimal solutions of (P) and (D) respectively if and only if

$$\bar{y}^T(A\bar{x} - b) = 0 \quad \text{and} \quad \bar{x}^T(A^T\bar{y} - c) = 0;$$

   these two conditions are called complementarity slackness.

5.2 Matrix games and minmax theorem

1. Let e^j denote the following vector: $e^j_i = 0$ for all $i \neq j$ and $e^j_j = 1$.

2. Every $A \in \mathbf{R}^{m\times n}$ defines a game for two; let these two be Ron (the row player) and Claude (the column player). Ron selects one of the rows $i = 1, 2, \ldots, m$, and Claude selects one of the columns, $j = 1, 2, \ldots, n$ (their pure strategies). The resulting payoff for Ron is a_{ij} (Ron pays if a_{ij} is negative).

3. Let Ron and Claude choose row i and column j with probability x_i and y_j, respectively (their mixed strategies). Note that $x, y \geq 0$ and $\mathbf{1}^T x = \mathbf{1}^T y = 1$; we call such vectors stochastic (and subsequently, all x's and y's are assumed to be stochastic vectors). Then the average payoff for Ron is $x^T Ay$.

4. For a given x, the *guaranteed* payoff for Ron is $\min_y x^T Ay$.

5. We note that $\min_y x^T Ay = \min_j A^T x$:

$$x^T Ay = y^T A^T x \geq \min_j A^T x,$$

and on the other hand, $\min_y x^T Ay \leq (A^T x)_j$ for each j.

6. The optimal (mixed) strategy for Ron is found by $\max_x \min_j (A^T x)_j$, or in an LP form:

$$\max_{z,x} z \quad \text{subject to:} \quad z - e_j^T A^T x \leq 0, \quad j = 1, \ldots, n. \tag{5.12}$$

7. Similarly, Claude likes to $\min_y \max_i (Ay)_i$, or in an LP form:

$$\min_{w,y} w \quad \text{subject to:} \quad w - e_i^T Ay \geq 0, \quad i = 1, \ldots, m. \tag{5.13}$$

8. **Theorem 5.3** *For every matrix $A \in \mathbf{R}^{m\times n}$ there are stochastic vectors $\bar{x}$ and $\bar{y}$ such that*

$$\min_y \bar{x}^T Ay = \max_x x^T A\bar{y}.$$

Proof: The LP's (5.12) and (5.13) are feasible and duals of each other; now apply Theorem 5.1. □

5.3 Approximations and data fitting

1. Note that the dual of $\min_x \{c^T x \mid Ax \geq b\}$ is $\max_y \{b^T y \mid A^T y = c, y \geq 0\}$.
2. The ∞-norm and the 1-norm of $x \in \mathbf{R}^n$ are respectively, $\|x\|_\infty := \max_i |x_i|$ and $\|x\|_1 := \sum_i |x_i|$.
3. Chebyshev approximation: Given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, solve $\min_{x\in\mathbf{R}^n} \|Ax - b\|_\infty$.
4. Note that the least squares problem: Given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, solve $\min_{x\in\mathbf{R}^n} \|Ax - b\|_2$, has the unique solution $x = (A^T A)^{-1} A^T b$, provided that A has linearly independent columns.
5. First, put the problem in LP form:

$$(P): \quad \min_z c^T z \quad \text{subject to} \quad Dz \geq d, \tag{5.14}$$

where

$$D := \begin{bmatrix} \mathbf{1} & -A \\ \mathbf{1} & A \end{bmatrix}, \quad d := \begin{bmatrix} -b \\ b \end{bmatrix}, \quad c := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad z := \begin{bmatrix} t \\ x \end{bmatrix}.$$

Then the dual of (5.14) is

$$(D): \quad \max_w d^T w \quad \text{subject to} \quad D^T w = c,\ w \geq 0. \tag{5.15}$$

After eliminating the redundant variables and setting $q := w_2 - w_1$, (5.15) assumes the form

$$(D): \quad \max b^T q \quad \text{subject to} \quad \sum_i |q_i| \leq 1,\ A^T q = 0. \tag{5.16}$$

6. Let $\mathcal{R}(A)$ be range space of the matrix A.

7. (P) is then the following problem: given $b \in \mathbf{R}^n$, find an element of $\mathcal{R}(A)$ closest to b in the ∞-norm.

8. (D) is the following problem: find a linear functional on $\mathbf{R}^m$, with 1-norm at most one, that is identically zero on $\mathcal{R}(A)$ and is as large as possible at b. In other words, we are looking for a linear functional of the 1-norm at most one, which separates best of all the point b and the linear subspace $\mathcal{R}(A)$.

9. Duality leads to the following general statement: the p-norm distance of a point b to a subspace $\mathcal{R} \in \mathbf{E}$ is equal to the supremum of the quantity by which b can be separated from $\mathcal{R}$ by a linear functional of p^*-norm at most one; the p^*-norm is the conjugate norm of the p-norm:

$$\|y\|_{p^*} := \sup_x \{\langle y, x\rangle \mid \|x\|_p \leq 1\}.$$

10. Applications of LP to function approximation, filter synthesis, etc.

11. Applications of LP duality to the diet problem- the economic interpretation of the dual variables.

12. Check out Matlab's Optimization toolbox: `>> help toolbox/optim`, and in particular, linprog, quadprog, fminunc, and the demos.

<u>Reading:</u> [BV]: chapters 4 and 5
<u>Homework # 3:</u> (Due 2/20) [BV]: 3.7, 4.3, 4.5, 4.6, 4.9, 4.11, 4.38, 4.41, 5.1, 5.6

6 Week 6

Main ideas: Applications: Network flows, Max-cut and SDP, truss topology

6.1 Max Flow- Min-cut theorem

1. Max flow as an LP; Min-cut as an LP; their duality.

2. **Theorem 6.1** *In a network, the max flow between a source and a sink is equal to the minimal cut separating the source and the sink.*

   Proof: Using LP duality. □

6.2 Semidefinite programming

1. Let $\mathbf{E} := \mathcal{S}^n$, $\langle X, Y\rangle := X\cdot Y$ and $X \geq Y$ ($X > Y$) is equivalent to having $X - Y$ positive semi-definite (positive definite).

2. **Lemma 6.2** *Let $A_1, \ldots, A_n \in \mathcal{S}^n$. Then*

$$[\, x_1A_1 + \ldots + x_nA_n > 0\,] \quad \& \quad [\, A_i \cdot Y = 0 \ (i = 1, \ldots, n),\ Y \geq 0,\ Y \neq 0\,],$$

   are alternative systems (compare with 16.47).

3. **Lemma 6.3** *Consider the optimization problems*

$$\textit{(P):} \quad \min c^T x \quad \textit{subject to:} \quad x_1A_1 + \ldots + x_nA_n \geq B,$$

   and

$$\textit{(D):} \quad \max B\cdot Y, \quad \textit{subject to:} \quad A_i \cdot Y = c_i \ (i = 1, \ldots, n),\ Y \geq 0.$$

   Suppose that there is an x such that

$$x_1A_1 + \ldots + x_nA_n > B.$$

   Then, (1) for feasible x and Y, respectively, for (P) and (D), $c^T x \geq B\cdot Y$, (2) there exist $\bar{x}$ and $\bar{Y}$, feasible for (P) and (D) respectively, such that $c^T\bar{x} = B\cdot\bar{Y}$, and (3) if $\bar{x}$ and $\bar{Y}$ are primal and dual feasible, then $\bar{x}$ and $\bar{Y}$ are optimal solutions of (P) and (D) respectively, if and only if $\bar{Y}\cdot(\sum_i \bar{x}_iA_i - B) = 0$, or equivalently, if $\bar{Y}(\sum_i \bar{x}_iA_i - B) = 0$.

6.3 Max-cut

1. Let $G = (V, E)$ be a weighted graph of order n (the cardinality of V is n): $a_{ij} = a_{ij}$ is the nonnegative weight associated with edge ij.

2. We like to find the *cut* of G- a partition of its vertex set into two distinct subsets V_1 and V_2- with maximum weight. This MAX-CUT problem is computationally very difficult (NP-complete).

3. Let vector $x \in \mathbf{R}^n$ be such that $x_i = 1$ if $i \in V_1$ and $x_i = -1$ if $i \in V_2$. Then the weight of the cut (V_1, V_2) is $\frac{1}{4}\sum_{i,j} a_{ij}(1 - x_ix_j)$.

4. The MAX-CUT problem is thereby

$$\bar{\gamma} := \max_x \frac{1}{4} \sum_{i,j} a_{ij}(1 - x_i x_j) \quad \text{s.t.} \quad x_i^2 = 1 \ (i = 1, \ldots, n). \tag{6.16}$$

5. The SDP relaxation of (6.16) is

$$\bar{\eta} := \max_X \frac{1}{4} \sum_{i,j} a_{ij}(1 - X_{ij}) \quad \text{s.t.} \quad X \geq 0, \ X_{ii} = 1 \ (i = 1, \ldots, n). \tag{6.17}$$

6. We will denote by $\eta(X)$ the value of the objective functional for a feasible X in (6.17).

7. We need the following inequality shortly (see next page): when $-1 \leq x \leq 1$, one has

$$\cos^{-1}(x) \geq \frac{\pi}{2} \, (0.87856) \, (1 - x).$$

8. **Theorem 6.4** $0.87856 \, \bar{\eta} \leq \bar{\gamma} \leq \bar{\eta}$.

   Proof: It suffices to show the lower bound. We notice that since $X = LL^T$, we have $X_{ij} = v_i^T v_j$, for vectors $v_1, v_2, \ldots, v_n$, $\|v_i\| = 1$ for all i. Let $v \in \mathbf{R}^n$ be a random vector according to a uniform distribution. Let $V_1 := \{i \,|\, v^T v_i \geq 0\}$ be a random cut. The probability that vertices i and j are separated by this random cut is

$$\frac{2 \, \cos^{-1}(v_i^T v_j)}{2\pi} = \frac{\cos^{-1}(X_{ij})}{\pi}.$$

   The expected contribution of the vertices i and j to the objective functional in (6.17) is therefore

$$\frac{1}{4} \, (2) \, a_{ij} \cos^{-1}(X_{ij})/\pi.$$

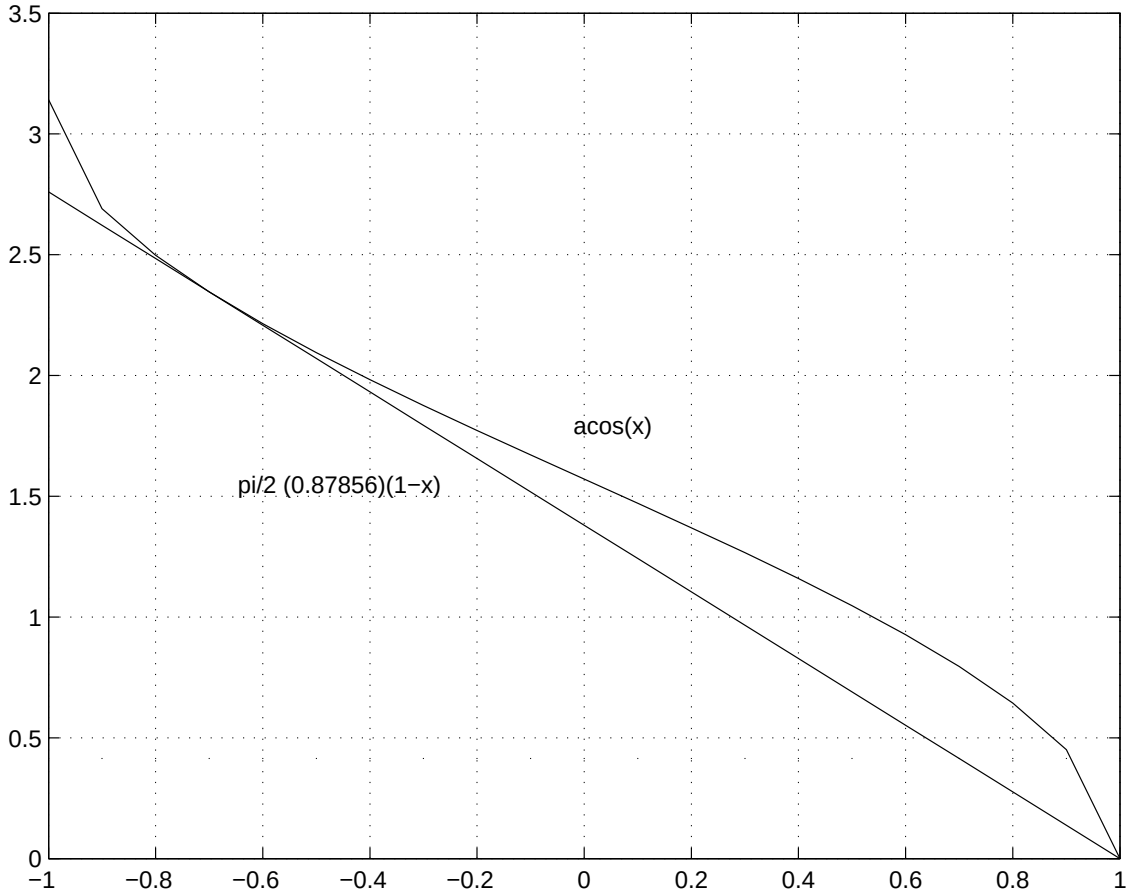
   Thus for every feasible X in (6.17), the expected weight of the corresponding cut is

$$\frac{1}{2} \sum_{i,j} a_{ij} \frac{\cos^{-1}(X_{ij})}{\pi} \geq 0.87856 \, \eta(X). \tag{6.18}$$

