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Fuzzy modifiers based on fuzzy relations

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Abstract

In this paper we introduce a new type of fuzzy modifiers (i.e. mappings that transform a fuzzy set into a modified fuzzy set) based on fuzzy relations. We show how they can be applied for the representation of weakening adverbs (*more or less*, *roughly*) and intensifying adverbs (*very*, *extremely*) in the inclusive and the non-inclusive interpretation. We illustrate their use in an approximate reasoning scheme.

Keywords: Fuzzy modifier; Fuzzy relation; Linguistic variable; Linguistic modifier; Approximate reasoning

1. Introduction

In fuzzy set theory, modifiers are often defined within the framework of linguistic variables. We prefer to tear modifiers from this context and consider them as technical tools operating on fuzzy sets, transforming one fuzzy set into another. After recalling some basic notions (Section 2), we will give a general definition of fuzzy modifiers and recall two popular types (Section 3). Grounding on the notion of "image of a fuzzy set under a fuzzy relation", we will then introduce a new class of powerful fuzzy modifiers (Section 4).

Computing with words becomes more and more important in science and technology [23]. A key role in this discipline is played by linguistic variables. In

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this framework fuzzy modifiers are used to model adverbs. First we will give a short overview of the (dis)advantages of the two popular types of fuzzy modifiers recalled in Section 3 for representing adverbs (Section 5). Then we will show how the new class of fuzzy modifiers introduced in Section 4 can be used to model weakening adverbs (*more or less*, *roughly*) and intensifying adverbs (*very*, *extremely*) in the inclusive (Section 6) and the non-inclusive interpretation (Section 7). We will also demonstrate the use of these new fuzzy modifiers in an approximate reasoning scheme.

2. Preliminaries

A *conjunctor* $\mathscr{C}$ is a non-decreasing [1] $[0,1]^2 \to [0,1]$ mapping that satisfies the boundary conditions $\mathscr{C}(1,1)=1$, $\mathscr{C}(1,0)=\mathscr{C}(0,1)=\mathscr{C}(0,0)=0$. A *semi-norm* $\mathscr{C}$ is a conjunctor that satisfies the boundary condition $(\forall x \in [0,1])$ $(\mathscr{C}(1,x)=\mathscr{C}(x,1)=x)$. A *triangular norm* $\mathscr{T}$ (or shortly t-norm) is an associative and commutative semi-norm. Popular t-norms are:

- The minimum operator M with $(\forall (x,y) \in [0,1]^2)(M(x,y)=\min(x,y))$.
- The algebraic product P with $(\forall (x,y) \in [0,1]^2)(P(x,y)=x \cdot y)$.
- The Łukasiewicz t-norm W with $(\forall (x,y) \in [0,1]^2)$ $(W(x,y)=\max(0,x+y-1))$.

A *disjunctor* $\mathscr{D}$ is a non-decreasing $[0,1]^2 \to [0,1]$ mapping that satisfies the boundary conditions $\mathscr{D}(1,1)=\mathscr{D}(1,0)=\mathscr{D}(0,1)=1$, $\mathscr{D}(0,0)=0$. A *triangular conorm* $\mathscr{S}$ (or shortly t-conorm) is an associative and commutative disjunctor that satisfies the boundary condition $(\forall x \in [0,1])(S(x,0)=x)$. Popular t-conorms are

- The maximum operator M^* with $(\forall (x,y) \in [0,1]^2)(M^*(x,y)=\max(x,y))$.
- The probabilistic sum P^* with $(\forall (x,y) \in [0,1]^2)(P^*(x,y)=x+y-xy)$.
- The bounded sum W^* with $(\forall (x,y) \in [0,1]^2)(W^*(x,y)=\min(1,x+y))$.

A *negator* $\mathscr{N}$ is a non-increasing $[0,1] \to [0,1]$ mapping that satisfies the boundary conditions $\mathscr{N}(0)=1$, $\mathscr{N}(1)=0$. An *involutive negator* $\mathscr{N}$ is a negator that satisfies $(\forall x \in [0,1])$ $(\mathscr{N}(\mathscr{N}(x))=x)$. A popular involutive negator is the standard negator $\mathscr{N}_s$ with $(\forall x \in [0,1])$ $(\mathscr{N}_s(x)=1-x)$.

[1] We recall that for a poset $(X, \leqslant)$ and f a $X \to \mathbb{R}$ mapping:

$$f \text{ is non-decreasing} \iff (\forall (x,y) \in X^2)\ (x \leqslant y \Rightarrow f(x) \leqslant f(y))$$

$$f \text{ is non-increasing} \iff (\forall (x,y) \in X^2)\ (x \leqslant y \Rightarrow f(x) \geqslant f(y))$$

An *implicator* $\mathscr{I}$ is an hybrid monotonic $[0,1]^2 \to [0,1]$ mapping (i.e. $(\forall x \in [0,1])$ ($\mathscr{I}(\cdot, x)$ is non-increasing and $\mathscr{I}(x, \cdot)$ is non-decreasing)) that satisfies the boundary conditions $\mathscr{I}(1,0) = 0$, $\mathscr{I}(1,1) = \mathscr{I}(0,1) = \mathscr{I}(0,0) = 1$. A *border implicator* $\mathscr{I}$ (or shortly B-implicator) is an implicator that satisfies the neutrality principle $(\forall x \in [0,1])$ $(\mathscr{I}(1,x) = x)$. Some B-implicators are:

- The Kleene–Dienes implicator $\mathscr{I}_{\mathrm{KD}}$ with $(\forall (x,y) \in [0,1]^2)$ $(\mathscr{I}_{\mathrm{KD}}(x,y) = \max(1-x, y))$.
- The Reichenbach implicator $\mathscr{I}_{\mathrm{R}}$ with $(\forall (x,y) \in [0,1]^2)$ $(\mathscr{I}_{\mathrm{R}}(x,y) = 1 - x + xy)$.
- The Standard Star implicator $\mathscr{I}_{\mathrm{g}}$ with $(\forall (x,y) \in [0,1]^2)$

$$\left(\mathscr{I}_{\mathrm{g}}(x,y) = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{elsewhere} \end{cases}\right)$$

x and y are called *zero divisors* of a conjunctor $\mathscr{C}$ iff $x > 0$ and $y > 0$ and $\mathscr{C}(x,y) = 0$. M and P do not have zero divisors; 0.25 and 0.5 are zero divisors of W.

A fuzzy set A on a universe X is characterized by its membership function

$$\begin{aligned} A: \quad & X \to [0,1] \\ & x \mapsto A(x), \quad \forall x \in X \end{aligned}$$

which maps every $x \in X$ onto the degree to which x belongs to A. We use $\mathscr{F}(X)$ to denote the class of all fuzzy sets on the universe X. For every $A \in \mathscr{F}(X)$ and $B \in \mathscr{F}(X)$ inclusion can be defined as follows:

$$A \subseteq B \iff (\forall x \in X)\ (A(x) \leqslant B(x))$$

Furthermore, if $\mathscr{C}$ is a conjunctor, $\mathscr{D}$ is a disjunctor and $\mathscr{N}$ is a negator then the $\mathscr{C}$-intersection $A \cap_{\mathscr{C}} B$ of A and B, the $\mathscr{D}$-union $A \cup_{\mathscr{D}} B$ of A and B and the $\mathscr{N}$-complement $\mathrm{co}_{\mathscr{N}}(A)$ of A are defined as: $(\forall x \in X)$ $(A \cap_{\mathscr{C}} B(x) = \mathscr{C}(A(x), B(x))$ and $A \cup_{\mathscr{D}} B(x) = \mathscr{D}(A(x), B(x))$ and $\mathrm{co}_{\mathscr{N}}(A)(x) = \mathscr{N}(A(x)))$. Finally the kernel $\ker(A)$ and the support $\mathrm{supp}(A)$ of A are defined as: $\ker(A) = \{x \mid x \in X \text{ and } A(x) = 1\}$ and $\mathrm{supp}(A) = \{x \mid x \in X \text{ and } A(x) > 0\}$.

If X and Y are universes, a fuzzy relation R from X to Y is a fuzzy set on $X \times Y$. The R-afterset (see [1]) of $x \in X$, denoted xR, is the $Y \to [0,1]$ mapping defined by $(\forall y \in Y)$ $(xR(y) = R(x,y))$. Analogously the R-foreset of $y \in Y$, denoted Ry, is the $X \to [0,1]$ mapping defined by $(\forall x \in X)$ $(Ry(x) = R(x,y))$. If $X = Y$ then R is called a fuzzy relation on X. In this case we say that R is reflexive iff $(\forall x \in X)$ $(R(x,x) = 1)$.

A set from boolean set theory is characterized by a $X \to \{0,1\}$ mapping and is therefore also considered a fuzzy set. We will refer to a fuzzy set that is also a boolean set as "crisp". The class of all crisp sets is denoted as usual by $\mathscr{P}(X)$.

3. Fuzzy modifiers

In this section we give some general definitions and properties of fuzzy modifiers and we recall the definition of two very popular types of fuzzy modifiers. For a more extensive overview of fuzzy modifiers developed during the last two decades, we refer to [12].

3.1. General

Definition 3.1. A fuzzy modifier m on X is a $\mathscr{F}(X) \to \mathscr{F}(X)$ mapping.

