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Rejoinder

Why fuzzy $\mathscr{T}$-equivalence relations do not resolve the Poincaré paradox, and related issues

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Abstract

In this paper, which is a rejoinder on the comments (Fuzzy Sets and Systems, 133 (2) (2003)), we will answer the questions whether approximate equality should satisfy transitivity, and whether fuzzy $\mathscr{T}$-equivalence relations resolve the Poincaré paradox. Furthermore, we will present an improved definition of resemblance relation and we will investigate its connection with nearness relations. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Taking into account the comments [1, 2, 13, 16], in this rejoinder we will answer the questions whether approximate equality should satisfy transitivity and whether fuzzy $\mathscr{T}$-equivalence relations resolve the Poincaré paradox. Furthermore, we will present a simplified definition of the concept of a resemblance relation. We will explain to the reader why we have taken an interest in modelling approximate equality in the first place, and which were to us the first signs that fuzzy $\mathscr{T}$-equivalence relations would not be suitable to do the job, forcing us to look for another kind of fuzzy relation. In this respect we want to stress that it was certainly never our intention to encourage the reader to dismiss fuzzy $\mathscr{T}$-equivalence relations at all, but rather to point out that these otherwise nice mathematical entities are not suitable to model approximate equality.

First of all, however, we will explain as clearly as possible what approximate equality is from the semantical point of view, since this is an important prerequisite for the further discussion as is pointed out rightly in [1, 2, 16].

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2. On the meaning of approximate equality

Just like “old”, “beautiful”, and “small”, “approximately equal” is a vague linguistic expression that is used to describe—in natural language—phenomena of the real world. Fuzzy set theory provides a framework for modelling these kinds of expressions in a mathematical way, namely by means of fuzzy sets that are characterized by membership functions. Note that the first three linguistic terms mentioned above are usually modelled by fuzzy sets on some suitable universe X, while “approximately equal” can be represented by a fuzzy set on a cartesian product $X \times X$, commonly called a fuzzy relation on X. The chosen fuzzy set should reflect the semantics of the represented linguistic expression as good as possible. Hence the modelling process can be divided into two steps:

Step 1: Describe the meaning of the linguistic expression as clearly as possible.

Step 2: Look for a fuzzy set that models this meaning in an/the most acceptable way.

In step 1 we are not yet looking for “a mathematical model what approximate equality means” [1]. Indeed: it will not be before step 2 that we start talking about a mathematical model. In this section we will deal with step 1; in further sections we will deal with (un)suitable mathematical models such as fuzzy $\mathscr{T}$-equivalence relations, nearness relations and resemblance relations (step 2).

To grasp the semantics of “approximate equality” we start by studying examples such as the following one:

Consider the children in a primary school. Those of the same class are usually approximately equal w.r.t. age (i.e. “of the same age”) although a very small minority of them is born on exactly the same day. Those of the first year and the final year are usually not approximately equal w.r.t. their age. Children in 2 successive years on the other hand might be considered as of the same age to some extent.

From this example and the ones given in [6, 16] we deduce that approximate equality can be described in the following way:

Approximate equality

(D.1) Approximate equality is a relation: roughly speaking it is a rule for combining objects.

(D.2) Approximate equality is a vague concept: the transition between *being* approximately equal and *not being* approximately equal is not abrupt but gradual. It is a matter of degrees.

(D.3) Every object is approximately equal to itself…

(D.4) … but two objects can also be approximately equal even if they are not exactly equal.

(D.5) Furthermore, the smaller the distance is between two objects the more they are approximately equal.