   In particular, one has

$$\bar{\eta} \geq \bar{\gamma} \geq \frac{1}{2} \sum_{i,j} a_{ij} \frac{\cos^{-1}(X_{ij})}{\pi} \geq 0.87856 \, \bar{\eta}.$$

$\square$


3.5
3
2.5
2
1.5
1
0.5
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
acos(x)
pi/2 (0.87856)(1−x)

7 Week 7

Main ideas: more on SDPs, applications in control

7.1 Some tricks of the trade

1. Schur complement formula: let $A > 0$ in the partitioned symmetric matrix $M := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$. Then
$$M \geq 0 \ (> 0) \quad \text{if and only if} \quad C - B^T A^{-1} B \geq 0 \ (> 0).$$
This can be proven by noting that M is congruent to the $\begin{bmatrix} A & 0 \\ 0 & C - B^T A^{-1} B \end{bmatrix}$ via the transformation SMS^T, where $S := \begin{bmatrix} I & 0 \\ B^T A^{-1} & I \end{bmatrix}$.

2. One has $x^T P x = \left\langle P, xx^T \right\rangle := P \cdot (xx^T)$.

3. The diagonal entries of a positive semi-definite matrix are non-negative; if zero, all entries on the corresponding column and row are zero. The diagonal entries of a positive definite matrix are positive.

4. If $A, B \geq 0$, then $A \cdot B = 0$ if and only if $AB = 0$.

5. $\mathcal{S}$-procedure: the implication
$$x^T Q_i x \geq 0 \quad (i = 1, \ldots, m) \Rightarrow x^T Q_0 x \geq 0$$
holds, if there exists $\lambda_i \geq 0$ $(i = 1, \ldots, m)$ such that
$$Q_0 - \sum_i \lambda_i Q_i \geq 0.$$

6. Let $A, B > 0$. Then $A \leq B$ if and only if $A^{-1} \geq B^{-1}$: $A \leq B$ if and only if $B^{-1/2} A B^{-1/2} \leq I$. This on the other hand is equivalent to $B^{1/2} A^{-1} B^{1/2} \geq I$; now do yet another congruent transformation using $B^{-1/2}$.

7. Let $A \in \mathbf{R}^{n \times n}$ and $\|A\|$ denote the operator 2-norm. Then $\|A\| \leq \gamma$ if and only if $\begin{bmatrix} \gamma I & A \\ A^T & \gamma I \end{bmatrix} \geq 0$.
If the matrix A affinely depends on the variable x ($A(x)$, the problem of choosing x to minimize the $\|A(x)\|$ is an SDP.

8. Consider
$$\min \frac{(c^T x)^2}{d^T x}, \quad \text{s.t.} \quad Ax \leq b, \tag{7.19}$$
where $d^T x > 0$ for all feasible x. Since
$$\begin{bmatrix} \gamma & c^T x \\ c^T x & d^T x \end{bmatrix} \leq 0 \quad \text{and} \quad \gamma \geq \frac{(c^T x)^2}{d^T y}$$
the problem (7.19) is equivalent to
$$\min \gamma, \quad \text{s.t.} \quad Ax \leq 0, \ \begin{bmatrix} \gamma & c^T x \\ c^T x & d^T x \end{bmatrix} \geq 0.$$

9. The set defined by the inequality $x^T(A^TA)x - b^Tx - \alpha \geq 0$ is equivalent to one defined by

$$\begin{bmatrix} I & Ax \\ (Ax)^T & b^Ty + \alpha \end{bmatrix} \geq 0.$$

7.2 Stability analysis and state feedback synthesis

1. Suppose that the square matrix A in

$$\dot{x}(t) = Ax(t)$$

is uncertain; in particular it is only known to belong to the convex hull of a given set of matrices (polytopic uncertainty): $A \in \text{co}\,\{A_1, \ldots, A_n\}$. Then a sufficient condition for stability (A being Hurwitz) is the existence of a common Lyapunov function $V(x) := x^TPx^T$, $P > 0$, such that $A_i^TP + PA_i < 0$, for all i.

2. State feedback synthesis problems can be addressed in this same framework. Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$. The problem is to find $K \in \mathbf{R}^{m\times n}$ such that

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t),$$

is asymptotically stable, i.e., find matrices $P > 0$, and K that satisfy

$$(A + BK)^TP + P(A + BK) < 0, \quad P > 0.$$

An LMI is obtained by a change the variables of the form $Y = KQ$ where $Q = P^{-1}$.

3. The LMI framework is not adequate for the output feedback synthesis problems. However a *small* augmentation of the LMI with a rank constraint will do. The rank constraint makes the original "easy" LMI problem to a very difficult one!

4. Matrix elimination lemma.

5. output feedback via min rank problems (I will add this piece soon).

7.3 Optimal control- LQR theory

(this will be revised soon)

1. Consider the Semi-Definite Program (SDP):

$$\max_Y \langle F, Y\rangle \quad \text{s.t.} \quad \langle G_i, Y\rangle = c_i \ (i = 1, \ldots, m),\ Y \geq 0. \tag{7.20}$$

The dual of (7.20) is

$$\inf_x c^Tx \quad \text{s.t.} \quad F + \sum_i x_iG_i \leq 0. \tag{7.21}$$

2. Slater's condition for the feasible set of (7.21) translates to having $F + \sum_i \tilde{x}_iG_i < 0$, for some $\tilde{x}$.

3. Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, and

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0,\ (A, B) \text{ controllable}. \tag{7.22}$$

4. Let $x(0) \in \mathbf{R}^n$ be a random vector such that $\mathbf{E}\{x(0)x(0)^T\} = I$. We consider the problem of finding $u(t)$, $t \in [0, \infty)$, for which

$$J := \mathbf{E} \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \quad \rightarrow \quad \text{minimized.} \tag{7.23}$$

We further assume that $Q \geq 0$, $R > 0$, and (Q, A) is observable.

5. If for some $P \in \mathcal{S}^n$ and all $t \geq 0$, the time rate of the change of $x(t)^T P x(t)$ over trajectories of (7.22) for which $x(t) \rightarrow 0$, strictly bounds from the above the quantity $-\{x(t)^T Q x(t) + u(t)^T R u(t)\}$, then

$$\langle I, P \rangle \leq \mathbf{E} \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt.$$

Therefore

$$\sup_P \langle I, P \rangle, \quad \text{s.t.} \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} \geq 0 \tag{7.24}$$

provides a lower bound for the optimum cost (7.23).

6. Let $\dot{x}(t) = Ax(t)$, where all eigenvalues of A have negative reals parts. Then

$$X := \int_0^\infty e^{A^T t} Q e^{At}\, dt \quad \text{if and only if} \quad A^T X + XA + Q = 0.$$

7. Thus when $u(t) = Kx(t)$, then the cost

$$\mathbf{E}\{\int_0^\infty x(t)^T \left(Q + K^T R K\right) x(t)\, dt\}$$

can be found by

$$\inf_{\tilde{Z}, K} \left\langle Q + K^T R K, \tilde{Z} \right\rangle, \quad \text{s.t.} \quad (A + BK)\tilde{Z} + \tilde{Z}(A + BK)^T + I = 0.$$

This turns out to be equivalent to the problem

$$\inf_Z \langle Q, Z_{11} \rangle + \langle R, Z_{22} \rangle, \quad \text{s.t.} \quad AZ_{11} + Z_{11}A^T + BZ_{12}^T + Z_{12}B^T + I = 0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \geq 0, \tag{7.25}$$

with

$$Z = \begin{bmatrix} I \\ K \end{bmatrix} \tilde{Z} \begin{bmatrix} I & K^T \end{bmatrix}.$$

The program (7.25) thus provides an upper bound on the optimal cost. However, (7.24) and (7.25) are duals of each other. The observability and controllability of (Q, A) and (A, B) implies, respectively, the strict feasibility of the primal and dual problems. Thus the optimal control assumes the form of a linear state feedback.

8. Continuing along this line, the complementarity slackness condition is equivalent to having $K = -R^{-1}B^T P$, where P satisfies the algebraic Riccati equation,

$$A^T P + PA + Q - PBR^{-1}B^T P = 0.$$

Reading: SDP review article (BV), [BV]:
Homework # 4: (due 3/6) [BV]: 4.45,

A LITTLE MORE ON THEORY

8 Week 8

Main ideas: Subgradients and Lagrange Multipliers

1. Let $C \subseteq \mathbf{E}$. We will allow the function f on C to be not necessary differentiable and extended-valued, either assuming values in $(-\infty, +\infty]$ or $\bar{\mathbf{R}} := [-\infty, +\infty]$. In the former case, it is better to talk about the convexity of f in terms of the convexity of its epigraph (to avoid having to consider what in the world $\infty - \infty$ could mean). Recall the definitions of the domain of such functions, whose nonemptyness is required for the function to be *proper*.

   Although we can get away with the notion of the interior of a set, we will make things more general, since it does not take much to do that.

2. The point $x \in C$ is said to belong to the core of C, core(C), if $x \in C$ and for any $d \in \mathbf{E}$ and all sufficiently small positive t, $x + td \in C$. Note that **int** $C \subseteq$ core(C); to see the difference between the two consider the set $\{x \in \mathbf{R}^2 \,|\, x_2 = 0, |x_2| \geq x_1^2\}$ that contains the origin in its core but not its interior.

3. The subdifferential of f at a point $\bar{x}$ is the *set*

$$\partial f(\bar{x}) := \{\phi \in \mathbf{E} \,|\, \langle \phi, x - \bar{x} \rangle \leq f(x) - f(\bar{x})\};$$

   it is convenient to think of the subdifferential as $\partial f : \mathbf{E} \to \mathbf{E}$, i.e., as a set-valued map. Each element of the subdifferential (at a point) is called a subgradient (at that point). If $\bar{x} \notin$ dom f, we let $\partial f = \emptyset$; dom ∂f is the set of points for which it is nonempty.

4. ∂f is a closed convex set.

   The following is immediate but notationally really cute.

5. For a proper function $f : C \to (-\infty, +\infty]$, $\bar{x} \in C$ is a (global) minimizer of f on C, if and only if $0 \in \partial f(\bar{x})$.

6. Let $f : C \to (-\infty, +\infty]$ be convex and $\bar{x} \in$ dom$f$. Then $\phi \in \partial f(\bar{x})$ if and only if for all $d \in \mathbf{E}$, $\langle \phi, d \rangle \leq f'(\bar{x}; d)$.

   I did not prove the following in class. But we provided few insights into why it is true. Note how a newly defined object (∂f) is related to an older one.

7. **Theorem 8.1** ***(Max formula)*** *Let $f : \mathbf{E} \to (-\infty, +\infty]$ be convex, $\bar{x} \in$ core$($dom $f)$, and $d \in \mathbf{E}$. Then*

$$f'(\bar{x}; d) = \max_{\phi \in \partial f(\bar{x})} \langle \phi, d \rangle .$$

8. Theorem 8.1 implies that if f is convex and $x \in$ core $($dom$f)$ then f admits a subgradient at x.

9. Theorem 8.1 also implies that if f convex and $\bar{x} \in$ core$($dom $f)$, then f is (Gâteaux) differentiable at $\bar{x}$ if and only if f has a unique subgradient at $\bar{x}$.