Definition 3.2. A fuzzy modifier m on X is called

(1) *expansive* iff $(\forall A \in \mathscr{F}(X))\ (A \subseteq m(A))$;
(2) *restrictive* iff $(\forall A \in \mathscr{F}(X))\ (m(A) \subseteq A)$;
(3) a *fuzzy modifier with pure premodification* iff a $X \to [0,1]$ mapping t exists with $(\forall A \in \mathscr{F}(X))\ (m(A) = A \circ t)$. t is called a premodifier of m;
(4) a *fuzzy modifier with pure postmodification* iff a $[0,1] \to [0,1]$ mapping r exists with $(\forall A \in \mathscr{F}(X))\ (m(A) = r \circ A)$. r is called a postmodifier of m.

Proposition 3.1. *Let $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(X)$, m a fuzzy modifier on X, $\mathscr{C}$ a conjunctor, $\mathscr{D}$ a disjunctor, $\mathscr{N}$ a negator.*

(1) *If m is restrictive then*:
  (a) $m(A) \cap_{\mathscr{C}} m(B) \subseteq A \cap_{\mathscr{C}} B$,
  (b) $m(A) \cup_{\mathscr{D}} m(B) \subseteq A \cup_{\mathscr{D}} B$,
  (c) $m(\mathrm{co}_{\mathscr{N}}(A)) \subseteq \mathrm{co}_{\mathscr{N}}(m(A))$.
(2) *If m is expansive then*:
  (a) $A \cap_{\mathscr{C}} B \subseteq m(A) \cap_{\mathscr{C}} m(B)$,
  (b) $A \cup_{\mathscr{D}} B \subseteq m(A) \cup_{\mathscr{D}} m(B)$,
  (c) $\mathrm{co}_{\mathscr{N}}(m(A)) \subseteq m(\mathrm{co}_{\mathscr{N}}(A))$.

Proof is straightforward.

3.2. Powering modifiers

In the early 1970s Zadeh [20] introduced a class of powering modifiers that has become very popular.

Definition 3.3. For $\alpha \in [0, +\infty[, P_\alpha$ is a $\mathscr{F}(X) \to \mathscr{F}(X)$ mapping defined by, for $A \in \mathscr{F}(X)$,

$$P_\alpha(A) : X \to [0,1]$$
$$x \mapsto (A(x))^\alpha, \quad \forall x \in X$$

The powering modifiers are fuzzy modifiers with pure postmodification.

Proposition 3.2. *For X a universe, $\mathscr{C}$ a conjunctor, $\mathscr{D}$ a disjunctor, $\mathscr{N}$ a negator, $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(X)$, $\alpha \in [0, +\infty[$:*

(P1) *Expansiveness and restrictiveness*
if $\alpha \leqslant 1$ then $A \subseteq P_\alpha(A)$;
if $\alpha \geqslant 1$ then $P_\alpha(A) \subseteq A$.
(P2) *Interaction with $\mathscr{C}$-intersection*
if $\alpha \leqslant 1$ then $A \cap_\mathscr{C} B \subseteq P_\alpha(A) \cap_\mathscr{C} P_\alpha(B)$;
if $\alpha \geqslant 1$ then $P_\alpha(A) \cap_\mathscr{C} P_\alpha(B) \subseteq A \cap_\mathscr{C} B$.
(P3) *Interaction with $\mathscr{D}$-union*
if $\alpha \leqslant 1$ then $A \cup_\mathscr{D} B \subseteq P_\alpha(A) \cup_\mathscr{D} P_\alpha(B)$;
if $\alpha \geqslant 1$ then $P_\alpha(A) \cup_\mathscr{D} P_\alpha(B) \subseteq A \cup_\mathscr{D} B$.
(P4) *Interaction with $\mathscr{N}$-complement*
if $\alpha \leqslant 1$ then $\mathrm{co}_\mathscr{N}(P_\alpha(A)) \subseteq P_\alpha(\mathrm{co}_\mathscr{N}(A))$;
if $\alpha \geqslant 1$ then $P_\alpha(\mathrm{co}_\mathscr{N}(A)) \subseteq \mathrm{co}_\mathscr{N}(P_\alpha(A))$.
(P5) *Monotonicity*
if $A \subseteq B$ then $P_\alpha(A) \subseteq P_\alpha(B)$.
(P6) *Effect on the universe*
$P_\alpha(X) = X$.
(P7) *Effect on the empty set*
$P_\alpha(\emptyset) = \emptyset$.
(P8) *Behaviour w.r.t. the kernel*
$\ker(P_\alpha(A)) = \ker(A)$.
(P9) *Behaviour w.r.t. the support*
$\mathrm{supp}(P_\alpha(A)) = \mathrm{supp}(A)$.

Proof is straightforward. Note that (P2)–(P4) follow immediately from Proposition 3.1 and (P1). See also [11] for specific properties of P_2 and $P_{\frac{1}{2}}$.

3.3. Shifting modifiers

Another type of fuzzy modifiers, called shifting modifiers, was already informally suggested by Lakoff [14] in the 1970's. Hellendoorn [8] and Bouchon [2] used it in a more formal manner. Since the shifting is an operation on objects of the universe (and not an operation on their degree of membership like the powering is), it is only applicable for fuzzy sets in a universe equipped with such an operation. The set of real numbers $\mathbb{R}$, very popular in practical applications, is such a universe.

Definition 3.4. For $\alpha \in \mathbb{R}$, S_α is a $\mathscr{F}(\mathbb{R}) \to \mathscr{F}(\mathbb{R})$ mapping defined by, for $A \in \mathscr{F}(\mathbb{R})$,

$$\begin{aligned} S_\alpha(A) : \mathbb{R} &\to [0,1] \\ x &\mapsto A(x-\alpha), \quad \forall x \in \mathbb{R} \end{aligned}$$

The shifting modifiers are fuzzy modifiers with pure premodification.

Proposition 3.3. *For $\mathscr{C}$ a conjunctor, $\mathscr{D}$ a disjunctor, $\mathscr{N}$ a negator, $A \in \mathscr{F}(\mathbb{R})$, $B \in \mathscr{F}(\mathbb{R})$, $\alpha \in \mathbb{R}$:*

(S1) *Restrictiveness and expansiveness*
if A is non-decreasing and $\alpha \geqslant 0$ then $S_\alpha(A) \subseteq A$;
if A is non-increasing and $\alpha \leqslant 0$ then $S_\alpha(A) \subseteq A$;
if A is non-decreasing and $\alpha \leqslant 0$ then $S_\alpha(A) \supseteq A$;
if A is non-increasing and $\alpha \geqslant 0$ then $S_\alpha(A) \supseteq A$.
(S2) *Interaction with $\mathscr{C}$-intersection*
$S_\alpha(A \cap_{\mathscr{C}} B) = S_\alpha(A) \cap_{\mathscr{C}} S_\alpha(B)$.
(S3) *Interaction with $\mathscr{D}$-union*
$S_\alpha(A \cup_{\mathscr{D}} B) = S_\alpha(A) \cup_{\mathscr{D}} S_\alpha(B)$.
(S4) *Interaction with $\mathscr{N}$-complement*
$\mathrm{co}_{\mathscr{N}}(S_\alpha(A)) = S_\alpha(\mathrm{co}_{\mathscr{N}}(A))$.
(S5) *Monotonicity*
if $A \subseteq B$ then $S_\alpha(A) \subseteq S_\alpha(B)$.
(S6) *Effect on the universe*
$S_\alpha(X) = X$.
(S7) *Effect on the empty set*
$S_\alpha(\emptyset) = \emptyset$.
(S8) *Behaviour w.r.t. the kernel*
$\neg(\forall E \in \mathscr{F}(\mathbb{R}))\ (\forall \alpha \in \mathbb{R})\ (\ker(S_\alpha(E)) = \ker(E))$.
(S9) *Behaviour w.r.t. the support*
$\neg(\forall E \in \mathscr{F}(\mathbb{R}))\ (\forall \alpha \in \mathbb{R})\ (\mathrm{supp}(S_\alpha(E)) = \mathrm{supp}(E))$.

Proof is straightforward. See also [11].

4. A new type of fuzzy modifiers based on fuzzy relations

Definition 4.1. For X a universe, R a fuzzy relation on X, $\mathscr{C}$ a conjunctor and $\mathscr{I}$ an implicator, we define the fuzzy modifiers $R^{\clubsuit}_{\mathscr{C}}$ and $R^{\heartsuit}_{\mathscr{I}}$ on X with for $A \in \mathscr{F}(X)$

$$R^{\clubsuit}_{\mathscr{C}}(A) : X \to [0,1]$$
$$y \mapsto \sup_{x\in X} \mathscr{C}(R(x,y), A(x)), \quad \forall y \in X$$
$$R^{\heartsuit}_{\mathscr{I}}(A) : X \to [0,1]$$
$$y \mapsto \inf_{x\in X} \mathscr{I}(R(x,y), A(x)), \quad \forall y \in X$$

Remark. Usually $R^{\clubsuit}_{\min}(A)$ is called the direct image of A under R [11].