(D.1) and (D.3) are beyond dispute in [1, 2, 13, 16]. The second criterion (D.2) is questioned in [2] where the author claims that the relation $\approx$ on $\mathbb{R}$ defined by

$$x \approx y \quad \text{if and only if } |x - y| < \varepsilon$$

for all x and y in $\mathbb{R}$, and the so-called error bound ε in $]0, +\infty[$, is “an approximate equality which is certainly not vague”, and that it “provides a basic, natural and intuitive example of a crisp approximate equality”. He continues, however, by rightly admitting “the general impression left by such an example is that, although the associated approximate equality is a truly crisp relation, its nature is not". In this example, step 1 and step 2 described above are mixed up. On one hand we are dealing with a concept of approximate equality which is indeed truly vague in its nature (step 1), while on the other hand we are considering the crisp relation $\approx$ as a mathematical model (step 2). In traditional set theory, the concept "young" is usually modelled by means of some crisp interval $[0, 20]$ but this does not imply that "young" itself is crisp. In fact we even agree that $[0, 20]$ is not a good model for "young" since the linguistic term is vague and the model is not able to reflect this. The same applies for the example given in [2]: using an unsatisfactory crisp model to represent a linguistic expression does not imply that this expression is not vague.

(D.5) reflects the duality between approximate equality and distance, which is widely accepted. As is pointed out rightly in [2, 16] "we have to be careful about a naive use of distance", and the chosen mathematical model for approximate equality should "preserve in some way the metric structure inherent to the problem under study". We will return to this issue in Sections 3 and 4.

The fourth criterion (D.4) seems to be the most provocative one, leading to the greater part of the discussions published in this issue. Although many real-life examples, such as the one concerning the car prices given in [16], support (D.4), in [2] this criterion is called "an anomaly" that is referred to as "partial metric collapse", i.e. "pairs whose levels of proximity are distinguishable to some extent in the metric context become pairs with completely indistinguishable levels of similarity". In the same paper (D.4) is also described as a "counterintuitive situation in which a qualitative part of the information provided by the pseudometric is deliberately lost". But this is exactly the way in which people mostly reason! Although the customer knows that there is a difference of \$1 between the car price in his mind and the actual price, he will consider them to be approximately equal [16]. Furthermore, this is also the way in which people often exchange information: although a mother knows for sure that her daughter is one month and one day older than the child's younger niece, she might still say in some situation that the children are of the same age (i.e. "approximately equal w.r.t. their age"). In deliberatedly "conveying" precise knowledge behind a vague linguistic expression, people are often following at least one of the rules of conversation explicited by Grice [10], namely the maxim of quantity:

(i) make your contribution as informative as is required for the current purposes of exchange
(ii) do not make your contribution more informative than is required.

When modelling human reasoning and when building human-like interfaces, it is highly desirable to reflect this behaviour. Furthermore, sometimes one knows that there is some difference in the metric context but not the size of the difference. In this case one also uses a vague linguistic expression to convey ones lack of precise knowledge: "Chris and Krista are approximately of the same height, though I believe he is a few inches taller, so they will make a nice couple."

Example 4 in [6] should also be seen in this respect: for the reasons given above the similarity between the ages is intentionally expressed by means of a linguistic expression, represented by a fuzzy relation. The fact that the precise ages are given as well, might be misleading [16]; we only added them to make the reader who does not know Alberik, Bart and Chris, capable of checking the validity of the example. Probably the example would have been more convincing if we had given pictures of them without mentioning their precise age.

Summarizing, we can say that (D.4) cannot be called an anomaly. Moreover, the author himself in [2] later on admits "situations like these arise naturally in experimental environments".

Now that the semantics of “approximate equality” is explicited, we can go to step 2. (D.1)–(D.3) suggest to model “approximate equality” by means of a reflexive fuzzy relation. In the next section we will evaluate the use of fuzzy $\mathcal{T}$-equivalence relations for this purpose, in answering the question whether fuzzy $\mathcal{T}$-equivalence relations resolve the Poincaré paradox. In Section 4, we return to resemblance relations as an alternative mathematical model.