10. Let $f, g_1, \ldots, g_m : C \to \bar{\mathbf{R}}$, be convex with dom$f \subseteq \cap_idomg_i$. We consider again the problem

$$\inf_x f(x) \quad \text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m. \tag{8.26}$$

    Define the *value function* $v : \mathbf{R}^m \to \bar{\mathbf{R}}$ for (8.26) via $v(b) := \inf\{f(x) \,|\, g(x) \leq b\}$.

11. The value function for the convex program (8.26) is convex (one way to see this is to use the convexity of the indicator function of the constrained set).

12. Recall that $\bar{\lambda} \in \mathbf{R}^m_+$ is called a Lagrange multiplier for feasible $\bar{x}$ in (8.26) if $\bar{x}$ minimizes $L(x, \bar{\lambda}) := f(x) + \bar{\lambda}^T g(x)$ over $\mathbf{E}$, and $\bar{\lambda}^T g(\bar{x}) = 0$.

13. We say that the constraint set of (8.26) satisfies the Slater's condition if there exists $\hat{x} \in \text{dom} f$ such that $g_i(\hat{x}) < 0$, for all i.

14. If the (8.26) satisfies the Slater's condition, $0 \in \text{core}(\text{dom} f)$ and $v : \mathbf{R}^m \to (-\infty, +\infty]$: clearly $v(0) > -\infty$, and $0 \in \text{core}\ (\text{dom} v)$. To establish a contradiction, suppose that $v(y) = -\infty$ for $y \in \mathbf{E}$. Since the origin is in the core of th e domain of v, there is a real $t > 0$ such that $0 + t(0 - y) \in \text{dom} v$. Thus there exists a real β such that $(0 + t(0 - y), \beta) \in \text{epi}(v)$. But for any real γ, $(y, \gamma) \in \text{epi}(v)$, and thus

$$\frac{1}{1+t}(-ty, \beta) + \frac{t}{t+1}(y, \gamma) = (0, \frac{\beta + t\,\gamma}{1+t}) \in \text{epi}\ (v).$$

Let $\gamma \to -\infty$: given that $0 \in \text{core}\ (\text{dom}\ v)$, we thus arrive at a contradiction.

We prove the following familiar result in the next lecture- this time using the fact that a convex function admits a subgradient in the core of its domain- much like a differentiable function admitting a derivative in the interior of its domain.

15. **Theorem 8.2** ***(*Lagrangian Necessary Conditions: revisited*)*** *Suppose that the point $\bar{x} \in dom f$ is optimal for the convex program and that the Slater's condition holds. Then $\bar{x}$ admits a Lagrange multiplier.*

Proof: (Theorem 8.2) The value function v is convex and $0 \in \text{core}\ (\text{dom} v)$, and as shown in the previous lecture, v never attains the value $-\infty$. Thus it admits a subgradient at the origin- call it $-\bar{\lambda}$. First we show that $\bar{\lambda}$ is a nonnegative vector: let $b \in \mathbf{R}^m_+$, one has $v(b) \leq f\bar{x})$, and

$$f(\bar{x}) = v(0) \leq v(b) + \bar{\lambda}^T b \leq f(\bar{x}) + \bar{\lambda}^T b \Rightarrow \bar{\lambda}^T b \geq 0 \quad \text{for all } b \geq 0.$$

Thus $\bar{\lambda} \in \mathbf{R}^m_+$. Now for $x \in \text{dom} f$,

$$f(x) \geq v(g(x)) \geq v(0) - \bar{\lambda}^T g(x) = f(\bar{x}) - \bar{\lambda}^T g(x). \tag{8.27}$$

First, let $x = \bar{x}$. Since $\bar{\lambda} \geq 0$ and $g(\bar{x}) \leq 0$ one has $\bar{\lambda}^T g(\bar{x}) = 0$. Moreover (8.27) implies that every $x \in \text{dom} f$ satisfies

$$f(x) + \bar{\lambda}^T g(x) \geq f(\bar{x}) = f(\bar{x}) + \bar{\lambda}^T g(\bar{x}),$$

i.e., $\bar{x}$ minimizes $L(.; \bar{\lambda})$. $\square$

A GLIMPSE OF ALGORITHMS

9 Week 9

Main ideas: A glimpse into the world of algorithms (this lecture corresponds to chapter 12 of the reader)

9.1 Center of Gravity Algorithm

1. Let $f : \mathbf{R} \to \mathbf{R}$, be convex and differentiable. Suppose that the minimizer of f is known to belong the interval C_0. The bisection algorithm for finding $\bar{x}$ proceeds as follows: (1) let $i = 0$, (2) evaluate f' at the midpoint of C_i; if this f' is negative, *discard* the left-half piece, and let $C_i \leftarrow C_i \cap$ (the right hand piece), (3) go to (2). After $k := \log(\text{length of } C_0/\epsilon)$ steps an ϵ-approximation to $\bar{x}$ is obtained (o.k., I did not put a termination rule!).

2. The Ellipsoid algorithm for a convex optimization problem, i.e., finding $\inf f(x) \,|\, x \in C\}$, C convex set and $f :\to \mathbf{R}$, is built on very similar principles as the bisection method. Suppose that $\bar{x}$ is known to lie in an ellipsoidal region E_0. The algorithm proceeds to evaluate the *subgradient* of f, at the center of E_0, x_0; call this subgradient $g(x_0)$.

3. We now discard the half-space $\{x \,|\, g(x_0)^T(x - x_0) \geq 0\}$, since for all such x, one has

$$f(x) \geq f(x_0) + g(x_0)^T(x - x_0) \Rightarrow f(x) \geq f(x_0),$$

   and given that we are looking for the $\inf f$ over C, there is no point to include such points in our search!

4. At each step k, the Ellipsoid algorithm proceeds by finding the minimum volume ellipsoid E_{k+1} that contains the region

$$E_k \cap \{x \,|\, g(x_k)^T(x - x_k) \leq 0\},$$

   and going to its center.

5. The volume of these ellipsoids shrinks pretty fast: vol $(E_{k+1}) < e^{-1/2n}$vol (E_k). However, we still need to show that these iterates converge to the optimal, how to construct the centers fast, what is the stopping criteria, etc.

6. We talked about the center of gravity algorithm and pointed out that in-spite of its conceptual elegance, it is computationally difficult to realize (the sub-problems turn out to be as, or even more difficult as, compared to the original problem!).

9.2 Ellipsoid Algorithm

Before taking a closer look at the Ellipsoid algorithm, we review few facts about ellipsoids and optimization problems over them.

1. Denote by $B(0, 1)$ the unit ball in $\mathbf{R}^n$, with center at the origin.

2. An ellipsoid $E(a, A) \subseteq \mathbf{R}^n$, $a \in \mathbf{R}^n$, $A \in \mathcal{S}^n_{++}$, is the set

$$E(a, A) := \{x \in \mathbf{R}^n \,|\, (x - a)^T A^{-1}(x - a) \leq 1\} = A^{1/2}B(0, 1) + a.$$

3. The volume of $E \subseteq \mathbf{R}^n$ is defined as vol$(E) = \int_{x \in E} dx$.

4. It is known that vol $(E(a,A)) = \sqrt{\det A}\, V_n$, where V_n is the volume of the unit ball,

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad (x > 0).$$

5. A rough estimate for V_n is $n^{-n} \le V_n \le 2^n$.

6. It is easy to maximum a linear functional over an ellipsoid. Start by observing that

$$\max\{c^T x \,|\, x \in B(0,1)\} = c^T(a + \frac{c}{\|c\|}).$$

Note that $A^{-1/2}E(a,A) = B(0,1) + A^{-1/2}a = B(A^{-1/2}a, 1)$. Now we observe that

$$\begin{aligned}
\max\{c^T x \,|\, x \in E(a,A)\} &= \max\{c^T A^{1/2} A^{-1/2} x \,|\, A^{-1/2}x \in A^{-1/2}E(a,A)\} \\
&= \max\{c^T A^{1/2} y \,|\, y \in B(A^{-1/2}a, 1)\} \\
&= c^T A^{1/2} \frac{1}{\|A^{1/2}c\|} A^{1/2}c + c^T A^{1/2} A^{-1/2} a \\
&= c^T \frac{1}{\sqrt{c^T A c}} A c + c^T a = \sqrt{c^T A c} + c^T a.
\end{aligned}$$

7. Generally, optimization algorithms are expected to find an ϵ-approximation ($\epsilon > 0$) of the optimal value (not the optimal solution!)- a point x such that $f(\bar{x}) \ge f(x) + \epsilon$. However, in some special cases (LP) we go for the optimal solution itself.

8. The last observation in Lecture 28 leads to a (simple) stopping criteria for the Ellipsoid algorithm: suppose that at the k-th step, $\sqrt{g(x_k)A_k g(x_k)} \le \epsilon$, where $E_k = E(x_k, A_k)$, and $g(x_k)$ is the subgradient of f at x_k. Given that, as we will show, $\bar{x} \in E_k$,

$$\begin{aligned}
f(\bar{x}) &\ge f(x_k) + g(x_k)^T(\bar{x} - x_k) \ge f(x_k) + \inf_{x \in E_k} g(x_k)^T(x - x_k) \\
&= f(x_k) + \sqrt{g(x_k)A_k g(x_k)} \ge f(x_k) + \epsilon.
\end{aligned}$$

9. Let $0 \ne a \in \mathbf{R}^n$ and $E_k = E(x_k, A_k)$. The intersection of E and $H := \{x \,|\, a^T(x - x_k) \le 0\}$ is called a half-ellipsoid. Let

$$x_{k+1} = x_k - \frac{1}{n+1}\frac{A_k a}{\sqrt{a^T A_k a}}, \quad A_{k+1} = \frac{n^2}{n^2-1}(A_k - \frac{2}{n+1}\frac{A_k a a^T A_k}{a^T A_k a}).$$

Then $A_{k+1} \in \mathcal{S}^n_{++}$ and the ellipsoid $E_{k+1} = E(x_{k+1}, A_{k+1})$ satisfies

$$E_k \cap H \subseteq E_{k+1} \quad \text{and} \quad \text{vol}\,(E_{k+1}) < e^{-1/2(n+1)}\text{vol}(E_k).$$

The above result is the basic geometric construct for the various versions of the Ellipsoid algorithm- we shall describe few below. We first consider the generic case, but the second version is targeted toward LP as we want to say something about the complexity theoretic aspects of LP.

10. We assume that for the given closed bounded convex $C \subseteq \mathbf{E}$, over which we want to find $\inf f$, there is a subroutine which does the following: supplied with a point $x \in \mathbf{E}$, if $x \in C$, it returns a subgradient of f at x, and if $x \notin C$, it returns (the parameters of) the hyperplane separating x from C.

11. Let $E_0 = E(x_0, A_0)$ be such that $C \subseteq E_0$.

12. If $x_0 \notin C$, then the black box returns $a \neq 0$ such that $\sup_{y \in C} a^T y \leq a^T x_0$. Thus we consider the minimum volume ellipsoid containing the half-ellipsoid $E_0 \cap \{x \mid a^T x \leq a^T x_0\}$. On the other hand if $x_0 \in C$, then black box returns $g(x_0)$ and we consider the minimum volume ellipsoid containing the half-ellipsoid $E_0 \cap \{x \mid g(x_0)^T (x - x_0) \leq 0\}$, if $g(x_k) \neq 0$. If $g(x_k) = 0$, we have found the exact minimizer and the algorithm terminates. Note that this algorithm has to converge to the (set of) optimal solution(s).

9.3 Complexity theory

1. We now go through the Ellipsoid algorithm for LP with a view toward finding bounds on the running time of the algorithm and highlight the complexity theoretic implications.

2. First few definitions related to complexity theory: an optimization problem is the collection of its instances: an LP *instance* is a particular realization of the LP *problem*. The size of an instance is the number of bits required to represent it. This is true for a model of computation that we can call the bit-model or the Turing model. Given an LP instance defined by the triplet (c, A, b), $c \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$, and $b \in \mathbf{R}^m$ what is its size? Well, o.k., if we want to use the bit-model, we have to let the entries of these vectors and matrix be integers (or rationals).