Proposition 4.1. *Let X be a universe, R a fuzzy relation on X, $\mathscr{C}$ a conjunctor, $\mathscr{I}$ an implicator, $\mathscr{N}$ a negator:*

(R1) *Expansiveness and restrictiveness*
  (1) *if R is reflexive and $\mathscr{C}$ is a semi-norm then $R^{\clubsuit}_{\mathscr{C}}$ is expansive;*
  (2) *if R is reflexive, $\mathscr{C}$ is a semi-norm and $\mathscr{N}$ is an involutive negator then $\mathrm{co}_{\mathscr{N}} \circ R^{\clubsuit}_{\mathscr{C}} \circ \mathrm{co}_{\mathscr{N}}$ is restrictive;*
  (3) *if R is reflexive and $\mathscr{I}$ is a border implicator then $R^{\heartsuit}_{\mathscr{I}}$ is restrictive;*
  (4) *if R is reflexive, $\mathscr{I}$ is a border implicator and $\mathscr{N}$ is an involutive negator then $\mathrm{co}_{\mathscr{N}} \circ R^{\heartsuit}_{\mathscr{I}} \circ \mathrm{co}_{\mathscr{N}}$ is expansive.*

Proof. As an example we prove part (4). Proofs of the other parts are similar. For $A \in \mathscr{F}(X)$ and $y \in X$:

$$\begin{aligned}
(\mathrm{co}_{\mathscr{N}} \circ R^{\heartsuit}_{\mathscr{I}} \circ \mathrm{co}_{\mathscr{N}})(A)(y) &= (\mathrm{co}_{\mathscr{N}}(R^{\heartsuit}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}}(A))))(y) \\
&= \mathscr{N}((R^{\heartsuit}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}}(A)))(y)) \\
&= \mathscr{N}\left(\inf_{x\in X} \mathscr{I}(R(x,y), \mathrm{co}_{\mathscr{N}}(A)(x))\right) \\
&= \mathscr{N}\left(\inf_{x\in X} \mathscr{I}(R(x,y), \mathscr{N}(A(x)))\right) \\
&\geqslant \mathscr{N}(\mathscr{I}(R(y,y), \mathscr{N}(A(y)))) \\
&= \mathscr{N}(\mathscr{N}(A(y))) = A(y) \qquad \square
\end{aligned}$$

Corollary 4.1. *Let X be a universe, R a fuzzy relation on X, $A \in \mathscr{F}(X)$, $\mathscr{C}$ a conjunctor, $\mathscr{I}$ an implicator.*

(1) *If R is reflexive and $\mathscr{C}$ is a semi-norm then* $\ker(A) \subseteq \ker(R^{\clubsuit}_{\mathscr{C}}(A))$ *and* $\mathrm{supp}(A) \subseteq \mathrm{supp}(R^{\clubsuit}_{\mathscr{C}}(A))$.
(2) *If R is reflexive and $\mathscr{I}$ is a border implicator then* $\ker(R^{\heartsuit}_{\mathscr{I}}(A)) \subseteq \ker(A)$ *and* $\mathrm{supp}(R^{\heartsuit}_{\mathscr{I}}(A)) \subseteq \mathrm{supp}(A)$.

Proposition 4.2. *Let X be a universe, R a fuzzy relation on X, $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(X)$. $\mathscr{C}$ and $\mathscr{C}_1$ are conjunctors, $\mathscr{I}_1$ is an implicator, $\mathscr{D}$ is a disjunctor, $\mathscr{N}$ is a negator*:

(R2) *Interaction with $\mathscr{C}$-intersection*
if R is reflexive and $\mathscr{C}_1$ is a semi-norm then

$$A \cap_{\mathscr{C}} B \subseteq R^{\clubsuit}_{\mathscr{C}_1}(A) \cap_{\mathscr{C}} R^{\clubsuit}_{\mathscr{C}_1}(B)$$

if R is reflexive and $\mathscr{I}_1$ is a border implicator then

$$R^{\heartsuit}_{\mathscr{I}_1}(A) \cap_{\mathscr{C}} R^{\heartsuit}_{\mathscr{I}_1}(B) \subseteq A \cap_{\mathscr{C}} B$$

(R3) *Interaction with $\mathscr{D}$-union*
if R is reflexive and $\mathscr{C}_1$ is a semi-norm then

$$A \cup_{\mathscr{D}} B \subseteq R^{\clubsuit}_{\mathscr{C}_1}(A) \cup_{\mathscr{D}} R^{\clubsuit}_{\mathscr{C}_1}(B)$$

if R is reflexive and $\mathscr{I}_1$ is a border implicator then

$$R^{\heartsuit}_{\mathscr{I}_1}(A) \cup_{\mathscr{D}} R^{\heartsuit}_{\mathscr{I}_1}(B) \subseteq A \cup_{\mathscr{D}} B$$

(R4) *Interaction with $\mathscr{N}$-complement*
if R is reflexive and $\mathscr{C}_1$ is a semi-norm then

$$\mathrm{co}_{\mathscr{N}}(R^{\clubsuit}_{\mathscr{C}_1}(A)) \subseteq R^{\clubsuit}_{\mathscr{C}_1}(\mathrm{co}_{\mathscr{N}}(A))$$

if R is reflexive and $\mathscr{I}_1$ is a border implicator then

$$R^{\heartsuit}_{\mathscr{I}_1}(\mathrm{co}_{\mathscr{N}}(A)) \subseteq \mathrm{co}_{\mathscr{N}}(R^{\heartsuit}_{\mathscr{I}_1}(A))$$

Proof. Proof of these properties is straightforward from Propositions 3.1 and 4.1. □

Proposition 4.3. *Let X be a universe, R, R_1 and R_2 fuzzy relations on X, $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(X)$. $\mathscr{C}$, $\mathscr{C}_1$ and $\mathscr{C}_2$ are conjunctors, $\mathscr{I}$, $\mathscr{I}_1$ and $\mathscr{I}_2$ are implicators.*

(R5) *Monotonicity*
(1) *w.r.t. the fuzzy set being modified*
$A \subseteq B \Rightarrow R^{\clubsuit}_{\mathscr{C}}(A) \subseteq R^{\clubsuit}_{\mathscr{C}}(B)$
$A \subseteq B \Rightarrow R^{\heartsuit}_{\mathscr{I}}(A) \subseteq R^{\heartsuit}_{\mathscr{I}}(B)$
(2) *w.r.t. the fuzzy relation defining the modifier*
$R_1 \subseteq R_2 \Rightarrow R_1{}^{\clubsuit}_{\mathscr{C}}(A) \subseteq R_2{}^{\clubsuit}_{\mathscr{C}}(A)$
$R_1 \subseteq R_2 \Rightarrow R_1{}^{\heartsuit}_{\mathscr{I}}(A) \supseteq R_2{}^{\heartsuit}_{\mathscr{I}}(A)$

(3) *w.r.t. the conjunctor/implicator defining the modifier*
$\mathscr{C}_1 \subseteq \mathscr{C}_2 \Rightarrow R^{\clubsuit}_{\mathscr{C}_1}(A) \subseteq R^{\clubsuit}_{\mathscr{C}_2}(A)$
$\mathscr{I}_1 \subseteq \mathscr{I}_2 \Rightarrow R^{\heartsuit}_{\mathscr{I}_1}(A) \subseteq R^{\heartsuit}_{\mathscr{I}_2}(A)$

Proof. All proofs of (R5) being similar we give as an example the proof of property (R5)(1). Let $y \in X$ then

$$\begin{aligned} A \subseteq B &\Rightarrow (\forall x \in X)(A(x) \leqslant B(x)) \\ &\Rightarrow (\forall x \in X)(\mathscr{C}(R(x,y),A(x)) \leqslant \mathscr{C}(R(x,y),B(x))) \\ &\Rightarrow \sup_{x\in X} \mathscr{C}(R(x,y),A(x)) \leqslant \sup_{x\in X} \mathscr{C}(R(x,y),B(x)) \\ &\Rightarrow R^{\clubsuit}_{\mathscr{C}}(A)(y) \leqslant R^{\clubsuit}_{\mathscr{C}}(B)(y) \end{aligned}$$

$$\begin{aligned} A \subseteq B &\Rightarrow (\forall x \in X)(A(x) \leqslant B(x)) \\ &\Rightarrow (\forall x \in X)(\mathscr{I}(R(x,y),A(x)) \leqslant \mathscr{I}(R(x,y),B(x))) \\ &\Rightarrow \inf_{x\in X} \mathscr{I}(R(x,y),A(x)) \leqslant \inf_{x\in X} \mathscr{I}(R(x,y),B(x)) \\ &\Rightarrow R^{\heartsuit}_{\mathscr{I}}(A)(y) \leqslant R^{\heartsuit}_{\mathscr{I}}(B)(y) \end{aligned}$$

Hence if $A \subseteq B$ then for all $y \in X$: $R^{\clubsuit}_{\mathscr{C}}(A)(y) \leqslant R^{\clubsuit}_{\mathscr{C}}(B)(y)$ and $R^{\heartsuit}_{\mathscr{I}}(A)(y) \leqslant R^{\heartsuit}_{\mathscr{I}}(B)(y)$. Therefore $R^{\clubsuit}_{\mathscr{C}}(A) \subseteq R^{\clubsuit}_{\mathscr{C}}(B)$ and $R^{\heartsuit}_{\mathscr{I}}(A) \subseteq R^{\heartsuit}_{\mathscr{I}}(B)$. □

Proposition 4.4. *Let X be a universe, R a fuzzy relation on X, $\mathscr{C}$ a conjunctor, $\mathscr{I}$ an implicator.*

(R6) *Effect on the universe*

$$R^{\heartsuit}_{\mathscr{I}}(X) = X$$

(R7) *Effect on the empty set*

$$R^{\clubsuit}_{\mathscr{C}}(\emptyset) = \emptyset$$

Proof. For every $y \in X$

$$R^{\heartsuit}_{\mathscr{I}}(X)(y) = \inf_{x\in X} \mathscr{I}(R(x,y),X(x)) = \inf_{x\in X} 1 = 1 = X(y)$$

which proves (R6). The proof of (R7) is similar. □

Proposition 4.5. *Let X be a universe, R a fuzzy relation on X, $A \in \mathscr{F}(X)$, $y \in X$, $\mathscr{C}$ a conjunctor, $\mathscr{I}$ an implicator.*