3. Do fuzzy $\mathcal{T}$-equivalence relations resolve the Poincaré paradox?

As we demonstrated in [7, Example 3] and as is affirmed in [1, 13], (D.4) and $\mathcal{T}$-transitivity do not go together when modelling approximate equality without leading to strange paradoxes. In other words:

Statement. A fuzzy $\mathcal{T}$-equivalence relation that is not a fuzzy $\mathcal{T}$-equality, i.e. a fuzzy $\mathcal{T}$-equivalence relation that does not satisfy the separation condition

$$E(x, y) = 1 \quad \text{iff } x = y, \tag{1}$$

in general, cannot model approximate equality in an acceptable way.

The reason for this comes down to the following: if $(x_1, x_2, x_3, \ldots, x_n)$ is an n-tuple of distinct elements such that two successive elements are approximately equal to degree 1 (i.e. $E(x_i, x_{i+1}) = 1$, i in $\{1, 2, \ldots, n-1\}$), one can prove that x_1 and x_n are approximately equal to degree 1 when modelling E by means of a fuzzy $\mathcal{T}$-equivalence relation. Normally this is not desired. Imagine, e.g. that the elements of the chain are girls that can be ranked in beauty from the ugliest one to the prettiest one. Although girls standing next to each other might be approximately equal to degree 1 in beauty, it is likely that there is a big difference between the first and the last one. This phenomenon is caused by the following proposition which was proved in [8] and is a consequence of $\mathcal{T}$-transitivity:

Proposition 1. *If E is a fuzzy $\mathcal{T}$-equivalence relation on X, then for all x, y and z in X*

$$E(x, y) = 1 \quad \textit{implies } E(x, z) = E(y, z).$$

Hence approximate equality, or indistinguishability, can only be modelled by means of a fuzzy $\mathcal{T}$-equivalence relation when (D.4) does not occur. Successful applications testify that this is indeed possible, but the t-norm should be chosen with great care. In this respect Example 1 of [6] was a bad choice [1, 2]. Examples of well-considered choices for P and W are given in [2], while one for M can be found in [16]. Furthermore, in [2] it is stated that the t-norm is actually not chosen but induced by the metric structure inherent to the problem, which results in the prerequisite of finding the proper (pseudo)-metric d. Given a suitable d, in [16] the fuzzy $\mathcal{T}$-equivalence relation E defined by

$$E(x, y) = 1 - d(x, y)$$

for all x and y in X, is considered. Note that whenever d is a genuine pseudo-metric (which is not a metric), i.e. whenever there exist distinct objects x and y such that $d(x, y) = 0$, E does not satisfy the separation condition and can therefore not model approximate equality in an acceptable way.

The Poincaré paradox was recalled in [1, 2, 6, 16]. We add one more example (quote from [20]):

> On a observé, par exemple, qu'un poids A de 10 grammes et un poids B de 11 grammes produisaient des sensations identiques, que le poids B ne pouvait non plus être discerné d'un poids C de 12 grammes, mais que l'on distinguait facilement le poids A du poids C. [1]

This example is a clear case of (D.4): A and B as well as B and C are not exactly equal but still indistinguishable. If we represent approximate equality by means of a reflexive fuzzy relation E, we have that $E(A,B)=1$ although $A\neq B$, and $E(B,C)=1$ although $B\neq C$. Clearly E does not satisfy the separation condition, and hence because of the statement mentioned above, E cannot be a fuzzy $\mathscr{T}$-equality.

For suppose that we model the approximate equality by means of a fuzzy $\mathscr{T}$-equivalence relation E. Following Statement (1) given at the beginning of this section, E should satisfy the separation condition which is a vital prerequisite. Since A and B as well as B and C are indistinguishable, we unavoidably have $E(A,B)=1$ and $E(B,C)=1$ (this is resulting from the experiment, nothing can be done about it!). The separation condition would imply

$$A = B \quad \text{and} \quad B = C.$$

The crisp equality is of course transitive, so also $A=C$, and presuming that E satisfies the separation condition also $E(A,C)=1$. Which absolutely contradicts the experimental result that A and C are distinguishable! This is why we say that fuzzy $\mathscr{T}$-equivalence relations do not resolve the Poincaré paradox.