3. Let U be a positive integer. An integer $r \leq U$, can represented as

$$r = a_k 2^k + a_{k-1} 2^{k-1} + \ldots + a_1 2^1 + a_0,$$

where $k \leq \lfloor \log U \rfloor$: r can be represented by the binary vector $(a_0, \ldots, a_k)$ and counting the sign 'bit,' we conclude that any integer with a magnitude of at most U can be represented by at most $\lfloor \log U \rfloor + 2$ bits. Thus an LP instance can be represented via $(mn + m + n)(\lfloor \log U \rfloor + 2)$ bits, where U is an upper bound on the largest (in magnitude) number appearing in c, A, or b.

4. Let $f : N \to N$, where N is the set of natural numbers. We write $f(n) = O(n^k)$ for a given positive integer k, if there exist positive numbers n_0 and c, such that $f(n) \leq cn^k$ for all $n \geq n_0$.

5. The running time an algorithm, in a particular model of computational, is the number of elementary operations that the algorithm requires to terminate on instances of the problem. Let $T_A(n)$ be the worst-case running time of the algorithm A over all instances of the problem of size n.

6. An algorithm A is a polynomial-time algorithms for a *problem* if there exists an integer k such that $T_A(n) = O(n^k)$.

7. A *problem* is called polynomial-time solvable if it admits a polynomial-time algorithm (for the solution of all of its instances).

8. The class of polynomial-time solvable problems is denoted by $\mathcal{P}$.

9. Is LP $\in \mathcal{P}$? Yes. Here is why ...

10. We start by first considering checking the feasibility of the polyhedron

$$F := \{x \in \mathbf{R}^n \mid Ax \geq b\}.$$

11. A polyhedron is called full-dimensional if it has a positive volume.

12. We first assume that the polyhedron F is either full-dimensional or empty, as well as bounded: there exists v and V such that $v <$ vol $(F) < V$ where V is the volume of the ball $B(0, R) = E(0, R^2 I)$ containing F. Furthermore, we assume that v and V are a priori known to the algorithm.

13. The Ellipsoid algorithm proceeds as follows: If $x_0 \in F$ then we are done! On the other hand, if $x_0 \notin F$ find $a_i^T x_0 < b_i$ and let $H_0 = \{x \in \mathbf{R}^n \mid a_i^T x \geq a_i^T x_t\}$. Now find the minimum volume ellipsoid E_1 containing $E_0 \cap H$. Note that $F \subseteq E_1$: if $x \in F$, then $x \in H_0$, $F \subseteq E_0$ and, and $E_0 \cap H \subseteq E_1$. Now we iterate but stop at the

$$k^* = \lceil 2(n+1)\log(V/v) \rceil \tag{9.28}$$

iteration.

14. Why does this work: if $x_k \in F$ for $k < k^*$ then the algorithm terminates and F is nonempty. Now suppose that all the points generated before $k^* - 1 \notin F$. Then, by induction, we know that $F \subseteq E_k$ for $k < k^*$, and after (9.28) iterations

$$\text{vol } (E_{k^*}) < V e^{-\lceil 2(n+1)\log(V/v)\rceil/(2(n+1))} \leq V e^{-\log V/v} = v,$$

which means that F has to be empty!

15. Now, we can in fact relax the bounded-ness and the full-dimensionality assumption via the following results (which we did not prove in class): (1) For an LP instance with integer entries, with maximum in magnitude entry less than U, the solution has to lie in the box $x_i \leq (nU)^n$, for all i, (2) we can always pose the feasibility problem on a full-dimensional polyhedron, and (3) for a full dimensional polyhedron, with A and b having integer entries, vol $(F) > n^{-n}(nU)^{-n^2(n+1)}$.

16. Thus we can let

$$v = n^{-n}(nU)^{-n^2(n+1)} \quad \text{and} \quad V = (2n)^n (nU)^{n^2},$$

and therefore the number of iterations in the Ellipsoid algorithm becomes $O(n^4 \log(nU))$. If we start with a unbounded and/or a not-full-dimensional polyhedron, it can shown that $O(n^6 \log(nU))$ iterations will suffice. Moreover, one can show that all numbers generated at each step of the Ellipsoid algorithm have polynomial number of bits (well, this is really a painful exercise but isn't this really wonderful!).

17. How about extensions of the algorithm to LP: we know that

$$\min c^T x \quad \text{s.t.} \quad Ax \geq b,$$

is equivalent to solving the checking the feasibility of the system of inequalities

$$b^T y = c^T x,\ Ax \geq b,\ A^T y = c,\ y \geq 0.$$

Thus LP $\in \mathcal{P}$ (for the bit-model of computation).

18. We say that problem Π_1 reduces to problem Π_2 in polynomial time, written as

$$\Pi_1 \overset{p}{\rightsquigarrow} \Pi_2,$$

if there exists an algorithm for Π_1 that consists of a polynomial number of computations, in addition to polynomial number of subroutine calls to an algorithm for problem Π_2.

19. A recognition problem is a problem that has only a "yes" or "no" answer; an example: given $A \in \mathbf{R}^{m\times n}$, $c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and a real number α, does there exist an x such that $Ax = b$, $x \geq 0$, and $c^T x \leq \alpha$?

20. For a recognition problem Π, we denote the "yes" instances by Π^Y.

21. For two recognition problems Π_1 and Π_2 we write

$$\Pi_1 \overset{p}{\rightsquigarrow} \Pi_2,$$

if there exists a polynomial time algorithm which given $I_1 \in \Pi^Y$ outputs an instance $I_2 \in \Pi_2$, such that $I_1 \in \Pi_1^Y$ if and only if $I_2 \in \Pi_2^Y$.

22. The class of problems for which there is a polynomial-time *verification* algorithm is denoted by $\mathcal{NP}$: if you are given a candidate x for the example in item 20 (assuming that you can do real number computation), you can *verify*, in polynomial-time, that it is in fact a solution.

23. LP with an additional constraints on the variables requiring them to be *integers* is called integer programming (IP).

24. If $\Pi_1 \overset{p}{\rightsquigarrow} \Pi_2$ and $\Pi_2 \in \mathcal{P}$, then $\Pi_1 \in \mathcal{P}$.

25. A problem Π is called $\mathcal{NP}$-hard if

$$IP \overset{p}{\rightsquigarrow} \Pi.$$

26. A problem Π is called $\mathcal{NP}$-complete if $\Pi \in \mathcal{NP}$ and $\Pi \in \mathcal{NP} -$ hard.

27. IP, MAX-CUT, traveling salesman problem (TSP), quadratic programming (not necessary convex), boolean programming, minimum rank problems, bilinear matrix inequality feasibility, ... are all examples of $\mathcal{NP}$-complete problems.

    Adding to the list of $\mathcal{NP}$-hard and $\mathcal{NP}$-complete problems is accomplished by using the following observation:

28. If $\Pi_1 \in \mathcal{NP} -$ hard and $\Pi_0 \overset{p}{\rightsquigarrow} \Pi_2$, then $\Pi_2 \in \mathcal{NP} -$ hard. The crucial step is therefore to come up with the first one! For this, take a course on computational complexity! The first problem shown to be $\mathcal{NP}$-complete was the satisfiability problem in propositional logic.

10 Week 10

Main ideas: Introduction to interior point methods for convex programming

1. Up to now, you have been kind enough to trust me when I said that LP, SDP, and convex quadratic programs can be solved efficiently. You might have asked yourself: why is this guy trying to put everything in terms of SDPs?! We now want to validate this. In this venue, we will see that duality, complementarity, and the necessary optimality conditions, come into the picture in an elegant way.

2. An iterative algorithm for an optimization problem is really a dynamical system with the solution as its equilibrium point- preferably a globally asymptotically stable one. Iterative algorithms are usually comprised of the an initialization procedure, update rule, and a termination criteria.

3. Recall that the Jacobian of a function $f : \mathbf{R}^n \to \mathbf{R}^n$ at a point $z \in \mathbf{R}^n$, is the matrix of partial derivatives: $J_{ij}(z) = \partial f_i(x)/\partial x_j$, evaluated at z.

4. Newton's method: Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable. It is desired to find $\bar{x}$ such that $f(\bar{x}) = 0$. Starting with an initial point x^0, we consider the Taylor series expansion of f around x^0: $f(x^0 + d) \approx f(x^0) + J(x^0)d$. The first Newton step $n(x^0)$ attempts to make this approximation vanish at $x^0 + n(x^0)$. When $J(x^0)$ is invertible, this means that $n(x^0) = -J(x^0)^{-1}f(x^0)$. In general, when x is the current point, we make a Newton update

$$x^+ = x + n(x),$$

where $n(x) := -J(x)^{-1}f(x)$. Then, well, iterate! For now, it suffices to say that when the x^0 is *close* to $\bar{x}$, Newton's method (algorithm) converges to the solution very, very, fast.

5. When $f : \mathbf{R}^n \to \mathbf{R}$ is twice differentiable and the objective is to minimize f, the Newton's step $x \in \mathbf{R}^n$ takes the form

$$n(x) = -H(x)^{-1}\nabla f(x),$$

provided, that of course, the Hessian H is invertible. Thus we have a candidate algorithm of unconstrained optimization.

6. We first consider the the primal path following algorithm for LP, and then generalize it to the general finite dimensional convex optimization.

7. We consider the LP in standard form,

$$P: \quad \min_x c^T x \quad Ax = b,\ x \geq 0 \quad \text{and} \quad D: \quad \max_y b^T y \quad \text{s.t.} \quad A^T y + s = c,\ s \geq 0. \tag{10.29}$$

8. The barrier functional for (P) is $b_\mu = c^T x - \mu \sum \log x_i$, where $\mu > 0$. The barrier problem for (P) is

$$\min_x b_\mu(x) \quad \text{s.t.} \quad Ax = b. \tag{10.30}$$

Similarly, the barrier problem for (D) is

$$\max_y b^T y + \mu \sum \log s_i \quad \text{s.t.} \quad A^T y + s = c.$$

9. Suppose that we could solve (approximately) the primal and dual barrier problems, obtaining the triplet $(x(\mu), y(\mu), s(\mu))$. The central path is defined to be $x(\mu)$ as $\mu \to 0$.

10. We will show in the next lecture that given an appropriate update rule for the barrier parameter μ, the central path will converge, in a rather efficient manner, to the optimal solution of (10.29). This will follow by observing that the algorithm produces primal and dual feasible points with duality gap $x(\mu)^T s(\mu)$ converges to zero.

11. We start from the end- the termination criteria is supplied by the duality gap: if x and y are primal and dual feasible points such that $c^T x - b^T y < \epsilon$, then
$$c^T \bar{x} \leq c^T x \leq c^T \bar{y} + \epsilon \quad \text{and} \quad b^T \bar{y} - \epsilon \leq b^T y \leq b^T \bar{y}.$$

12. We will use the following convention just for the purpose of this lecture: if $x \in \mathbf{R}^n$, then X is the diagonal matrix such that $X_{ii} = x_i$ for all i; similarly for vectors s and d, and square matrices S and D.

13. We assume that the matrix A in (10.29) has full row rank.

14. We assume, for now, that initial strictly feasible primal and dual points, x^0 and (y^0, s^0), have been supplied to us; along with a desired optimality tolerance $\epsilon > 0$. We talk more about this below (initialization).

15. We start from current primal feasible vector x^k and take one Newton step in the null space of A, in order to minimize B_μ. To obtain the expression for this projected Newton step, we first consider the first three terms of the Taylor series expansion of $b_\mu(x^k + d)$ around x^k, as
$$b_\mu(x^k) + \nabla b_\mu(x)^T d + \frac{1}{2} d^T H(x) d, \quad \text{where} \quad \nabla b_\mu = c - \mu X^{-1}\mathbf{1},\ H(x) = \mu X^{-2}.$$

16. Next we consider
$$\min_d (c - \mu X^{-1}\mathbf{1})^T d + \frac{1}{2}\mu d^T X^{-2} d, \quad \text{s.t.} \quad Ad = 0.$$
This can be solved explicitly; introduce the Lagrange multiplier y and consider,
$$c - \mu X^{-1}\mathbf{1} + \mu X^{-2} d - A^T y = 0, \quad \text{and} \quad Ad = 0; \tag{10.31}$$
This is a system of linear equations in d and y. We now let $\mu^{k+1} = \alpha\mu^k$, $x^{k+1} = x^k + d$, $y^{k+1} = y$, and $s^{k+1} = c - A^T y$ and, well, iterate till $(x^k)^T s^k < \epsilon$. But why should we ever get to that?