(R8) *Behaviour w.r.t. the kernel*
(1) $\ker(A) \cap \ker(Ry) \neq \emptyset \Rightarrow y \in \ker(R^{\clubsuit}_{\mathscr{C}}(A))$
(2) *If $\mathscr{I}$ is a border implicator then*

$$\mathrm{co}(\ker(A)) \cap \ker(Ry) \neq \emptyset \Rightarrow y \notin \ker(R^{\heartsuit}_{\mathscr{I}}(A))$$

(3) *If* $\mathscr{I}$ *satisfies the boundary condition* $(\forall(x,y)\in[0,1]^2)$ $(x\leqslant y\Rightarrow \mathscr{I}(x,y)=1)$ *then* $Ry\subseteq A\Rightarrow y\in\ker(R^{\heartsuit}_{\mathscr{I}}(A))$

(R9) *Behaviour w.r.t. the support*

(1) *if* $\mathscr{C}$ *is a conjunctor with no zero divisors then* $\text{supp}(A)\cap\text{supp}(Ry)\neq\emptyset\Rightarrow y\in\text{supp}(R^{\clubsuit}_{\mathscr{C}}(A))$

(2) *if* $\mathscr{I}$ *satisfies the boundary condition* $(\forall x\in]0,1])(\mathscr{I}(x,0)=0)$ *then* $\text{co}(\text{supp}(A))\cap\text{supp}(Ry)\neq\emptyset\Rightarrow y\notin\text{supp}(R^{\heartsuit}_{\mathscr{I}}(A))$

Proof. As an example we prove (R8)(2).

$$\begin{aligned}\text{co}(\ker(A))\cap\ker(Ry)\neq\emptyset &\Rightarrow (\exists x\in X)\ (x\in\text{co}(\ker(A))\cap\ker(Ry))\\ &\Rightarrow (\exists x\in X)\ (x\notin\ker(A)\text{and } x\in\ker(Ry))\\ &\Rightarrow (\exists x\in X)\ (R(x,y)=1\text{and } A(x)<1)\\ &\Rightarrow (\exists x\in X)\ (\mathscr{I}(R(x,y),A(x))<1)\\ &\Rightarrow \inf_{x\in X}\mathscr{I}(R(x,y),A(x))<1\\ &\Rightarrow R^{\heartsuit}_{\mathscr{I}}(A)(y)<1\\ &\Rightarrow y\notin\ker(R^{\heartsuit}_{\mathscr{I}}(A))\qquad\square\end{aligned}$$

Proposition 4.6. *Let* X *be a universe. If* E *is the* **crisp** *equality on* X, *i.e.* $(\forall(x,y)\in X^2)$ $(E(x,y)=1$ *iff* $x=y)$, *and* $\mathscr{C}$ *is a semi-norm then* $(\forall A\in\mathscr{F}(X))$ $(E^{\clubsuit}_{\mathscr{C}}(A)=A)$.

Proof. For $y\in X$

$$\begin{aligned}E^{\clubsuit}_{\mathscr{C}}(A)(y) &= \sup_{x\in X}\mathscr{C}(E(x,y),A(x)) = \max\left(\sup_{x\neq y}\mathscr{C}(0,A(x)),\mathscr{C}(1,A(y))\right)\\ &= A(y)\end{aligned}$$

Hence a modifier "built on" the crisp equality does not cause a modification, its effect is neutral. $\square$

Proposition 4.7. *If* $\mathscr{C}$ *is a semi-norm*, m *is a fuzzy modifier with pure premodification and the premodifier is* t, *and* R *is the inverse relation of* t *i.e.*

$$R(x,y)=\begin{cases}1 & \text{if } x=t(y)\\ 0 & \text{otherwise}\end{cases}$$

Then $m=R^{\clubsuit}_{\mathscr{C}}$.

Proof. For $A \in \mathscr{F}(X)$, $y \in X$

$$
\begin{aligned}
R_{\mathscr{C}}^{\clubsuit}(A)(y) &= \sup_{x \in X} \mathscr{C}(R(x,y), A(x)) \\
&= \max\left(\sup_{x \neq t(y)} \mathscr{C}(0, A(x)), \mathscr{C}(1, A(t(y)))\right) \\
&= A(t(y))
\end{aligned}
$$

Hence $R_{\mathscr{C}}^{\clubsuit}(A) = A \circ t = m(A)$. $\square$

5. Modifiers in natural language

5.1. Linguistic terms

A linguistic variable [22] is a variable which values are linguistic terms. The set of all values of a linguistic variable is called the term set. In general a linguistic variable corresponds to a noun. The term set then contains a *primary term* (which is an adjective) and most often also its *antonym* and an *average term*. [2] From these base terms other terms can be constructed using the following scheme:

⟨conjunction⟩: = and;
⟨disjunction⟩: = or;
⟨negation⟩: = not;
⟨linguistic modifier⟩: = more or less | roughly| very | extremely | at least | at most;
⟨base term⟩: = ⟨primary term⟩|⟨antonym⟩|⟨average term⟩;
⟨term⟩:= ⟨base term⟩|⟨term⟩⟨conjunction⟩⟨term⟩|⟨term⟩⟨disjunction⟩ ⟨term⟩| ⟨negation⟩⟨term⟩|⟨modifier⟩⟨term⟩;

E.g. *long*, *short*, more or less *long*, not *long* and not very *short*, extremely *short*,... are values of the linguistic variable *Length* with *long* being the primary term and *short* the antonym. [3] The meaning of each term can be represented by a fuzzy set. For simplicity we will make no notational distinction between a fuzzy set A, its membership function A and the term A represented by that fuzzy set.

[2] Novák [17] a.o. call these three terms a linguistic trichotomy.
[3] We want to emphasize that the concepts *primary term* and *antonym* are not interchangeable. The linguistic variable *Length* for instance has *long* as a primary term and *short* as its antonym (and not vice-versa) because "How long is it?" is a neutral question while "How short is it?" is not.

Once the meaning of the base terms is known, the meaning of all other terms can be deduced. If A and B are two fuzzy sets representing the meaning of two linguistic terms then "A and B", "A or B" and "not A" can be represented by $A \cap_{\mathscr{T}} B$, $A \cup_{\mathscr{S}} B$ and $\text{co}_{\mathscr{N}}(A)$, respectively for $\mathscr{T}$ a t-norm, $\mathscr{S}$ a t-conorm and $\mathscr{N}$ a negator.

Furthermore if the meaning of a linguistic term is denoted by the fuzzy set $A \in \mathscr{F}(X)$, then the meaning of the term modified by a linguistic modifier (an adverb), is denoted by $m(A)$ where m is a fuzzy modifier representing the adverb in question. More or less and roughly are called weakening modifiers, very and extremely are intensifying adverbs and at least and at most are ordering based modifiers.

5.2. Linguistic modifiers: classical representations

In fuzzy set theoretical literature and applications two main, fundamentally different, interpretations can be found regarding the representation of base terms and terms constructed from them using weakening and intensifying modifiers.

5.2.1. Inclusive interpretation

In the inclusive interpretation [19] it is assumed that semantic entailment holds (e.g. [11,13,14,16,20]): for X a universe, $A \in \mathscr{F}(X)$ and $x \in X$: x is extremely $A \Rightarrow x$ is very $A \Rightarrow x$ is $A \Rightarrow x$ is more or less $A \Rightarrow x$ is roughly A. Representing linguistic terms by means of fuzzy sets, this corresponds to:

$$\text{extremely } A \subseteq \text{very } A \subseteq A \subseteq \text{more or less } A \subseteq \text{roughly } A \tag{1}$$

In the early 1970s Zadeh [20] proposed to model more or less and very using powering modifiers. For $A \in \mathscr{F}(X)$: more or less $A = P_{\frac{1}{2}}(A)$ and very $A = P_2(A)$. extremely and roughly can be modelled in a similar way: roughly $A = P_{\frac{1}{4}}(A)$ and extremely $A = P_4(A)$. Proposition 3.2 (P1) guarantees that for these representations formula (1) holds. (P8) and (P9) on the other hand state that powering modifiers keep the kernel and the support, which is considered to be an important disadvantage by numerous authors [5,8,11,14]. It is easy to see that (P8) and (P9) indeed lead to counterintuitive results. E.g., if very is represented by P_2 then every age that is considered to be *old* to degree 1 is also very *old* to degree 1. According to most people's intuition however a person that is 80 can be called *old* to degree one, but very *old* only to a lower degree, e.g. degree 0.7. Furthermore (P8) and (P9) imply that for A crisp, $P_{\frac{1}{2}}(A) = A$. In other words: a crisp concept A (e.g. *rectangular*) and the modified concept more or less A (e.g. more or less *rectangular*) get the same representation. In a psycholinguistic experiment Hersh and Caramazza [9] also noticed that P_2 increases the slope of an increasing membership function for *large* while the slopes of the functions for *large* and very *large* resulting from the experiment were approximately equal.

Another possibility is to model weakening and intensifying adverbs by means of shifting modifiers. For $A \in \mathscr{F}(\mathbb{R})$:

A is non-decreasing	A is non-increasing
roughly $A = S_{-4}(A)$	roughly $A = S_4(A)$
more or less $A = S_{-2}(A)$	more or less $A = S_2(A)$
very $A = S_2(A)$	very $A = S_{-2}(A)$
extremely $A = S_4(A)$	extremely $A = S_{-4}(A)$

Unfortunately Proposition 3.3 (S1) only guarantees formula (1) for non-decreasing and non-increasing membership functions A ([11]). This means that they cannot be used to model the membership function for a concept like more or less *about 20h00*, since the membership function for *about 20* is partly non-decreasing and partly non-increasing (see Fig. 5). In this case an artificial and quite complicated solution to the problem can be found by dividing the membership function into its non-decreasing and non-increasing parts and applying a different shift to each part. Furthermore the shifting modifiers are only applicable in a universe equipped with a shifting operation.