4. Concerning the definition of resemblance relation

In [6] we proposed to model approximate equality by means of a resemblance relation, i.e. a fuzzy relation characterized by 3 criteria. As rightly noted in [16] the third criterion induces the second one, namely symmetry. Furthermore, we agree with [16] that the mapping g makes the definition complicated from the theoretical point of view, but as noted in [1] it may help improve clarity in applications. Taking all of this into account we therefore present the simplified definition:

Definition 1. A fuzzy relation E on a pseudo-metric space (X,d) is called a resemblance relation (w.r.t. d) iff for all x, y, z and u in X:

(R.1) $E(x,x) = 1$.
(R.2) $d(x,y)\leqslant d(z,u)$ implies $E(x,y)\geqslant E(z,u)$.

If it is not straightforwardly clear which meaningful pseudo-metric is available on the universe, a mapping g from X to a pseudometric space $(\mathscr{M},d)$ might do the trick, replacing condition (R.2) by

$$d(g(x),g(y)) \leqslant d(g(z),g(u)) \quad \text{implies } E(x,y) \geqslant E(z,u).$$

[1] One has observed, for instance, that a weight A of 10 g and a weight B of 11 g produced identical sensations, that the weight B could not be distinguished from a weight C of 12 g, but that one easily distinguished the weight A from the weight C.

From now on (R.1) and (R.2) refer to the conditions stated in the definition above. In the original definition in [6], the last criterion was added to the demand for reflexivity and symmetry, to reflect the duality with distance, which distincts resemblance relations from fuzzy tolerance relations in general. In [16], however, it was proven that for every fuzzy tolerance relation a pseudo-metric d can be found such that E is a resemblance relation w.r.t. d. Although the presented pseudo-metric is not really the best imaginable one from the intuitive point of view ($d(x,x)=0$ while $d(x,y)\geqslant 1$ for $x\neq y$), this suggests that the definition of resemblance relation is too broad in the sense that fuzzy relations that are (in some situations) unsuitable to model approximate equality still comply with the definition of resemblance relation. Proposition 3 of [6] also points in this direction: all fuzzy $\mathcal{T}$-equivalence relations E (for $W\subseteq\mathcal{T}$) are resemblance relations w.r.t. the pseudo-metric that coincides with their complement $\mathrm{co}_{\mathcal{N}_s}E$. Nevertheless, in Section 3 we have stated once more that fuzzy $\mathcal{T}$-equivalence relations are usually unsuitable to model approximate equality if (D.4) occurs. On the other hand, we want to stress that (R.2) was not added to the definition with the purpose to exclude transitive relations, or to model non-transitivity as is said in [2].

In [2] the definition of resemblance relation is modified into that of a so-called type I resemblance relation, which coincides in a sense with fuzzy $\mathcal{T}$-equivalence relation. One of the changes to the original definition resulting in this coincidence is

$$d(g(x),g(y)) \leqslant d(g(z),g(u)) \quad \text{iff } E(x,y) \geqslant E(z,u),$$

i.e. demanding equivalence instead of one-way implication on (R.2). It is exactly this modification which causes type I resemblance relations to satisfy Proposition 1. Indeed, let x, y and z be elements of X. Since E is reflexive, we have that $E(x,x)=1$. Assuming that $E(x,y)=1$ we therefore have $E(x,y)\geqslant E(x,x)$ which implies $d(x,y)\leqslant d(x,x)=0$ due to the definition of type I resemblance relation and that of pseudo-metric. The triangle inequality then yields

$$d(x,z) \leqslant d(x,y)+d(y,z) \leqslant d(y,z),$$
$$d(y,z) \leqslant d(y,x)+d(x,z) \leqslant d(x,z).$$

Hence $d(x,z)=d(y,z)$ and $E(x,z)=E(y,z)$. This means that for a type I resemblance relation E on X

$$E(x,y)=1 \quad \text{implies } E(x,z)=E(y,z)$$

holds for all x, y, and z in X (see also Proposition 1), straightforwardly leading to the problems discussed in Section 3. Resemblance relations were exactly created to overcome this problem! In [2] proposed modifications of the definition of resemblance relation undo the benefits of the original definition. Type I resemblance relations are as useless as fuzzy $\mathcal{T}$-equivalence relations when (D.4) occurs.