17. Let $\beta < 1$, and $\alpha = 1 - \{(\sqrt{\beta} - \beta)/(\sqrt{\beta} + \sqrt{n})\}$. For a given μ^0, first we let $\mu^k = \alpha^k \mu^0$ (note that $\mu^k \to 0$).

18. From (10.31) it follows that $\mu^{k+1} d^T X_k^{-2} d = d^T(\mu^{k+1} X_k^{-1}\mathbf{1} - c)$. Thus
$$\begin{aligned}
\|X_k^{-1} d\|^2 &= d^T X_k^{-2} d = (X_k^{-1}\mathbf{1} - \frac{1}{\mu^{k+1}} c)^T d = (X_k^{-1}\mathbf{1} - \frac{1}{\mu^{k+1}}(s^k + A^T y^k))^T d \\
&= (X_k^{-1}\mathbf{1} - \frac{1}{\mu^{k+1}}(s^k)^T d = -(\frac{1}{\mu^{k+1}} X_k S_k \mathbf{1} - \mathbf{1})^T X_k^{-1} d \leq \|\frac{1}{\mu^{k+1}} X_k S_k \mathbf{1} - \mathbf{1}\| \, \|X_k^{-1} d\|.
\end{aligned}$$
But
$$\begin{aligned}
\|\frac{1}{\mu^{k+1}} X_k S_k \mathbf{1} - \mathbf{1}\| &= \|\frac{1}{\alpha\mu^k} X_k S_k \mathbf{1} - \mathbf{1}\| = \|\frac{1}{\alpha}(\frac{1}{\mu^k} X_k S_k \mathbf{1} - \mathbf{1}) + (\frac{1-\alpha}{\alpha})\mathbf{1}\| \\
&= \frac{1}{\alpha}\|\frac{1}{\mu^k} X_k S_k \mathbf{1} - \mathbf{1}\| + (\frac{1-\alpha}{\alpha})\|\mathbf{1}\| \leq \frac{\beta}{\alpha} + \frac{1-\alpha}{\alpha}\sqrt{n} = \sqrt{\beta}
\end{aligned}$$
Thus $\|X_k^{-1} d\| \leq \sqrt{\beta} < 1$.

19. We note that $x^{k+1} = x^k + d = X_k(e + X_k^{-1}d) > 0$ and $s^{k+1} = c - A^T y^{k+1} = \mu^{k+1} X_k^{-1}(e - X_k^{-1}d) > 0$. By construction $Ax^{k+1} = b$ and $A^T y^{k+1} + s^{k+1} = c$. Thereby, the algorithm always generate primal and dual feasible points.

20. Let $x \in \mathbf{R}^n$. Then $\|x\| \le \|x\|_1 = \sum |x_i|$.

21. We now note that since $x_i^{k+1} = x_i^k(1 + d_j/x_j^k)$ and $s_i^{k+1} = (\mu^{k+1}/x_j^k)(1 - d_j/x_j^k)$, one has

$$(1/\mu^{k+1}) x_j^{k+1} s_j^{k+1} - 1 = -(d_j/x_j^k)^2.$$

Thus

$$\begin{aligned} \|\frac{1}{\mu^{k+1}} X_{k+1} S_{k+1} \mathbf{1} - \mathbf{1}\| &= \|X_k^{-2} D^2 \mathbf{1}\| \le \|X_k^{-2} D^2 \mathbf{1}\|_1 = \mathbf{1}^T X_k^{-2} D^2 \mathbf{1} \\ &= \mathbf{1}^T D X_k^{-2} D \mathbf{1} = d^T X_k^{-2} d = \|X_k^{-1} d\|^2 \le (\sqrt{\beta})^2 = \beta. \end{aligned}$$

22. But this means that

$$-\beta \le \frac{1}{\mu^k} x_i^k s_i^k - 1 \le \beta, \quad \text{and thus} \quad n\mu^k(1-\beta) \le (s^k)^T x^k \le n\mu^k(1+\beta).$$

Moreover

$$\mu^k = \alpha^k \mu^0 = (1 - \frac{\sqrt{\beta} - \beta}{\sqrt{\beta} + \sqrt{n}})^k \, \mu^0 \le e^{-k(\sqrt{\beta}-\beta)/(\sqrt{\beta}+\sqrt{n})} \, \mu^0.$$

23. Define

$$K := \lceil \frac{\sqrt{\beta} + \sqrt{n}}{\sqrt{\beta} - \beta} \log \frac{(s^0)^T x^0 (1+\beta)}{\epsilon(1-\beta)} \rceil \ge \lceil \frac{\sqrt{\beta} + \sqrt{n}}{\sqrt{\beta} - \beta} \log \frac{\mu^0 n (1+\beta)}{\epsilon} \rceil;$$

after K iterations, one has $(s^K)^T x^K \le \epsilon$.

24. How about initialization: the strategy is to convert the given LP to one which has a trivial initialization. Here is one strategy that works best with primal path following method. It can be shown that if A, b, c have integer entries, with absolute values bounded by the number U, the we can always confine ourselves to the

$$\min_x c^T x \quad \text{s.t.} \quad Ax = b, \ \mathbf{1}^T x \le n(mU)^m, \ x \ge 0$$

which can be also be written as

$$\min_x c^T x \quad \text{s.t.} \quad Ax = \bar{b}, \ \mathbf{1}^T x \le n + 2, \ x \ge 0.$$

Now let M to be a large positive number and consider the primal and dual LP's:

$$\min c^T x + M x_{n+1} \quad \text{s.t.} \quad Ax + (\bar{b} - A\mathbf{1}) x_{n+1} = \bar{b}, \ \mathbf{1}^T x + x_{n+1} + x_{n+2} = n + 2, x \ge 0, x_{n+1}, x_{n+2} \ge 0$$

and

$$\begin{aligned} \max b^T y + y_{m+1}(n+2) \quad \text{s.t.} \quad & A^T y + y_{m+1}\mathbf{1} + s = c, \ (\bar{b} - A\mathbf{1})^T y + y_{m+1} + s_{n+1} = M, \\ & y_{m+1} + s_{n+2} = 0, s \ge 0, s_{n+1}, s_{n+2} \ge 0. \end{aligned}$$

25. Let $\mu^0 = 4(\|c\|^2 + M^2)^{1/2}$. We note that

$$(x^0, x^0_{n+1}, x^0_{n+2}) = (\mathbf{1}, 1, 1), \quad \text{and} \quad (p^0, p^0_{m+1}, s^0, s^0_{n+1}, s^0_{n+2}) = (0, -\mu^0, c + \mu^0\mathbf{1}, M + \mu^0, \mu^0),$$

are primal and dual feasible vectors and

$$\|\frac{1}{\mu^0} X_0 S_0 \mathbf{1} - \mathbf{1}\| = \frac{1}{4}.$$

By choosing M large, we can guarantee that $x_{n+1} = 0$. Thus we can let $\beta = 1/4$ above.

11 Week Eleven

Main ideas: Student Presentations, Epilogue

11.1 Student presentations

11.2 Epilogue

I finish with a few words of ... I am not sure what to call them:

1. Optimization theory generally does not help you choose the *right* objective for your problem. However, it does assist you in formulating problems that are meaningful, relevant, yet, solvable.
2. Optimization theory is a language- it has a tremendous expressive power- however, use it with care- it might not be the right language to formulate or solve your (engineering) problem.
3. Pay attention to algorithms- people (engineers) are generally not too enthusiastic about problem formulations that nobody can solve or can solve but have to wait for a long long time for their solution.
4. On linear analysis and its relation with optimization. zeros and cones, subspaces and convex sets, linearity and convexity

"A good teacher does not teach facts,
he or she teaches enthusiasm,
open mindedness, and values."

Gian-Carlo Rota

Check back occasionally for updates and do send me your suggestions[1].

[1]Email: *mesbahi@aa.washington.edu*

EXTRAS

12 Problems

Main ideas: Problems and their solutions

13 Extra 1

Main ideas: More on PSD representability

1. What is the dual cone of the positive semi-definite (psd) matrices (with respect to the trace inner product in the Euclidean space of symmetric matrices)? Recall that for every $x \in \mathbf{R}^n$, xx^T is psd, and so is the sum of such outer products. In fact, every psd matrix, X, admits a representation of the form $\sum_i \lambda_i x_i x_i^T$, $\lambda \geq 0$. Thus if $X, Y \in \mathcal{S}^n_+$, $\langle X, Y \rangle = \sum \lambda_i x_i^T Y x_i$, i.e., $\mathcal{S}^n_+ \subseteq (\mathcal{S}^n_+)^*$. On the other hand, if $Y \in (\mathcal{S}^n_+)^*$ it has to be the case that $\left\langle Y, xx^T \right\rangle = x^T Y x \geq 0$, for all $x \in \mathbf{R}^n$. We thus conclude that the psd cone is self-dual.

2. In many areas of engineering it is desired to minimize the sum of the k-th largest eigenvalues of the a matrix that is a function of a decision vector x. Examples include filter design and structural optimization.

3. An extreme point of a convex set $C \subseteq \mathbf{E}$, x, is a point of C such that $C \backslash \{x\}$ is convex. This means that x can not be expressed as a convex combination of two distinct points of C.

4. Let $C_1 := \{xx^T \mid x \in \mathbf{R}^n,\ \|x\| = 1\}$ and $C_2 := \{X \mid X \in \mathcal{S}^n,\ 0 \leq X \leq I,\ \textbf{Trace}\ X = 1\}$. Then C_1 is the set of extreme points of C_2 and C_2 is the convex hull of C_1.

5. Let $A \in \mathcal{S}^n$. The maximum eigenvalue of A is characterized by

$$\max_{\|x\|=1} x^T A x = \max\{A \cdot X \mid X \geq 0, \textbf{Trace}\ X = 1,\ 0 \leq X \leq I\}. \tag{13.32}$$

   This follows from the observation that the optimum on the right hand side of (13.32) occurs at an extreme point.

6. Similarly, let $C_1 := \{XX^T \mid X \in \mathbf{R}^{n \times k},\ X^T X = I\}$ and $C_2 := \{Z \mid Z \in \mathcal{S}^n,\ 0 \leq Z \leq I,\ \textbf{Trace}\ Z = k\}$. Then C_1 is the set of extreme points of C_2 and C_2 is the convex hull of C_1.

7. **Lemma 13.1** *The sum of k-th largest eigenvalues of $A \in \mathcal{S}^n$, can be found by*

$$\bar{\alpha} := \max_X A \cdot X \quad \textit{s.t.} \quad \textbf{Trace}\ X = k,\ 0 \leq X \leq I. \tag{13.33}$$

   Proof: We use the Courant-Fischer variational characterization of the k-th largest eigenvalue of a $A \in \mathcal{S}^n$; it goes like this: let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$. Then for $k = 1, 2, \ldots, n$ one has

$$\lambda_k(A) = \max\{q_k^T A q_k \mid \|q_k\| = 1,\ q_k^T \bar{q}_i = 0\ (i = 1, 2, \ldots, k-1)\},$$

   where $\bar{q}_i$'s are the optimal solutions for the same optimization problem when calculating $\lambda_i(A)$. Thus

$$\sum_{i=1}^{k} \bar{q}_i^T A \bar{q}_i = \textbf{Trace}\ (Q^T A Q) = QQ^T \cdot A, \quad \text{where} \quad Q = [\, q_1, \ldots, q_n \,].$$

   Since QQ^T belongs to the feasible set of (13.33) by (5), $\bar{\alpha}$ is an upper bound for the sum of the k-th largest eigenvalues of A; it is also a lower bound: suppose that the optimal solution to (13.33) is $\bar{X}$. Since $\bar{X}$ can be chosen to be an extreme point of the feasible region (13.33), $\bar{\alpha}$ = QQ$^T \cdot A = \sum_i q_i^T A q_i$, where $Q^T Q = I$. $\square$

8. Suppose now that A is affinely dependent on a decision varilable, x, i.e., $A = A_0 + \sum_{i=1}^{m} x_i A_i$. We like to choose x as to minimize the sum of the k-th largest eigenvalue of $A(x)$. Note that the inclusion of x in (13.33) makes the objective of the corresponding min-max problem *bilinear*.