A suggestion to use both a non-trivial pre- and postmodifier at once for the inclusive interpretation was made by Novák [15]. Although it is an improval on the solutions discussed in the previous paragraphs, it can only be applied for a special kind of membership functions and it also involves a process of division of the membership function in increasing and decreasing parts.

5.2.2. Non-inclusive interpretation

In the non-inclusive interpretation a term modified by a weakening or an intensifying linguistic modifier does not denote a subset neither a superset of the original term. Powering modifiers are therefore not applicable. The original term and the modified term denote two different (possibly overlapping) categories. This interpretation is often used in fuzzy control applications (e.g. [10]). In this case the universe is the set of real numbers $\mathbb{R}$ and the membership functions for the base terms are mountain shaped.

Definition 5.1. A fuzzy set B on $\mathbb{R}$ is said to be **mountain shaped** for $(a,b,c) \in \{(u,v,w)\,|\,(u,v,w) \in \mathbb{R}^3 \text{ and } u < v < w\}$ iff B is continuous and for all x and y in $\mathbb{R}$:

(M1) $x \leqslant a \Rightarrow B(x) = 0$;
(M2) $a \leqslant x \leqslant y \leqslant b \Rightarrow B(x) \leqslant B(y)$;
(M3) $B(b) = 1$;
(M4) $b \leqslant x \leqslant y \leqslant c \Rightarrow B(x) \geqslant B(y)$;
(M5) $x \geqslant c \Rightarrow B(x) = 0$.

Typical examples of mountain shaped fuzzy sets are triangular, trapezoidal, and bell-shaped (or Gaussian-shaped) fuzzy sets. The weakening modifier more or less basically moves the membership function of the primary term (respectively, antonym) in the direction of the antonym (resp. primary term). The adverb roughly moves it even further in the same direction. The intensifying modifier very on the other hand, moves it in the opposite direction. These three linguistic modifiers can be approximately represented by means of shifting fuzzy modifiers. Depending on the nature of the term B, the terms roughly B, more or less B and very B can be modelled by the representations in the first or the second column.

roughly $B = S_{-4}(B)$	roughly $B = S_4(B)$
more or less $B = S_{-2}(B)$	more or less $B = S_2(B)$
very $B = S_2(B)$	very $B = S_{-2}(B)$

Unlike the other linguistic modifiers, extremely does not only cause a shift, but also a change of the shape of the original membership function for the atomic term. While the membership function for the base term B is mountain shaped, the membership function for extremely B will be either increasing or decreasing. This is due to the fact that in fuzzy control applications normally the whole universe is "covered": for all $x \in \mathbb{R}$ there is at least one fuzzy set A on $\mathbb{R}$ involved in the application such that $A(x) > 0$. Neither powering modifiers, neither shifting modifiers are suitable to model extremely.

Remark. Many applications use only one intensifying modifier, namely very (e.g. [4]). In this case very takes the responsibility of extremely described above, namely making sure that the whole universe is covered. Therefore in this case **very** should cause both a shift and a change.

We will now illustrate how the new type of fuzzy modifiers defined in Definition 4.1 provide a general framework that can be used to model extremely, very, more or less and roughly in both the inclusive and the non-inclusive interpretation. These fuzzy modifiers do not keep the kernel and the support in general and they do not demand the universe should be equipped with a specific operation (like shifting e.g.).

6. Weakening and intensifying modifiers: inclusive interpretation

In the inclusive interpretation weakening adverbs are modelled by expansive fuzzy modifiers, and intensifying adverbs by restrictive fuzzy modifiers. In order to define such modifiers within the framework of the new type of modifiers introduced above, we need to choose a suitable fuzzy relation. We propose a relation that models approximate equality, namely a pseudo-metric [4] based resemblance relation [6].

Definition 6.1 (*Resemblance relation*). For a universe X, a pseudo-metric space $(\mathscr{M}, d)$, a $X \to \mathscr{M}$ mapping g, a fuzzy relation E on X is a (g, d)-resemblance relation on X iff for all (x, y, z, u) in X^4

(RES.1) $E(x,x) = 1$;
(RES.2) $E(x,y) = E(y,x)$;
(RES.3) $d(g(x), g(y)) \leqslant d(g(z), g(u)) \Rightarrow E(x,y) \geqslant E(z,u)$.

Remark
(1) If X is already equipped with a pseudo-metric, then g can be the identical mapping on X, i.e. $(\forall x \in X)\ (\mathbb{I}_X(x) = x)$.
(2) If E is a (g, d)-resemblance relation and thus represents approximate equality then for $y \in X$

$$(\forall x \in X)((Ey)(x) = E(x,y) = \text{the degree to which } x \text{ resembles } y)$$

This means that Ey is the fuzzy set of objects of X that resemble y.

Definition 6.2. For a universe X, a fuzzy relation E on X is a resemblance relation on X iff there exists a pseudo-metric space ($\mathscr{M}$, d) and a $X \to \mathscr{M}$ mapping g such that E is a (g, d)-resemblance relation on X.

A resemblance relation E on X is reflexive. Hence for a semi-norm $\mathscr{C}$ and a border implicator $\mathscr{I}$, $E^{\clubsuit}_{\mathscr{C}}$ is an expansive modifier and $E^{\heartsuit}_{\mathscr{I}}$ is a restrictive modifier (cf. Proposition 4.1). We will now show how $E^{\clubsuit}_{\mathscr{C}}$ can be used to model linguistic modifiers like more or less and roughly, while $E^{\heartsuit}_{\mathscr{I}}$ is suitable for adverbs like very and extremely.

[4] We recall that for a universe $\mathscr{M}$, a $\mathscr{M}^2 \to [0, +\infty[$ mapping d is a pseudo-metric on $\mathscr{M}$ iff for all (x,y,z) in $\mathscr{M}^3$

(PM.1) $d(x,x) = 0$;
(PM.2) $d(x,y) = d(y,x)$;
(PM.3) $d(x,y) + d(y,z) \geqslant d(x,z)$.

The couple ($\mathscr{M}$, d) is called a pseudo-metric space. E.g. ($\mathbb{R}$, $d_{|\cdot|}$) is a pseudo-metric space for $d_{|\cdot|}$ defined as $(\forall (x,y) \in \mathbb{R}^2)\ (d_{|\cdot|}(x,y) = |x - y|)$.

6.1. More or less

Let E denote a crisp resemblance relation on a universe X, i.e. E is a crisp relation satisfying (*RES.1*), (*RES.2*) and (*RES.3*). Note that a person can be called more or less *adult* if he/she resembles in appearance to an adult. Therefore, in general, for P in $\mathscr{P}(X)$ and y in X, we can define

$$\begin{aligned} y \in \text{ more or less } P &\iff \text{some element of } P \text{ resembles to } y \\ &\iff (\exists x \in X)(E(x,y) = 1 \wedge x \in P) \\ &\iff Ey \cap P \neq \emptyset \end{aligned}$$

Hence y belongs to more or less P if the intersection of Ey and P is not empty, or in other words if the set of objects resembling y overlaps with P. This underlying meaning can be generalized for a resemblance relation R and a fuzzy set A on X, using a semi-norm $\mathscr{C}$:

$$\text{more or less } A = E^{\clubsuit}_{\mathscr{C}}(A)$$

Indeed for $y \in X$

$$\begin{aligned} (\text{more or less } A)(y) &= \sup_{x\in X} \mathscr{C}(E(x,y), A(x)) \\ &= \sup_{x\in X} \mathscr{C}((Ey)(x), A(x)) \\ &= \sup_{x\in X} (Ey \cap_{\mathscr{C}} A)(x) \end{aligned}$$

This last expression can be interpreted as the degree to which the fuzzy set of all objects of X that resemble y overlaps with A. Using the minimum t-norm M, in Fig. 1, y_1 is more or less A to degree t, while y_2 does not belong to more or less A. Notice that if A is a crisp singleton, i.e. $A = \{a\}$ $(a \in X)$ then for $x \in X$

$$\begin{aligned} E^{\clubsuit}_{\mathscr{C}}(\{a\})(x) &= \sup_{y\in X} \mathscr{C}(E(y,x), \{a\}(y)) \\ &= \max\left(\mathscr{C}(E(a,x),1), \sup_{y\neq a} \mathscr{C}(E(y,x),0)\right) \\ &= E(a,x) \end{aligned}$$

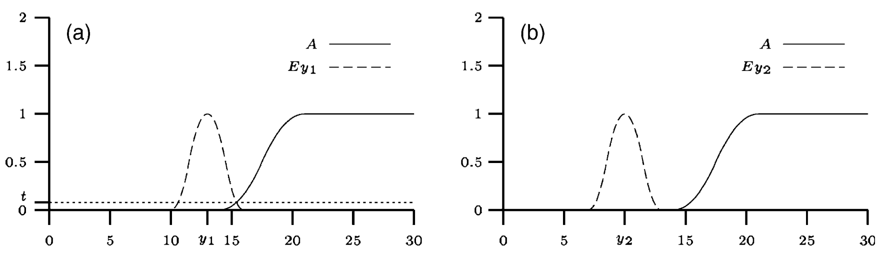

Fig. 1. y_1 is more or less A to degree t; y_2 is more or less A to degree 0.