In [13] it is argued that resemblance relations are nearness relations, which are usually defined on $\mathbb{R}$. The third criterion of Definition 1 in [13]

$$t \text{ is between } x \text{ and } y \quad \text{implies } E(x,t) \geqslant E(x,y) \tag{2}$$

is sometimes also formulated (see [14], and the definition of "equivalence" in [9]) as

$$x \leqslant s \leqslant t \leqslant y \quad \text{implies } E(s,t) \geqslant E(x,y) \tag{3}$$

for x, s, t and y in $\mathbb{R}$. This criterion is related to (R.2) though not the same. While (R.2) expresses (D.5), criterion (3) expresses only part of it, namely: given two objects x and y such that $x \leqslant y$, any two objects s and t of the interval $[x, y]$ are "approximately equal" to the same or to a higher degree than x and y. Note, however, that (3) does not say anything about the approximate equality of two objects s' and t' such that either s' or t' or both do not belong to $[x, y]$. The fuzzy relation E on $[0, 1]$ defined by

$$E(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0.9 & \forall (x, y) \in [0.4, 0.8]^2, x \neq y, \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

is a nearness relation. Furthermore, $E(0.2, 0.3) = 0$ while $E(0.4, 0.8) = 0.9$ which obviously does not make E very suitable to model approximate equality.[2] Note that E is not a resemblance relation w.r.t. $d_{|.|}$ which is a very "natural" ordering on numerical universes such as $[0, 1]$.

A possible aid to this problem might be the addition of another criterion to the definition of nearness relations as was done in [14], namely that of continuity mentioned in [13].

$$(\forall \alpha \in]0, 1[)\ (\forall x \in \mathbb{R})\ (\exists! (x_{-\alpha}, x_\alpha) \in \mathbb{R}^2) \quad (x_{-\alpha} < x < x_\alpha \text{ and } E(x, x_{-\alpha}) = E(x, x_\alpha) = \alpha).$$

Indeed example (4) does not satisfy this condition. The condition is based on the notions of left and right which makes it hard to generalize it to arbitrary universes X such as the one of fairytale characters described in [6]. Maybe we could demand that for every x in X and α in $]0, 1]$ the set $\{y \mid E(x, y) = \alpha\}$ contains exactly two elements, but then again: why two? Furthermore, the condition implies that for every nearness relation E (on a non-empty universe X) it holds that $]0, 1[\subseteq range(E)$ which can never be achieved on a discrete and finite universe X.

In [8] a definition of nearness relation on an arbitrary universe is presented, but criterion (3) is replaced by a new one that implies

$$E(x, y) = 1 \quad \text{implies } E(x, z) = E(y, z)$$

for every nearness relation E. As we already stated in Section 3 this behaviour supports the Poincaré paradox.

5. Motivation behind our conjecture

Although in the past the definition of similarity relations (i.e. fuzzy min-equivalence relations) already caused us to raise our eyebrows several times, the thorough thinking process concerning the representation of approximate equality started when we were preparing [3, 5]. In these papers a fuzzy relation based model for the representation of linguistic hedges such as roughly, more or less, very and extremely is proposed. The new framework was designed to overcome the shortcomings of traditional approaches. As is commonly known in the fuzzy community an important drawback

[2] It should be stressed that the concept of equivalence is not introduced in [9] to this purpose.