9. Here is a trick: the dual of (13.33) is

$$\min_{\alpha, Y} k\alpha + \textbf{Trace}\, Y \quad \text{s.t.} \quad \alpha I + Y \geq A,\ Y \geq 0. \tag{13.34}$$

This latter program nicely accommodates the inclusion of an affine dependency of A on decision vector, while maintaining linearity:

$$\min_{\alpha, x, Y} k\alpha + \textbf{Trace}\, Y \quad \text{s.t.} \quad \alpha I + Y \geq A_0 + \sum_i x_i A_i,\ Y \geq 0. \tag{13.35}$$

We observe that the dual of (13.35) is

$$\max_X A_0 \cdot X \quad \text{s.t.} \quad \textbf{Trace}\, X = k,\ A_i \cdot Y = 0\ (i = 1, \ldots, m),\ 0 \leq X \leq I. \tag{13.36}$$

10. On algorithms, efficiency, and what it means to be exponential-time.

14 Extra 2

Main ideas: Theorems of alternative on $\mathbf{R}^n$

1. In this and the next few lectures we will let $\mathbf{E} = \mathbf{R}^n$ and $\langle x, y\rangle$ is our familiar $x^T y$.

2. The theory of inequalities is the cornerstone of optimization theory. It is important for us to be able to efficiently address consistency of a system of inequalities, the geometry of the feasible sets, etc.

3. The set defined by a system of inequalities is called its feasible set. If the feasible set is empty we call the system of inequalities infeasible.

4. We first consider inequalities that are defined by linear functionals: we like to understand the properties of sets defined by the intersections of hyperplanes and halfspaces. For this purpose we look into extending ideas from linear algebra.

   A quick review of two important results in linear algebra ...

5. Let $S \subseteq \mathbf{R}^n$. Then the orthogonal complement of S in $\mathbf{R}^n$ is

$$S^\perp := \{y \in \mathbf{R}^n \,|\, x^T y = 0, \text{ for all } x \in S\}. \tag{14.37}$$

6. If S is a subspace then $S^{\perp\perp} = S$.

7. One of the key relations in linear algebra is the following: for every matrix A, $R(A) = N(A^T)^\perp$, where $R(A)$ is the range space of A (the space spanned by the columns of the matrix A) and $N(A)$ is the null space of A (the set of vectors orthogonal to all rows of A).

8. Suppose that we are given $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. How can one check whether $Ax = b$ does not have a solution- ever? From (8) it follows that $Ax = b$ has a solution if and only if for all $y \in \mathbf{R}^m$ such that $A^T y = 0$, $b^T y = 0$, i.e., $Ax = b$ is feasible if and only if the pair $A^T y = 0$, $b^T y > 0$ is infeasible. We call

$$[\,Ax = b\,] \quad \text{and} \quad [\,A^T y = 0,\, b^T y > 0\,],$$

   systems of alternative built on the matrix A and vector b.

9. Let $\mathbf{A} := \{a^1, a^2, \ldots, a^n\}$, $a^i \in \mathbf{R}^m$. Then

$$\textbf{pos } \mathbf{A} := \{\sum_{i=1}^n \lambda_i a^i,\ \lambda_i \geq 0\} = \textbf{pos } A = \{Ax \,|\, x \in \mathbf{R}^n,\ x \geq 0\},$$

   where $A \in \mathbf{R}^{m\times n}$ and $x \geq 0$ abbreviates having $x_i \geq 0$ for all i.

10. For $A \in \mathbf{R}^{m\times n}$, **pos** A is a closed cone (we will say more its closedness later).

11. The (negative) polar cone of a set $C \subseteq \mathbf{E}$

$$C^- := \{y \in \mathbf{E} \,|\, \langle x, y\rangle \leq 0, \text{ for all } x \in C\}.$$

12. The polar cone of a set is always a convex cone.

    We are now ready for our first *theorem of alternative* ...

13. **Theorem 14.1** *Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$.Then*

$$\underbrace{[\,Ax = b,\ x \geq 0\,]}_{\textit{system 1}} \quad \textit{and} \quad \underbrace{[\,y^T A \leq 0,\ y^T b > 0\,]}_{\textit{system 2}}$$

are systems of alternative.

Proof: We note that both systems can not be feasible since if they did then

$$0 < y^T b = y^T (Ax) = (y^T A)\, x \leq 0.$$

Now suppose that system 1 is not feasible, i.e., $b \notin \mathbf{pos}\ A$. Then by Theorem 2.2 there exists $y \in \mathbf{R}^m$ such that $y^T b > 0$ and $y^T a^i \leq 0$ ($i = 1, \ldots, n$), which is equivalent to the feasibility of system 2. $\square$

14. By Farkas Lemma we usually refer to a re-statement of the above theorem: for $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$, if $y^T b \leq 0$ for all $y^T A \leq 0$, then there exists $x \geq 0$ such that $Ax = b$: if b is away from all vectors in the polar cone generated by the columns of A it has no choice but be in the cone generated by them!

15. Another way to look at Farkas Lemma, and to remember it, is in terms of the generalization of (7).

16. **Theorem 14.2** *Let $C \subseteq \mathbf{E}$ be a closed convex cone. Then $C^{--} = C$.*

Proof: Let $x \in C$. Then $\langle x, y\rangle \leq 0$ for all $y \in C^-$ and therefore, $x \in C^{--}$ and $C \subseteq C^{--}$. Now let $x \in C^{--}$. Consider the projection of x on C, $\widetilde{x}$; we know that $\langle x - \widetilde{x}, y - \widetilde{x}\rangle \leq 0$, for all $y \in C$ (this is the restatement of $-\nabla f(\widetilde{x}) \in N_C(\widetilde{x})$ when f is (half of) the distance function; see Theorem 2.2). Let $y = 0$ and $y = 2\,\widetilde{y}$ above; we conclude that

$$\langle x - \widetilde{x}, \widetilde{x}\rangle = 0, \tag{14.38}$$

which implies that $\langle x - \widetilde{x}, y\rangle \leq 0$ for all $y \in C$, i.e., $x - \widetilde{x} \in C^-$. But $x \in C^{--}$ and thereby $\langle x - \widetilde{x}, x\rangle \leq 0$. Adding this last inequality to (14.38) we obtain $x = \widetilde{x}$, i.e., $x \in C$. Thus $C^{--} \subseteq C$. $\square$

Reading: [BV]: 1.5, 2.1, 6.1
Exercises: [BV]: 1.6, 1.7, 1.10, 1.16, 1.19

15 Extra 3

Main ideas: LP duality and applications - I

1. Farkas Lemma is only one of the many instances of the theorems of alternative. Here are few more examples; again, our focus will be on the case when $\mathbf{E} := \mathbf{R}^n$; keep in mind however that very similar statements hold in arbitrary Euclidean spaces.

2. Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. Then the following are instances of systems of alternative

$$[\,Ax \geq b\,] \quad \& \quad [\,y^T A = 0,\, y \geq 0,\, y^T b > 0\,], \tag{15.39}$$

$$[\,Ax \geq b,\, x \geq 0\,] \quad \& \quad [\,y^T A \leq 0,\, y \geq 0,\, y^T b > 0\,], \tag{15.40}$$

$$[\,Ax > 0\,] \quad \& \quad [\,y^T A = 0,\, y \geq 0,\, y \neq 0\,], \tag{15.41}$$

$$[\,Ax > 0,\, x > 0\,] \quad \& \quad [\,y^T A \leq 0,\, y \geq 0,\, y \neq 0\,], \tag{15.42}$$

$$[\,Ax \geq 0,\, Ax \neq 0\,] \quad \& \quad [\,y^T A = 0,\, y > 0\,]. \tag{15.43}$$

Reading: [BV]: 1.4, 1.6, 3.1, 3.2 (except 3.2.5), 3.3, 3.4
Exercises: [BV]: 1.22, 1.23, 1.28, 1.29, 3.3, 3.4

16 Extra 4

Main ideas: LP duality and applications - II

1. The dual of $\min_x\{c^T x \mid Ax \geq b\}$ is $\max_y\{b^T y \mid A^T y = c, y \geq 0\}$.
2. The ∞-norm and the 1-norm of $x \in \mathbf{R}^n$ are respectively, $\|x\|_\infty := \max_i |x_i|$ and $\|x\|_1 := \sum_i |x_i|$.
3. Chebyshev approximation:

$$\text{Given} \quad A := \begin{bmatrix} (a^1)^T \\ (a^2)^T \\ \vdots \\ (a^m)^T \end{bmatrix}, \quad \text{and} \quad b \in \mathbf{R}^m, \quad \text{solve} \quad \min_{x \in \mathbf{R}^n} \|Ax - b\|_\infty.$$

4. First, put the problem in LP form:

$$(P): \quad \min c^T z \quad \text{subject to} \quad Dz \geq d, \tag{16.44}$$

where

$$D := \begin{bmatrix} 1 & a_1^T \\ 1 & -a_1^T \\ \vdots & \vdots \\ 1 & a_m^T \\ 1 & -a_m^T \end{bmatrix}, \quad d := \begin{bmatrix} b_1 \\ -b_1 \\ \vdots \\ b_m \\ -b_m \end{bmatrix}, \quad c := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad z := \begin{bmatrix} t \\ x \end{bmatrix}.$$

Then the dual instance (D) will be

$$\max d^T w \quad \text{subject to} \quad D^T w = c,\ w \geq 0. \tag{16.45}$$

Let $v_i = w_i - w_{i+1}$ and $u_i = w_i + w_{i+1}$, for $i = 1, 3, 5, \ldots, 2m-1$. Then (D), after eliminating the u-variables, assumes the form

$$(D): \quad \max b^T v \quad \text{subject to} \quad \sum_i |v_i| \leq 1,\ \sum_i v_i a^i = 0. \tag{16.46}$$

5. Let $\mathcal{L}$ be the linear subspace spanned by the columns of A.
6. (P) is then the following problem: given $b \in \mathbf{R}^n$, find an element of $\mathcal{L}$ closest to b in the ∞-norm.
7. (D) is the following problem: find a linear functional on $\mathbf{R}^m$ of the 1-norm not exceeding one, which separates best of all the point b and the linear subspace $\mathcal{L}$.
8. Duality says the following: the ∞-norm distance from b to $\mathcal{L}$ is equal to the maximum quantity by which b can be separated from $\mathcal{L}$ by a linear functional of 1-norm at most 1.
9. Check out `>>help linprog` at the Matlab prompt.
10. Two optimal control problems reducible to LP.

    I also want to say a few words on generalized inequalities- we will re-visit this again later ...

11. Let $K \subseteq \mathbf{E}$ be a closed convex cone, **int** $K \neq \emptyset$ (solid), and if $x, -x \in K$ then $x = 0$ (pointed); we call it a proper cone for short. K defines an ordering in (for) $\mathbf{E}$: $x \leq_K y$ if and only if $y - x \in K$. Notice that $\leq_K$ is reflexive, nonnegative homogeneous, anti-symmetric (from pointed-ness), is preserved under limits (from closed-ness), and is preserved under addition. The solid-ness of being proper is used to facilitate using strict inequalities if we need to.

12. The dual cone of $C \subseteq \mathbf{E}$ is

$$C^* := \{y \in \mathbf{E} \mid \langle x, y \rangle \geq 0, \quad \text{for all } x \in C\}.$$

13. If $K \subseteq \mathbf{E}$ is proper then K^* is proper; thus K^* also defines an ordering on $\mathbf{E}$.

14. Systems of alternative can be expressed using inequalities induced by proper cones, e.g., with proper cone $K \subseteq \mathbf{R}^n$, (16.47) reads:

$$[\, Ax >_K 0 \,] \qquad \& \qquad [\, A^T y = 0,\ y \geq_K 0,\ y \neq 0 \,], \tag{16.47}$$

are alternative systems.

17 Extra 5

Main ideas: Minimax theorem, SDP Duality

Convexity is the key ingredient that makes duality statements and strong necessary conditions possible for LP and SDP.

Two recurrent themes in optimization is the appearance of multipliers and the notion of complementarity.

Let $\mathbf{E}$ and $\mathbf{Y}$ be two Euclidean spaces and $A : \mathbf{E} \to \mathbf{Y}$ be a linear map. The adjoint of A, A^* is the map $A^* : \mathbf{Y} \to \mathbf{E}$ such that for all $x \in \mathbf{E}$ and $y \in \mathbf{Y}$, one has $\langle A^* y, x\rangle = \langle y, Ax\rangle$.