If $E^{\clubsuit}_{\mathscr{C}}$ is used to represent more or less, the equality $E^{\clubsuit}_{\mathscr{C}}(\{a\})(x) = E(a,x)$ reads as "x is more or less $\{a\}$ to the degree to which x resembles a", which is totally according to our intuition. Furthermore, for most E, $E^{\clubsuit}_{\mathscr{C}}(\{a\}) \neq \{a\}$, while always $P_{\frac{1}{2}}(\{a\}) = \{a\}$.

6.2. Roughly

Suppose more or less is represented by $E1^{\clubsuit}_{\mathscr{C}_1}$ for E_1 a resemblance relation and $\mathscr{C}_1$ a semi-norm. Since roughly is also a weakening adverb, we will need a resemblance relation and a semi-norm to model it. In [16] it is assumed that roughly is more weakening than more or less (see also formula (1)). If we want to respect this, there are several possibilities to model *roughly*:

- $E2^{\clubsuit}_{\mathscr{C}_1}$ where E_2 is a resemblance relation, $E_1 \subseteq E_2$;
- $E1^{\clubsuit}_{\mathscr{C}_2}$ where $\mathscr{C}_2$ is a semi-norm, $\mathscr{C}_1 \subseteq \mathscr{C}_2$;
- $E2^{\clubsuit}_{\mathscr{C}_2}$ where E_2 is a resemblance relation, $E_1 \subseteq E_2$, $\mathscr{C}_2$ is a semi-norm, $\mathscr{C}_1 \subseteq \mathscr{C}_2$.

The properties regarding monotonicity (cf. Proposition 4.3 (R5)) w.r.t. the defining relation and semi-norm guarantee that the representation for roughly will indeed be more weakening than the one for more or less, in either one of the three cases.

Let us consider the situation where a second resemblance relation E_2 is introduced. More or less is modelled by means of $E1^{\clubsuit}_{\mathscr{C}_1}$ while roughly is modelled by means of $E2^{\clubsuit}_{\mathscr{C}_1}$. In Fig. 2, y_1 is more or less A to degree t and roughly A to a higher degree s, whereas y_2 belongs to roughly A to some degree greater than 0, but not to more or less A.

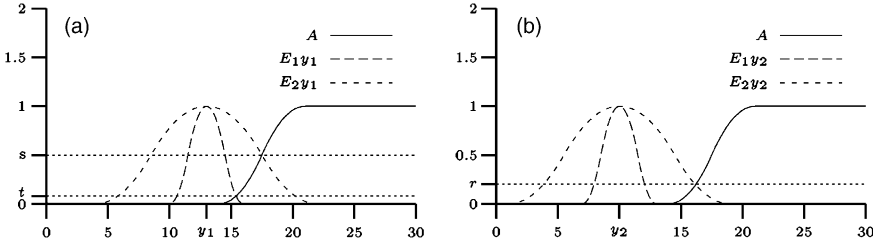

Fig. 2. y_1 is more or less A to degree t, and roughly A to degree s; y_2 is more or less A to degree 0 and roughly A to degree r.

6.3. Very

For a crisp resemblance relation E on X, P in $\mathscr{P}(X)$ and y in X, we can define

$$
\begin{aligned}
y \in \text{ very } P &\iff \text{all elements that resemble to } y \text{ belong to } P\\
&\iff (\forall x \in X)(E(x,y) = 1 \Rightarrow x \in P)\\
&\iff Ey \subseteq P
\end{aligned}
$$

In natural language, the application of an intensifying hedge like very to a crisp concept usually has little or no meaning. However, for a fuzzy concept, such as *beautiful* it is clear that e.g. a woman must be very *beautiful* if everybody whom she resembles to is *beautiful*. More formally we define for A a fuzzy set on X, R a resemblance relation on X, and $\mathscr{I}$ a border implicator

$$\text{very } A = E^{\heartsuit}_{\mathscr{I}}(A)$$

Indeed for $y \in X$

$$
\begin{aligned}
(\text{very } A)(y) &= \inf_{x \in X} \mathscr{I}(E(x,y), A(x))\\
&= \inf_{x \in X} \mathscr{I}((Ey)(x), A(x))
\end{aligned}
$$

This last expression is known as the degree to which Ey is a subset of A (see e.g. [18]).

6.4. Extremely

Suppose very is represented by $E_{1\,\mathscr{I}_1}^{\heartsuit}$ with E_1 a resemblance relation and $\mathscr{I}_1$ a border implicator. Since extremely is also an intensifying adverb, we will need a resemblance relation and a border implicator to model it. In [16] it is assumed that extremely is more intensifying than very (see also formula (1)). If we want to respect this, there are several possibilities to model roughly:

- $E_{2\,\mathscr{I}_1}^{\heartsuit}$ where E_2 is a resemblance relation, $E_1 \subseteq E_2$;
- $E_{1\,\mathscr{I}_2}^{\heartsuit}$ where $\mathscr{I}_2$ is a border implicator, $\mathscr{I}_1 \supseteq \mathscr{I}_2$;
- $E_{2\,\mathscr{I}_2}^{\heartsuit}$ where E_2 is a resemblance relation, $E_1 \subseteq E_2$, $\mathscr{I}_2$ is a border implicator, $\mathscr{I}_1 \supseteq \mathscr{I}_2$.

The properties regarding monotonicity w.r.t. the defining relation and border implicator (cf. Proposition 4.3 (R5)) guarantee that the representation for extremely will indeed be more intensifying than the one for very, in each of the three cases.

Let us consider the situation where a second resemblance relation E_2 is introduced. Very is modelled by means of $E_{1\,\mathscr{I}_1}^{\heartsuit}$, while extremely is modelled by means of $E_{2\,\mathscr{I}_1}^{\heartsuit}$.

6.5. Examples

Example 1. The linguistic term *warm* can be represented by the fuzzy set $A \in \mathscr{F}(\mathbb{R})$ defined by [5]$(\forall x \in \mathbb{R})$ $(A(x) = S(x; 19, 20, 21))$. The fuzzy relation E on $\mathbb{R}$ defined by $(\forall (x,y) \in \mathbb{R}^2)$ $(E(x,y) = \max(1 - |x-y|, 0))$ is an $(\mathbb{I}_{\mathbb{R}}, d_{|\cdot|})$-resemblance relation on $\mathbb{R}$. Fig. 3(a) shows the effect of $E^{\clubsuit}_{\mathscr{C}}$ on A for $\mathscr{C}$ respectively the minimum t-norm M and the product t-norm P. Since $E^{\clubsuit}_{\mathscr{C}}$ is expansive it is evident that the kernel does not become smaller (see Corollary 4.1). However, we would like it to really grow. (R8) (1) gives a clue how to achieve this: if R is a fuzzy relation on X then for an object y in X to belong to $\ker(R^{\clubsuit}_{\mathscr{C}}(A))$, it is sufficient that the crisp intersection of the kernel of A and the kernel of the R-foreset of y is not empty. In other words: for y to be "more or less A" to degree 1, it is sufficient that y is approximately equal to degree 1 to an arbitrary object x that is A to degree 1. For the resemblance relation E chosen above to model approximate equality however $(\forall y \in \mathbb{R})$ $(\ker(Ey) = \{y\})$. The kernel of Ey is too small for property (R8)(1) to do its work properly. For the $(\mathbb{I}_{\mathbb{R}}, d_{|\cdot|})$-resemblance relation E_1 defined by $(\forall (x,y) \in \mathbb{R}^2)(E_1(x,y) = \min(1, \max(1.2 - |x-y|, 0)))$ it holds that $\ker(E_1 y) = [y - 0.2, y + 0.2]$. Fig. 3(b) shows the effect of $E_1{}^{\clubsuit}_{\mathscr{C}}$ on A for $\mathscr{C}$ respectively M and P. A temperature is *warm* to degree 1 iff it is higher than or equal to 21. Since 20.9 is approximately equal to 21 (to degree 1); therefore 20.9 belongs to the fuzzy set more or less *warm* to degree 1.

Fig. 4(a) shows the effect of $E^{\heartsuit}_{\mathscr{I}}$ on $A =$ warm for $\mathscr{I}$ respectively $\mathscr{I}_{\mathrm{KD}}$ and $\mathscr{I}_{\mathrm{R}}$. To reduce the support (R9)(2) suggests we should use an implicator that satisfies the condition $(\forall x \in]0,1])$ $(\mathscr{I}(x, 0) = 0)$, e.g. $\mathscr{I}_{\mathrm{g}}$. Fig. 4(b) shows the effect of $E^{\heartsuit}_{\mathscr{I}_{\mathrm{g}}}$ on A. Since $\mathscr{I}_{\mathrm{g}}$ also satisfies the condition stated in (R8)(2) (i.e. it is a border implicator), we use the resemblance relation E_1 where the foresets have a bigger kernel: Fig. 4(b) shows clearly that $E_1{}^{\heartsuit}_{\mathscr{I}_{\mathrm{g}}}$ does not keep the kernel of A.

Example 2. Let the $(\mathbb{I}_{\mathrm{R}}, d_{|\cdot|})$-resemblance relation E_2 be defined by $(\forall (x, y) \in \mathbb{R}^2)$ $(E_2(x,y) = \min(1, \max(1.6 - |x-y|, 0)))$. Fig. 5 shows the effect of $(E_1)^{\clubsuit}_{\mathscr{C}}$ (more or less) and $(E_2)^{\clubsuit}_{\mathscr{C}}$ (roughly) on the triangular shaped membership

[5] The S-function is one of the general shape functions used for modelling membership functions for fuzzy sets on $\mathbb{R}$. For $(\alpha, \beta, \gamma) \in \mathbb{R}^3$, $\alpha < \gamma$ and $\beta = \frac{\alpha+\gamma}{2}$

$$
\begin{aligned}
S(\cdot; \alpha, \beta, \gamma):\quad & \mathbb{R} \to [0,1] \\
& x \mapsto 0 \quad \forall x \in]-\infty, \alpha[\\
& x \mapsto 2\left(\frac{x-\alpha}{\gamma-\alpha}\right)^2 \quad \forall x \in [\alpha, \beta] \\
& x \mapsto 1 - 2\left(\frac{x-\gamma}{\gamma-\alpha}\right)^2 \quad \forall x \in [\beta, \gamma] \\
& x \mapsto 1 \quad \forall x \in [\gamma, +\infty[
\end{aligned}
$$

.