of the powering hedges P_α (for α in $]0,+\infty[$) proposed in [22] and defined by

$$P_\alpha(A)(x) = A(x)^\alpha$$

for all A in $\mathscr{F}(X)$ and all x in X is that they keep the kernel and the support [11, 15, 17]. As an immediate consequence, when deriving the representation of a linguistic term such as very old from the membership function of old by means of a powering hedge (often P_2 is used for this purpose), we cannot make a distinction between people who are old to degree 1 and people who are very old to degree 1, although we can think of situations in which we would call a man of 80 years old to degree 1 but very old only to a lower degree, say 0.8. What is usually overlooked is that the behaviour of the powering hedges w.r.t. the kernel and the support also has an influence on the shape of the whole membership function, as was indicated experimentally in [12]. A solution in this particular case is to derive very old from old by means of some shifting hedge S_α [11, 17] defined by

$$S_\alpha(A)(x) = A(x-\alpha)$$

for all A in $\mathscr{F}(\mathbb{R})$ and all x in $\mathbb{R}$ in which the real number α determines the direction and the size of the shift. Unfortunately, this approach only works in universes equipped with a shifting operation, such as $\mathbb{R}$, and even in this case only for increasing and for decreasing membership functions. The goal in [3, 5] was the development of a framework of fuzzy hedges that do not necessarily keep the kernel and the support, and that can be widely applied to all kinds of membership functions in all kinds of universes.

Note that in determining the degree to which y is very A, the powering hedges look at the degree to which y is A but completely ignore all other objects of the universe. On the other hand, the shifting hedges look at one object $y-\alpha$ in the context or neighbourhood of y but they ignore all other objects in the context of y, including y itself (unless in the trivial case where $\alpha = 0$). In [3, 5] it is suggested to wittingly consider the context of y when determining the degree to which y belongs to the modified fuzzy set. For the linguistic hedges more or less and very this context consists of the objects of X resembling y. We start from the observation that "a comment is more or less long iff it resembles a long comment". In general, assuming for a moment for simplicity that A and resemblance are crisp concepts, we have: "y is more or less A iff y resembles an x that is A". But of course a linguistic term like long is not crisp and neither is the concept of resemblance/approximate equality. So when representing the former by a fuzzy set A and the latter by means of a fuzzy relation R, we end up with the following

$$\text{more or less } A(y) = \sup_{x\in X} \mathscr{T}(R(x,y), A(x)) \tag{5}$$

($\mathscr{T}$ being a continuous triangular norm). Likewise we observe that "a woman is very beautiful iff all women she resembles are beautiful". In general: "y is very A iff all x resembling y are A", or the true "fuzzy version":

$$\text{very } A(y) = \inf_{x\in X} \mathscr{I}_{\mathscr{T}}(R(x,y), A(x)) \tag{6}$$

($\mathscr{I}_{\mathscr{T}}$ being the residual implicator induced by $\mathscr{T}$). The right-hand sides of the formulas (5) and (6) can be abbreviated to $R^{\clubsuit}(A)(y)$ and $R^{\heartsuit}(A)(y)$, i.e. more or less A is represented by $R^{\clubsuit}(A)$, also called the direct image of A under R while very A is modelled by means of $R^{\heartsuit}(A)$ which is related to the superdirect image of A under R (see [15]).

Proposition 2 (De Cock et al. [7], Orlowska and Radzikowska [19]). *For all R in $\mathscr{F}(X \times X)$:*

(1) *R is reflexive iff for all A in $\mathscr{F}(X)$*

$$R^{\heartsuit}(A) \subseteq A \subseteq R^{\clubsuit}(A)$$

(2) *R is $\mathscr{T}$-transitive iff for all A in $\mathscr{F}(X)$*

$$R^{\heartsuit}(A) \subseteq R^{\heartsuit}(R^{\heartsuit}(A)) \quad \textit{and} \quad R^{\clubsuit}(R^{\clubsuit}(A)) \subseteq R^{\clubsuit}(A).$$