Proposition 17.1 ***(First order condition for linear constraints)*** *Let $C \subseteq \mathbf{E}$, $f : C \to \mathbf{R}$ be differentiable, and $A : \mathbf{E} \to \mathbf{Y}$ be a linear map from one Euclidean space to another. Let $b \in Y$. Consider the optimization problem*

$$\inf f(x) \quad \textit{s.t.} \quad Ax = b,\ x \in C. \tag{17.48}$$

*Let $\bar{x} \in$ **int** C and $A\bar{x} = b$. Then if $\bar{x}$ is a local minimum of (17.48), $\nabla f(\bar{x}) \in A^*Y$. Furthermore, if f is convex, $\nabla f(\bar{x}) \in A^*Y$ guarantees that $\bar{x}$ is the global minimum of (17.48).*

Proof: We need to show that the normal cone to $\{x \mid Ax = b\}$ is A^*Y. □

Proposition 17.2 *Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable. Consider the optimization problem*

$$\inf f(x) \quad \textit{s.t.} \quad x \geq 0. \tag{17.49}$$

Let $\bar{x}$ be a local minimum of (17.49). Then

$$\nabla f(\bar{x}) \geq 0,\ \bar{x} \geq 0,\ \textit{and}\ \langle \bar{x}, \nabla f(\bar{x})\rangle = 0. \tag{17.50}$$

Furthermore, if f is convex, (17.50) is sufficient condition for global optimality.

Proof: What is the normal cone to the positive cone? □

Check out Matlab's Optimization toolbox: `>> help toolbox/optim`, and in particular, linprog, quadprog, fminunc, and the demos.
Reading: [BV] Chapter 3
Reminder: Midterm #1; next Thursday (take home)

18 Extra 6

Main ideas: Conjugation (We will talk about these more after the exam; read thru item 8)

1. $\nabla \log\det X = X^{-1}$, for $X \in \mathcal{S}^n_{++}$.
2. Convention: $\bar{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$, and $0 \times \infty = 0 = 0 \times (-\infty)$.
3. Let $h : \mathbf{E} \to \bar{\mathbf{R}}$. Define the Fenchel conjugate of h, $h^* : \mathbf{E} \to \bar{\mathbf{R}}$, through
$$h^*(\phi) = \sup_{x \in \mathbf{E}} \{\langle \phi, x\rangle - h(x)\}.$$
4. h^* is always a convex function since it is the pointwise sup of a parametrized family of linear functions. If dom$h \neq \emptyset$, then h^* never assumes the value $-\infty$.
5. The conjugation operation is order preserving: if $f, g : \mathbf{E} \to \bar{\mathbf{R}}$ and $g \leq f$, then $f^* \leq g^*$.
6. Here are some examples ($g(y) := f^*(y)$):
$$f(x) = ax + b,\ \text{dom} f = \mathbf{R} \quad \longleftrightarrow \quad g(y) = \begin{cases} -b & \text{if} \quad y = a \\ +\infty & \text{if} \quad y \neq a \end{cases}$$
$$f(x) = -\log x,\ \text{dom} f = \mathbf{R}_{++} \quad \longleftrightarrow \quad g(y) = \begin{cases} +\infty & \text{if} \quad y \geq 0 \\ -\log(-y) - 1 & \text{if} \quad y < 0 \end{cases}$$
$$f(x) = e^x,\ \text{dom} f = \mathbf{R} \quad \longleftrightarrow \quad g(y) = \begin{cases} +\infty & \text{if} \quad y < 0 \\ y\log(y) - y & \text{if} \quad y \geq 0 \end{cases}$$
(Boltzmann-Shannon entropy)
$$f(x) = \log(1 + e^x),\ \text{dom} f = \mathbf{R} \quad \longleftrightarrow \quad g(y) = \begin{cases} 0 & \text{if} \quad y = 0 \text{ or } 1 \\ y\log y + (1-y)\log(1-y) & \text{if} \quad y \in (0,1) \\ +\infty & \text{if} \quad y < 0 \end{cases}$$
(Fermi-Dirac entropy)
$$f(x) = -\log(1 - e^x),\ \text{dom} f = \mathbf{R} \quad \longleftrightarrow \quad g(y) = \begin{cases} 0 & \text{if} \quad y = 0 \\ y\log y - (1+y)\log(1+y) & \text{if} \quad y > 0 \\ +\infty & \text{if} \quad y < 0 \end{cases}$$
(Bose-Einstein entropy)
7. We have the following transformation rules: the conjugate of $h(\alpha x)$, $\alpha \neq 0$, is $h^*(y/\alpha)$; the conjugate of $\alpha h(x)$, $\alpha > 0$, is $\alpha h^*(y/\alpha)$.
8. Let lb : $\mathbf{R}^n \to (-\infty, +\infty]$ and ld : $\mathcal{S}^n \to (-\infty, +\infty]$ be defined as, lb$(x) = -\sum \log x_i$ when $x \in \mathbf{R}^n_{++}$ and lb$(x) = +\infty$, otherwise; similarly, let ld$(X) = -\log\det X$ when $X \in \mathcal{S}^n_{++}$ and ld$(X) = +\infty$, otherwise. Then
$$\text{lb}^*(x) = \text{lb}(-x) - n, \quad \text{for all } x \in \mathbf{R}^n, \quad \text{and} \quad \text{ld}^*(X) = \text{ld}(-X) - n, \quad \text{for all } X \in \mathcal{S}^n.$$
9. For any cone $\delta_{K^-} = \delta^*_K$ and $\delta_{K^{--}} = \delta^{**}_K$.
10. Recall that the polar cone of a set $K \subseteq \mathbf{E}$ is the convex cone
$$K^- := \{\phi \in \mathbf{E} \mid \langle \phi, x\rangle \leq 0, \quad \text{for all } x \in K\};$$
we note that $N_C(x) = (C - x)^-$.

11. The polar cones of $\mathbf{R}^n_+$ and $\mathcal{S}^n_+$ are respectively, $-\mathbf{R}^n_+$ and $-\mathcal{S}^n_+$.

12. **Theorem 18.1** ***(Bipolar cone)*** *The bipolar cone of any nonempty set $K \subseteq \mathbf{E}$ is given by $K^{--} =$ cl $\{$co $(\mathbf{R}_+ K)\}$.*

13. Let $T_C(x)$, the tangent cone to C at x, be defined by

$$T_C(x) := \text{cl}\, \mathbf{R}_+(C - x).$$

Then for $C \subseteq \mathbf{E}$, $\{N_C(x)\}^- = T_C(x)$ and $\{T_C(x)\}^- = N_C(x)$.

Definition 18.1 *Given an inner product on $\mathbf{R}^n$, and a linear map $L : \mathbf{R}^{n\times n} \to \mathbf{R}^n$, the adjoint of L, L^{adj} is the unique linear map such that,*

$$< u, Lv >=< L^{adj}u, v > .$$

Problem 18.2 *Let $P > 0$ and $< u, v >:= u^T P^{-1} v$. Show that for a given linear map A, A^{adj} with respect to this inner product is $PA^T P^{-1}$.*

Theorem 18.3 *Given $A \in \mathbf{R}^{n\times n}$, there exists an orthogonal matrix Q such that QAQ^T is upper triangular.*

Theorem 18.4 *Given $A \in \mathbf{R}^{n\times n}$, there exists a nonsingular T such that TAT^{-1} is upper triangular, has the eigenvalues of A on its diagonal, and the sum of the absolute values of all other elements is less that ϵ for any ϵ.*

19 On convexity, cones, etc.

19.1 Convex sets, cones, separating hyperplanes

19.2 Dual Cone

Consider ($\mathbf{R}^n, < ., . >$), and let $K \subseteq \mathbf{R}^n$.

Definition 19.1

$$K^* := \{y \in \mathbf{R}^n :< x, y >\geq 0,\ \forall x \in K\}$$

is called the dual of K (in $\mathbf{R}^n$ with respect to $< ., . >$).

Definition 19.2

$$K^\perp := \{y \in \mathbf{R}^n :< x, y >= 0,\ \forall x \in K\}$$

is called the orthogonal of K (in $\mathbf{R}^n$ with respect to $< ., . >$).

Problem 19.1 *If K is a subspace, show that $K^\perp = K^*$.*

Problem 19.2 *Show that K^* is always a closed convex cone.*

Problem 19.3

$$K_1 \subseteq K_2 \Rightarrow K_2^* \subseteq K_1^*$$

Problem 19.4 *$K \subseteq K^{**}$ and $K^* = K^{***}$.*

Theorem 19.5 *K is closed convex cones if and only if $K^* = K^{**}$.*

Proof: It suffices to show that $K^{**} \subseteq K$. Suppose that $x \in K^{**}$ but $x \notin K$. Then there exist a hyperplane that separates the point from the set, i.e., there exists a such that $a^T y \geq 0 > a^T x$ for all $y \in K$. But this means that there exists $a \in K^*$ such that for some $x \in K$, $a^T x < 0$ □

Definition 19.3 *A set is self-dual if $S = S^*$.*

Problem 19.6 *Why a self-dual set always a closed convex cone?*

Proposition 19.7 *Let K_1 and K_2 are two closed convex cones. Then $K_1^* \cap K_2^* = (K_1 + K_2)^*$*

Problem 19.8 *Prove the above proposition.*

Proposition 19.9 *Let K_1 and K_2 are two closed convex cones. The $\mathrm{cl}(K_1^* + k_2^*) = (K_1 \cap k_2)^*$.*

Proof:

$$(K_1 \cap K_2)^* = (K_1^{**} \cap K_2^{**})^* = (K_1^* + K_2^*)^{**} = \mathrm{cl}(K_1^* + K_2^*).$$

□

Our interest in self-dual cones in the context of semi-definite ordering stems from the fact that, for $(\mathcal{S}^n, \geq_{\mathcal{S}^n_+})$, equipped with the trace inner product $\cdot$, one has,

Theorem 19.10 *$\mathcal{S}^n_+$ is self-dual.*

The proof of this fact uses a new notion for matrix product and a accompanying beautiful theorem by Schur.

Definition 19.4 *$A, B \in \mathbf{R}^{n\times m}$;*

$$A \circ B = [(A \circ B)_{ij}] := [A_{ij}B_{ij}], \quad \forall i, j$$

is called the Hadamard product (or the Schur product) of A and B.

Recall that if $A \in \mathcal{S}^n_+$, and **rank** $(A) = k$, then there exists $v_1, \ldots, v_k \in \mathbf{R}^n$ such that,

$$A = \sum_{i=1}^{k} u_i u_i^T$$

where $u_i^T u_j = \delta_{ij}$; moreover, any matrix that has such a representation is in $\mathcal{S}^n_+$ and has rank k.

Theorem 19.11

$$A, B \in \mathcal{S}^n_+ \Rightarrow A \circ B \in \mathcal{S}^n_+$$

Proof: Note that $A \circ B$ can represented as

$$\sum_i \sum_j w_{ij} w_{ij}^T$$

where $w_{ij} = u_i \circ v_j$ and u_i's and v_i's are such that,

$$A = \sum_i u_i u_i^T, \quad B = \sum_i v_i v_i^T$$

□

We are now ready to prove the self-duality of the cone of positive semi-definite matrices (with respect to the trace inner product).

Proof: To show that $(\mathcal{S}^n_+)^* \subseteq \mathcal{S}^n_+$, let $X \in (\mathcal{S}^n_+)^*$, i.e.,

$$\begin{aligned} X \cdot A \geq 0,\ \forall A \geq 0 \\ \Rightarrow X \cdot aa^T \geq 0,\ \forall a \in \mathbf{R}^n \\ \Rightarrow a^T X a \geq 0,\ \forall a \in \mathbf{R}^n \end{aligned}$$

and thus $X \in \mathcal{S}^n_+$.

It now suffices to show that $\mathcal{S}^n_+ \subseteq (\mathcal{S}^n_+)^*$. Let $X \in \mathcal{S}^n_+$. For all $A \in \mathcal{S}^n_+$, $X \circ A \geq 0$ (Schur product theorem). Thus,

$$\mathbf{1}^T (X \circ A) \mathbf{1} \geq 0.$$

However,

$$0 \leq \mathbf{1}^T (X \circ A) \mathbf{1} = \sum_i \sum_j X_{ij} A_{ij} = X \cdot A;$$

thereby, $X \in (\mathcal{S}^n_+)^*$. □

Problem 19.12 *Let K be a pointed closed convex cone, then*

$$\textbf{int}\ K^* = \{y \in K^* :< x, y >> 0,\ \forall 0 \neq x \in K\}.$$

Problem 19.13 **int** $(\mathcal{S}^n_+) = \mathcal{S}^n_{++}$.