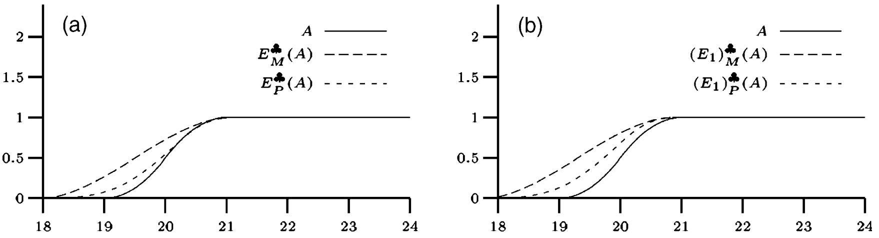

Fig. 3. (a) The fuzzy sets A (*warm*) and $E_M^{\clubsuit}(A)$, $E_P^{\clubsuit}(A)$ (more or less *warm*); (b) the fuzzy sets A (*warm*) and $E_{1}{}_M^{\clubsuit}(A)$, $E_{1}{}_P^{\clubsuit}(A)$ (more or less *warm*).

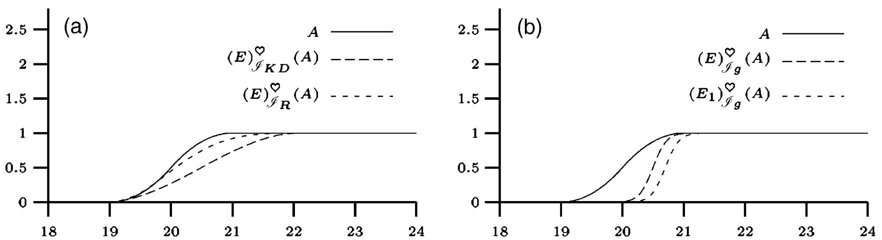

Fig. 4. (a) The fuzzy sets A (*warm*) and $E_{\mathscr{I}_{\mathrm{KD}}}^{\heartsuit}(A)$, $E_{\mathscr{I}_{\mathrm{R}}}^{\heartsuit}(A)$ (very *warm*); (b) the fuzzy sets A, $E_{\mathscr{I}_{\mathrm{g}}}^{\heartsuit}(A)$ and $E_{1}{}_{\mathscr{I}_{\mathrm{g}}}^{\heartsuit}(A)$.

function of the fuzzy set B on $[0, 24[$ representing *about* $20h00$. For $\mathscr{C}$ we choose, respectively, M and W. Another option to model more or less and roughly would be to use only one resemblance relation but two different conjunctors. Fig. 5 (a) and (b) show that $(E_1)_W^{\clubsuit}$ would be a possible representation for more or less, while $(E_1)_M^{\clubsuit}$ could serve to model roughly.

Using the resemblance relations E_3 and E_4 defined by $(\forall(x, y) \in \mathbb{R}^2)$ $(E_3(x, y) = \max(1 - 3|x - y|, 0))$ and $(\forall(x, y) \in \mathbb{R}^2)$ $(E_4(x, y) = \max(1 - 6|x -$

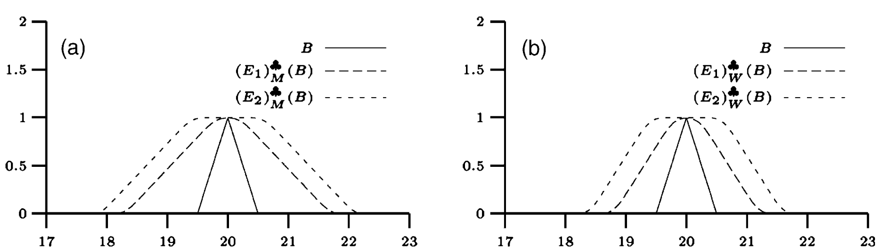

Fig. 5. (a) The fuzzy sets B (*about 20h00*), $E_{1}{}_M^{\clubsuit}(B)$ and $E_{2}{}_M^{\clubsuit}(B)$; (b) The fuzzy sets B (*about 20h00*), $E_{1}{}_W^{\clubsuit}(B)$ and $E_{2}{}_W^{\clubsuit}(B)$.

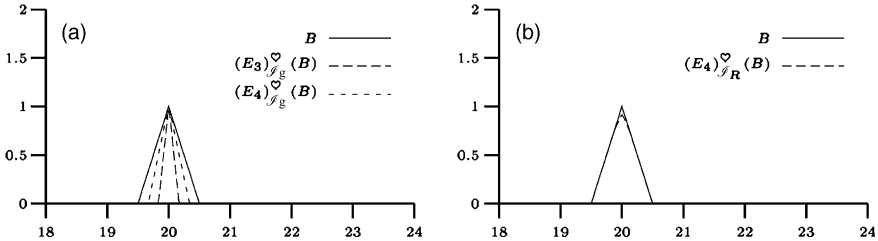

Fig. 6. (a) The fuzzy sets B, $(E_3)^{\heartsuit}_{\mathscr{I}_g}(B)$ and $(E_4)^{\heartsuit}_{\mathscr{I}_g}(B)$; (b) the fuzzy sets B and $(E_4)^{\heartsuit}_{\mathscr{I}_R}(B)$.

$y|, 0))$, $(E_3)^{\heartsuit}_{\mathscr{I}_g}$ can be used to model very while $(E_4)^{\heartsuit}_{\mathscr{I}_g}$ represents extremely. Fig. 6(a) shows the effect of $(E_3)^{\heartsuit}_{\mathscr{I}_g}$ and $(E_4)^{\heartsuit}_{\mathscr{I}_g}$ on the triangular shaped membership function B.

Example 3. In the universe of fairytale characters $X = \{$snowwhite, witch, wolf, dwarf, prince, little-red-riding-hood$\}$ the fuzzy sets *beautiful*, *average* and *ugly* are given.

	snowwhite	witch	wolf	dwarf	prince	red-hood
beautiful	1.00	0.00	0.00	0.10	0.80	0.50
average	0.00	0.30	0.00	0.70	0.20	0.50
ugly	0.00	0.70	1.00	0.20	0.00	0.00

For g the $X \to [0,1]^3$ mapping defined by $(\forall x \in X)\ (g(x) = (beautiful(x), average(x), ugly(x)))$ and d the pseudo-metric on $[0,1]^3$ defined by $(\forall((x_1,y_1,z_1),(x_2,y_2,z_2)) \in [0,1]^3 \times [0,1]^3)\ (d((x_1,y_1,z_1),(x_2,y_2,z_2)) = \max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|))$, we can model approximate equality by the (g, d)-resemblance relation E on X with the following matrix representation:

E	snowwhite	witch	wolf	dwarf	prince	red-hood
snowwhite	1.00	0.00	0.00	0.00	1.00	0.50
witch	0.00	1.00	1.00	0.50	0.00	0.00
wolf	0.00	1.00	1.00	0.00	0.00	0.00
dwarf	0.00	0.50	0.00	1.00	0.00	0.88
prince	1.00	0.00	0.00	0.00	1.00	1.00
red-hood	0.50	0.00	0.00	0.88	1.00	1.00

The membership degrees in the fuzzy sets representing some modified terms are:

	snow-white	witch	wolf	dwarf	prince	red-hood
$(E)_P^{\clubsuit}$(beautiful) more or less beautiful	1.00	0.05	0.00	0.44	1.00	0.50
$(E)_P^{\clubsuit}$(average) more or less average	0.25	0.35	0.30	0.70	0.50	0.61
$(E)_P^{\clubsuit}$(ugly) more or less ugly	0.00	1.00	1.00	0.35	0.00	0.18
$(E)_{\mathscr{I}_g}^{\heartsuit}$(beautiful) very beautiful	0.80	0.00	0.00	0.00	0.50	0.10
$(E)_{\mathscr{I}_g}^{\heartsuit}$(average) very average	0.00	0.00	0.00	0.30	0.00	0.00
$(E)_{\mathscr{I}_g}^{\heartsuit}$(ugly) very ugly	0.00	0.20	0.70	0.00	0.00	0.00

6.6. A note on the behaviour w.r.t. the kernel and the support

Let X be a universe, E a resemblance relation on X and $A \in \mathscr{F}(X)$.
For $\mathscr{C}$ a semi-norm we obtained

$$\ker(A) \subseteq \ker(E_{\mathscr{C}}^{\clubsuit}(A)) \quad \text{and} \quad \mathrm{supp}(A) \subseteq \mathrm{supp}(E_{\mathscr{C}}^{\clubsuit}(A))$$

(see Corollary 4.1(1)). However, this does not guarantee that

$$\ker(A) \subset \ker(E_{\mathscr{C}}^{\clubsuit}(A)) \quad \text{and} \quad \mathrm{supp}(A) \subset \mathrm{supp}(E_{\mathscr{C}}^{\clubsuit}(A))$$

i.e. we cannot say for sure that the kernel and the support are actually *changed* by the fuzzy modifier. (R8)(1) and (R9)(1) tell us how to really expand the kernel and the support.