Assuming (D.3) of Section 2, and hence the reflexivity of the fuzzy relation R that models approximate equality, in our representational framework Proposition 2 corresponds to

$$\text{very } A \subseteq A \subseteq \text{more or less } A$$

which is called semantic entailment [17] and often assumed in the literature (see e.g. [18]). As an immediate corollary if an object y belongs to $\ker(A)$ then it will also belong to $\ker(R^{\clubsuit}(A))$. Likewise if y does not belong to $\ker(A)$ it will also not belong to $\ker(R^{\heartsuit}(A))$. One of our main goals, however, was to actually *change* the kernel, in the sense that an object y that does not yet belong to $\ker(A)$, starts belonging to $\ker(R^{\clubsuit}(A))$, and that an object y that is in $\ker(A)$, does no longer belong to $\ker(R^{\heartsuit}(A))$. The following proposition presents a straightforward sufficient condition:

Proposition 3 (De Cock and Kerre [5]). *For all R in $\mathscr{F}(X \times X)$, for all A in $\mathscr{F}(X)$, for all y in X:*

(1) *If* $(\exists x \in X)(x \in \ker(A)$ *and* $R(x, y) = 1)$ *then* $y \in \ker(R^{\clubsuit}(A))$
(2) *If* $(\exists x \in X)(x \notin \ker(A)$ *and* $R(x, y) = 1)$ *then* $y \notin \ker(R^{\heartsuit}(A))$

In both cases the essence of the condition lies in the existence of a proper object x that is related to y to degree 1. Even if y does not belong to the kernel of A, it is sufficient that we can find such an x in the kernel of A for y to start belonging to the kernel of more or less A. And for y no longer to belong to the kernel of very A all we need is such an x that does not belong to the kernel of A. In both cases $R(x, y) = 1$ but $x \neq y$. (The latter stems from $y \notin \ker(A)$ and $x \in \ker(A)$, or vice versa.) This is a very natural situation (D.4), but as is demonstrated exuberantly above and in [6], we cannot model this kind of approximate equality in a meaningful way using a fuzzy $\mathscr{T}$-equivalence relation.

Furthermore, as can be seen from Proposition 2 $\mathscr{T}$-transitivity of relation R would correspond in our representational framework to more or less (more or less A) $\subseteq$ more or less A and very $A \subseteq$ very very A. Together with the reflexivity of R that we already assumed (D.3) we would even have

$$\text{more or less (more or less } A) = \text{more or less } A \quad \text{and} \quad \text{very } A = \text{very very } A$$

which does certainly not correspond to the meaning of the linguistic hedges. In fact, when modelling approximate equality by a suitable, non-transitive fuzzy relation, we can use $R^{\clubsuit}(R^{\clubsuit}(A))$

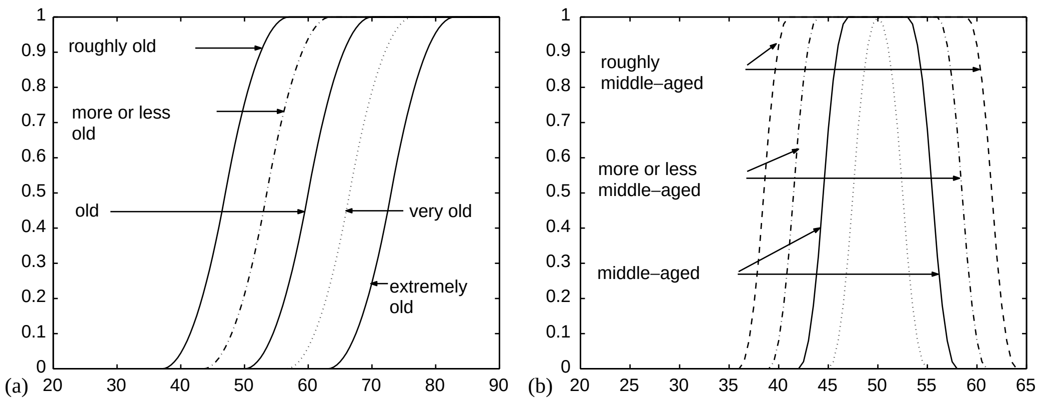

Fig. 1. (a) old and modified terms (b) middle-aged and modified terms.