19.3 General orderings

Definition 19.5 *Let $A, B \subseteq \mathbf{R}^n$;*

$$A \times B := \{x : \exists a \in A,\ b \in B \ni x = (a, b)\}.$$

Definition 19.6 *A relation R in $\mathbf{R}^n$ is any subset of $\mathbf{R}^n \times \mathbf{R}^n$.*

Definition 19.7 *A relation R is called antisymmetric if,*

$$\forall x, y \in \mathbf{R}^n : (x, y) \in R, (y, x) \in R \Rightarrow x = y.$$

Definition 19.8 *A partial order in $\mathbf{R}^n$ is a reflexive, antisymmetric, and transitive relation in $\mathbf{R}^n$, i.e.,*

$$\begin{aligned} \forall x \in \mathbf{R}^n : (x, x) \in R, \\ \forall x, y \in \mathbf{R}^n : (x, y) \in R, (y, x) \in R \Rightarrow x = y, \\ \forall x, y, z \in \mathbf{R}^n : (x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R. \end{aligned}$$

If R is a partial order, we write $x \geq y$ when $(x, y) \in R$; $y \geq x$ means $x \leq y$.

Proposition 19.14 *A pointed convex cones $K \in \mathbf{R}^n$ induces a partial order via,*

$$x \geq y \Leftrightarrow x - y \in K.$$

We shall denote the cone induced ordering by $\geq_K$.

Problem 19.15 *Prove the proposition; in particular why are we requiring the cone to be pointed?*

Corollary 19.16 *$\mathcal{S}^n_+$ induces a partial order in $\mathcal{S}^n$ (or in $\mathbf{R}^{\bar{n}}$).*

We call the ordering induced by $\mathcal{S}^n_+$ in $\mathcal{S}^n$ the psd ordering.

Definition 19.9 *If $(S, \geq)$ is such that*

$$\forall x, y \in S, x \leq y \text{ or } y \leq x,$$

then $\geq$ is called a total order on S; $(S, \geq)$ would then be called a chain.

Problem 19.17 *Is $(\mathcal{S}^n, \geq_{\mathcal{S}^n_+})$ a chain?*

Definition 19.10 *Consider $(S, \geq)$. If,*

$$\exists a \in S \ni a \leq x, \forall x \in S,$$

then a is called the least element of S; on the other hand, a is called minimal (in S) if for any $X \in S$,

$$x \leq a \Rightarrow x = a$$

(similarly for the greatest element and maximal element).

Problem 19.18 *Does the least element of a set (with respect to a given partial order) unique? How about the minimal element?*

Definition 19.11 *Consider $(X, \geq)$ and $E \subseteq X$. An element $a \in X$ is called a lower bound for E if for all $x \in E$, $a \leq X$; the set of all such a's shall be denoted by E_* (similarly, the notion of an upper bound and the set E^*).*

Problem 19.19 *Show that if $E \cap E_*$ is nonempty, then it is a singleton.*

Definition 19.12 *Consider $(X, \geq)$ and $E \subseteq X$, and $E_* \neq \emptyset$. If E_* has a greatest element it is called the infimum of E, $\inf E$ (similarly the notion of supremum of E, $\sup E$).*

Definition 19.13 *Consider $(X, \geq)$ and $E \subseteq X$. If E is such that,*

$$\forall x, y \in E, \inf\{x, y\} \in E \ \& \ \sup\{x, y\} \in E,$$

then E is called a lattice (in X with respect to $\geq$).

Problem 19.20 *Is $(\mathcal{S}^n, \geq_{\mathcal{S}^n_+})$ or $(\mathbf{R}^{\bar{n}}, \geq_{\mathcal{S}^n_+})$ a lattice?*

Problem 19.21 *Consider,*

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ C := \begin{bmatrix} 1+\epsilon & \theta \\ \theta & 1+\delta \end{bmatrix},$$

where θ, $\epsilon, \delta > 0$ are such that,

$$\theta \leq \min\{\epsilon(1+\delta), \delta(1+\epsilon).$$

Then $A \leq C$ and $B \leq C$. If $C \leq I$, then $\epsilon = \delta = \theta = 0$. What does this imply about the "latticity" of $(\mathcal{S}^n, \geq_{\mathcal{S}^n_+})$.

20 Mathematical programming over cones

Definition 20.1 *A system of inequalities in the variable x is called infeasible if there is no value for x which satisfies them.*

Theorem 20.1 *Let $L \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^n$ and K a solid closed convex cone. Then the following two statements are equivalent:*

1. $$Lx = b, \quad x \in \textbf{int}\, K$$

   is consistent.

2. *$b \in R(L)$ and $0 \neq L^{adj}y \in K^*$ implies $< b, y >> 0$.*

Proof:

1. (1) $\Rightarrow$ (2):
$$< b, y >=< Lx, y >=< x, L^{\text{adj}}y >> 0$$
if $0 \neq L^{\text{adj}}y \in K^*$.

2. (2) $\Rightarrow$ (1):
$$\begin{aligned} 0 \neq L^{\text{adj}}y \in K^* \quad &\Rightarrow \quad 0 << b, y >=< LL^\dagger b, y >=< L^\dagger b, L^{\text{adj}}y > \\ &\Rightarrow \quad L^\dagger b \in \textbf{int}\,[R(L^{\text{adj}}) \cap K^*]^* = \textbf{int}\,\{\text{cl}\{R(L^{\text{adj}})^* + K^{**}\}\} \\ &\Rightarrow \quad L^\dagger b \in \textbf{int}\,\{\text{cl}\{N(L) + K\}\} \\ &\Rightarrow \quad L^\dagger b + N(L) \cap \textbf{int}\, K \neq \emptyset, \end{aligned}$$
but $L^\dagger b + N(L)$ is exactly $\{x : Lx = b\}$.

□

Proposition 20.2 *Let $A \in \mathbf{R}^{n\times n}$, and define $L : \mathcal{S}^n \to \mathcal{S}^n$, by, $L_A(X) = A^T X + XA$. Then there exists $X > 0$ such that $L_A(X) = -I$ if and only if $-I \in R(L)$, $0 \neq L_A^{adj}(Y) \geq 0$ implies $\mathbf{Tr}Y < 0$.*

Theorem 20.3 *Let $A \in \mathbf{R}^{n\times n}$. Then A is stable if and only if $L_A(X) = -I$ has a positive definite solution.*

Proof: Handout. □

Given a linear map $L : \mathbf{R}^n \to \mathbf{R}^n$, and a cone $K \subseteq \mathbf{R}^n$, let,
$$L(K) := \{b : Ax = b,\ x \in K\}$$

Problem 20.4 *If K is closed convex cone and L is a linear map, show that $L(K)$ is a convex cone. Is $L(K)$ necessarily closed?*

Lemma 20.5 (Generalized Farkas Lemma) *Let K be a closed convex cone in $\mathbf{R}^n$ (with respect to the inner product $< ., . >$. Let L be a linear map on $\mathbf{R}^n$. Assume that $L(K)$ is closed. Then either*
$$Lx = b, \quad x \in K$$
is consistent, or,
$$L^{adj}y \in K^*, \quad < b, y >< 0,$$
but not both.

Proof: Suppose that there exist $x \in K$ such that $Lx = b$. Then $< b, y >=< Lx, y >=< x, L^{\text{adj}}y >\geq 0$.

Suppose now that $Lx = b$, $x \in K$ is infeasible, i.e., $L(K) \cap b = \emptyset$. Thus there exist a hyperplane separating b from $L(K)$, i.e., there exits a y such that $< b, y >< 0$ and $< z, y >\geq 0$ for all $z \in L(K)$; for any $x \in K$, $< Lz, y >=< z, L^{\text{adj}}y > 0$, that is $L^{\text{adj}}y \in K^*$. $\square$

Let us see what happens if the $L(K)$ is not closed. Let $K = \mathcal{S}^2_+$ and equip $\mathcal{S}$ with the trace inner product $\cdot$. Let E_{ij} be the matrix whose ij-th element is one and all other elements are zero. Consider the following two set of inequalities,

$$\begin{aligned} \frac{1}{2}(E_{12} + E_{21})\cdot X &= 1 \\ E_{11}\cdot X &= 0 \\ X &\geq 0 \end{aligned}$$

and,

$$\begin{aligned} \frac{1}{2}y_1(E_{12} + E_{21}) + y_2E_{11} &\geq 0 \\ y_1 &< 0 \end{aligned}$$

But the first set of inequalities translates to finding x such that $\begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix} \geq 0$ and the second one for a pair u, v such that $\begin{bmatrix} u & v \\ v & 0 \end{bmatrix} \geq 0$ and $v < 0$. Note that both set of inequalities are infeasible, however, an incorrect application of the generalized Farkas lemma guarantees that one of them should have a solution.

Lemma 20.6 *Given a linear map L on $\mathbf{R}^n$ a closed convex cone in $\mathbf{R}^n$, $L(K)$ is closed if and only if $N(L) + K$ is closed.*

Lemma 20.7 *Given a linear map L on $\mathbf{R}^n$, and a closed convex cones K, if $L^{adj}y \in K^*$ is strictly feasible then $L(K)$ is closed.*

1. For an extended-valued function, f is said to be convex if it is convex on its domain (similarly, concave functions, etc.).

2. The subdifferential of f at a point $\bar{x}$ is the set

$$\partial f(\bar{x}) := \{\phi \in \mathbf{E} \mid \langle \phi, x - \bar{x} \rangle \leq f(x) - f(\bar{x})\},$$

   each element of the subdifferential is called a subgradient. If $\bar{x} \notin \text{dom } f$ we let $\partial f = \emptyset$. dom ∂f is the set of points for which it is nonempty. ∂f is a closed convex set.

3. **Proposition 20.8** *For any proper function f, $\bar{x}$ is a global minimizer if and only if $0 \in \partial f(\bar{x})$.*

4. **Theorem 20.9** *(***Max Formula***) Let f be convex and $\bar{x} \in$ core dom f and $d \in \mathbf{E}$. Then*

$$f'(\bar{x}; d) = \max_{\phi \in \partial f(\bar{x})} \langle \phi, d \rangle .$$

5. **Corollary 20.10** *Let f be convex and $\bar{x} \in$ core dom f. The f is (Gâteaux) differentiable at $\bar{x}$ iff f has a unique subgradient at $\bar{x}$.*

6. Let C be open and Let $f : \text{cl } C \to$ twice continuously differentiable. The f is convex iff its Hessian matrix is positive semi-definite everwhere.

21 Week 8

Main ideas: Applications in estimation and signal processing

1. We consider the problem of communication without errors and its connections with SDP! We are given a set of symbols, say five, and the confusion graph, $G = (V, E)$, where V is the set of symbols, and there is an edge between vertices i and j, if the symbol i can be confused with symbol j during their transmission. We consider, for example, the 5-cycle, C_5.

2. An independent set of a graph is a subset of its vertices no two of which are connected. The maximum cardinality of the independent sets of the graph G is called its independence number, $\alpha(G)$. The best information rate associated with the confusion graph G is $\log \alpha(G)$.

3. How about if we allow longer strings of symbols, say of length two? Define the product of a graph $G = (V, E)$ with itself as follows: the vertex set of $G \times G$ will be

$$V \times V := \{(v_1, v_2) \,|\, v_1 \in V, v_2 \in V\};$$

there is an edge between distinct vertices (u_1, u_2) and (v_1, v_2) if $u_i = v_i$ or $u_i v_i \in E$ for $i = 1, 2$. The confusion graph of strings of two is thus G^2. The information rate of strings of lenght 2 per symbol is therefore

$$\frac{\log \alpha(G^2)}{2} = \log \sqrt{\alpha(G^2)}.$$

4. The information rate per symbol of strings of length n is similarly

$$\frac{\log \alpha(G^n)}{n} = \log \sqrt{{}^n\alpha(G^n)}.$$

5. We observe that $\alpha(G^n) \geq \alpha(G)^n$, since if U is the maximal independent set in G, then at least $\alpha(G)^n$ vertrices in G^n of the form $(u_1, \ldots, u_n)$ is independent.

6. The zero-error capacity of a graph G is given by

$$\Theta(G) = \sup_{n \geq 1} \sqrt{n}\alpha(G^n) = \lim_{n \geq 1} \sqrt{n}\alpha(G^n)$$