Analogously for $\mathscr{I}$ a border implicator we got: $\ker(E_{\mathscr{I}}^{\heartsuit}(A)) \subseteq \ker(A)$ and $\mathrm{supp}(E_{\mathscr{I}}^{\heartsuit}(A)) \subseteq \mathrm{supp}(A)$ (see Corollary 4.1(2)). However this does not guarantee that $\ker(E_{\mathscr{I}}^{\heartsuit}(A)) \subset \ker(A)$ and $\mathrm{supp}(E_{\mathscr{I}}^{\heartsuit}(A)) \subset \mathrm{supp}(A)$, i.e. we can not say for sure that the kernel and the support are actually *changed* by the fuzzy modifier. (R8)(2) and (R9) (2) tell us how to really restrict the kernel and the support.

Sometimes it is also preferable that the kernel is kept, e.g. in the case of a triangular membership function. Fig. 6(b) shows the effect of $E4^{\heartsuit}_{\mathscr{I}_R}$ on *about 20h00* where $\mathscr{I}_R$ does not satisfy the condition of (R8)(3).

6.7. Support fuzzification

Zadeh describes a process of point fuzzification which transforms a singleton $\{x\}$ of X into a fuzzy set of X that is concentrated around x. Following Kerre [11], we will name this fuzzy set after the process used to generate it, and thus define the concept of "point fuzzification centered around $x \in X$ w.r.t. d".

Definition 6.3. If (X, d) is a pseudo-metric space and $x \in X$, then xK is a *point fuzzification centered around x w.r.t. d* iff $xK \in \mathscr{F}(X)$ and $xK(x) = 1$ and $(\forall (a, b) \in X^2)\ (d(x, a) < d(x, b) \Rightarrow xK(a) \geqslant xK(b))$.

So xK is a decreasing function with respect to the distance from x. It can be considered as the K-afterset of x. This means the symbol K is not simply notational sugar anymore, but now denotes a fuzzy relation on X.

Definition 6.4. For (X, d) is a pseudo-metric space we call a fuzzy relation R on X a *point fuzzification of X w.r.t. d* iff $(\forall x \in X)(xR$ is a point fuzzification centered around x w.r.t. $d)$.

Support fuzzification is originally defined by Zadeh [20].

Definition 6.5. If (X, d) is pseudo-metric space and K is a point fuzzification of X w.r.t. d, then SF_K is a *support fuzzification with kernel K* iff SF_K is a $\mathscr{F}(X) \to \mathscr{F}(X)$ mapping with for $A \in \mathscr{F}(X)$:

$$\begin{aligned} SF_K A : X &\to [0, 1] \\ y &\mapsto \sup_{x \in X} K(x, y)A(x), \quad \forall y \in X \end{aligned}$$

It is clear that $SF_K = K_P^{\clubsuit}$. The notions point fuzzification and resemblance relation are related by the following proposition (see also [6]):

Proposition 6.1. *For* (X, d) *a pseudo-metric space, every* $(\mathbb{I}_X, d)$*-resemblance relation on X is a point fuzzification of X w.r.t. d.*

The reader should therefore not be surprised that Zadeh had the representation of more or less in mind when he developed support fuzzification.

7. Weakening and intensifying modifiers: non-inclusive interpretation

7.1. General scheme

We introduce a $(\mathbb{R} \times \mathscr{F}(\mathbb{R}^2)) \to \mathscr{F}(\mathbb{R}^2)$ mapping $\mathscr{G}$.

Definition 7.1. For $\alpha \in \mathbb{R}$ and $R \in \mathscr{F}(\mathbb{R}^2)$, $\mathscr{G}(\alpha, R)$ is defined by for $(x, y) \in \mathbb{R}^2$:

$$\mathscr{G}(\alpha, R)(x, y) = \begin{cases} 1 & \text{if } R(x, y - \alpha) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathscr{C}$ be a conjunctor, p a positive real number, $\mathscr{G}$ defined as above and B a mountain shaped fuzzy set on $\mathbb{R}$. Depending on the nature of the linguistic term B, the modified terms derived from B can be modelled by the representations proposed in the first or the second column:

roughly B $= \mathscr{G}(-2p, =)^{\clubsuit}_{\mathscr{C}}(B)$	roughly B $= \mathscr{G}(2p, =)^{\clubsuit}_{\mathscr{C}}(B)$
more or less B $= \mathscr{G}(-p, =)^{\clubsuit}_{\mathscr{C}}(B)$	more or less B $= \mathscr{G}(p, =)^{\clubsuit}_{\mathscr{C}}(B)$
very B $= \mathscr{G}(p, =)^{\clubsuit}_{\mathscr{C}}(B)$	very B $= \mathscr{G}(-p, =)^{\clubsuit}_{\mathscr{C}}(B)$
extremely B $= \mathscr{G}(2p, \leqslant)^{\clubsuit}_{\mathscr{C}}(B)$	extremely B $= \mathscr{G}(-2p, \geqslant)^{\clubsuit}_{\mathscr{C}}(B)$

For a particular choice of α and R, the fuzzy modifier $\mathscr{G}(\alpha, R)^{\clubsuit}_{\mathscr{C}}$ has a specific effect on the membership function of a mountain shaped fuzzy set B on $\mathbb{R}$.

α	R	Effect
$\alpha > 0$	$=$	Shift to the right
$\alpha > 0$	$\leqslant$	Shift to the right and change of shape
$\alpha < 0$	$=$	Shift to the left
$\alpha < 0$	$\geqslant$	Shift to the left and change of shape

Remark. For an arbitrary real number α the fuzzy modifiers $\mathscr{G}(\alpha, =)^{\clubsuit}_{\mathscr{C}}$ coincide with the shifting fuzzy modifiers.

7.2. Example

In the non-inclusive interpretation the atomic terms *warm* and *cold* can be represented by $B_1 \in \mathscr{F}(\mathbb{R})$ and $B_2 \in \mathscr{F}(\mathbb{R})$ respectively, defined by $B_1 = S(\cdot; 15, 16.25, 17.5) \cap_M \mathrm{co}_{\mathscr{N}_s}(S(\cdot; 17.5, 18.75, 20))$ and $B_2 = S(\cdot; 5, 6.25, 7.5) \cap_M \mathrm{co}_{\mathscr{N}_s}(S(\cdot; 7.5, 8.75, 10))$. The fuzzy sets $\mathscr{G}(-4, \geqslant)^{\clubsuit}_M(B_2)$ (extremely *cold*), $\mathscr{G}(-2, =)^{\clubsuit}_M(B_2)$ (very *cold*), B_2 (*cold*), $\mathscr{G}(2, =)^{\clubsuit}_M(B_2)$ (more or less *cold*), $\mathscr{G}(4, =)^{\clubsuit}_M(B_2)$ (roughly *cold*), $\mathscr{G}(-4, =)^{\clubsuit}_M(B_1)$ (roughly *warm*), $\mathscr{G}(-2, =)^{\clubsuit}_M(B_1)$ (more or less *warm*), B_1 (*warm*), $\mathscr{G}(2, =)^{\clubsuit}_M(B_1)$ (very *warm*), $\mathscr{G}(4, \leqslant)^{\clubsuit}_M(B_1)$ (extremely *warm*), are depicted in Fig. 7 from the left to the right.

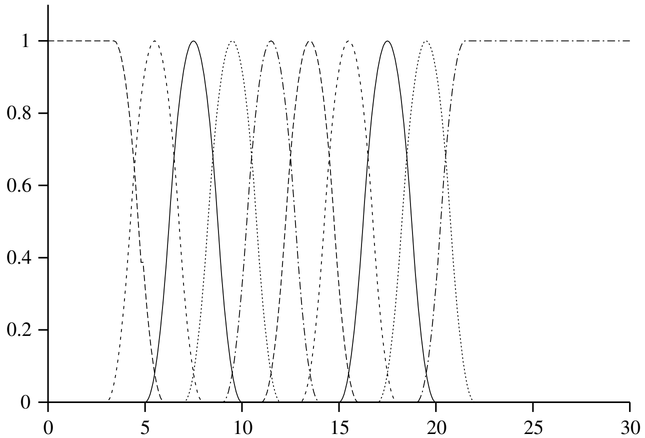

Fig. 7. Fuzzy sets for the terms derived from *cold* and *warm* in the non-inclusive interpretation.

8. Conclusion

In this paper we have introduced a new type of fuzzy modifiers which turn out to be a very powerful tool to model linguistic modifiers. We have illustrated how intensifying adverbs like very and extremely and weakening modifiers like more or less and roughly can be modelled in the inclusive and the non-inclusive interpretation within the *same* framework. Of course other weakening and intensifying adverbs like quite, fairly, highly, ... can be modelled analogously. Furthermore the suggestions made by Bodenhofer [3] to model at least and at most fit in this framework. All comes down to an appropriate choice of conjunctor, implicator and fuzzy relation.

Because the compositional rule of inference [21,7] is based on the same mechanism as the type of fuzzy modifiers defined in this paper, namely the direct image of a fuzzy set under a fuzzy relation, these new modifiers can be integrated smoothly in an approximate reasoning scheme. The inference becomes very straightforward in a lot of standard cases using fuzzy comparators. E.g. let A and E denote " handsome" and "approximately equal", respectively, then applying the CRI results in:

Lars is *handsome*.	$\rightarrow$	Lars's looks are A
Paul looks like Lars.	$\rightarrow$	Paul's looks and Lars's looks are E
Paul is more or less *handsome*.	$\leftarrow$	Paul's looks are $E_M^{\clubsuit}(A)$

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