and $R^{\heartsuit}(R^{\heartsuit}(A))$ as representations for roughly A and extremely A, respectively, as demonstrated in Figs. 1a and b. In both figures the Lukasiewicz t-norm W and implicator I_W was used, as well as the resemblance relation $R_{(p,q)}$ defined by

$$R_{(p,q)}(x,y) = \min(1, \max(0, p - q|x-y|, 0))$$

for all x and y in X. In Fig. 1a $p=3$ and $q=0.3$ while in Fig. 1b $p=2.5$ and $q=0.5$. Of course these parameters can be further adjusted according to the ones needs. Note that the kernels and the supports are changed and that we can apply the fuzzy modifiers based on fuzzy relations in the same way to an increasing as well as to a Π membership function. For an example in the universe of fairytale characters we refer to [4].

6. Final remarks and conclusion

Approximate equality is a relation. Furthermore, it is a vague concept. Every object is approximately equal to itself but two objects can also be approximately equal even if they are not exactly equal. Finally, the smaller the distance is between two objects, the more they are approximately equal.

Fuzzy $\mathscr{T}$-equivalence relations can model approximate equality in particular situations in which (D.4) does not occur. In these cases the t-norm (or the metric that induces the t-norm) must be chosen with special care. However, when (D.4) occurs, i.e. two objects can be approximately equal (to degree 1) although they are not exactly equal, the $\mathscr{T}$-transitivity gives rise to the paradoxal situation described in [6] Example 3, and we cannot model approximate equality by means of a fuzzy $\mathscr{T}$-equivalence relation. Combining this observation with the examples given in [6] of non-transitive fuzzy relations that model approximate equality in an acceptable way yields the answer to the title question of [16]: no, approximate equality should not satisfy transitivity.

Furthermore, the example in the Poincaré paradox is a clear case of (D.4). Therefore, fuzzy $\mathscr{T}$-equivalence relations do not resolve the paradox. Proposed patch ups are the separate treatment of objects with membership degree 1 [16] or demanding $\mathscr{T}$-transitivity only for some objects [21].

Our proposal was to drop the demand of transitivity completely and to replace it by a pseudo-metric based criterion. This criterion, however, does not *exclude* transitivity. In [1] the question was raised whether resemblance relations are strong enough to enforce meaningful concepts of indistinguishability? As Proposition 1 in [16] indicates, this highly depends on the meaningfulness of the pseudo-metric used in the definition. At least resemblance relations are not as restrictive as fuzzy $\mathcal{T}$-equivalence relations.

If we only consider the universe $\mathbb{R}$, resemblance relations on $\mathbb{R}$ are a special kind of nearness relations in the sense of Definition 1 in [13]. Criterion (3) of this definition only expresses part of (D.5), while condition (R.2) reflects (D.5) to the fullest. (R.2) is more restrictive than criterion (3), eliminating a class of fuzzy relations that are not acceptable models of approximate equality. Another option is the addition of the criterion of continuity to the definition of nearness relations, but can this criterion be formulated for arbitrary universes?

Even if it might be possible to approximate a resemblance relation by a fuzzy $\mathcal{T}$-equivalence relation [2], from the practical point of view this fuzzy $\mathcal{T}$-equivalence relation sometimes can really do no good (although stated to the contrary in [2]). Examples are the paradox with the compositional rule of inference, and the representation of linguistic hedges. The reason for this is that in the approximation process, the main advantage of resemblance relations over fuzzy $\mathcal{T}$-equivalence relations is lost. Indeed unlike resemblance relations, fuzzy $\mathcal{T}$-equivalence relations always satisfy

$$E(x,y) = 1 \quad \text{implies } E(x,z) = E(y,z)$$

Therefore, they cannot resolve the Poincaré paradox and we cannot use them to model approximate equality in any situation where this paradox may occur.

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