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Inverse Radiative Transfer Problems: A Review

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Abstract — *Recent solutions to inverse problems are reviewed for the estimation of optical properties or the thickness of a medium, or the presence of a spatially distributed source or an obscured object. Time-independent and time-dependent applications are considered, and both closed-form (explicit) and iterative (implicit) methods of solution are described.*

I. INTRODUCTION

In direct transport problems, the objective usually is to estimate the particle distribution within and emerging from a prescribed medium with specified ingoing boundary and initial conditions. Many measurement problems are inverse to the direct problem in that the objective is to determine the properties and/or the size of the medium or the boundary conditions (surface characteristics) of an obscured surface. Some imaging methods, such as computed tomography, take advantage of the fact that the medium is thin enough that only the attenuation of uncollided beams of photons need be measured; then, it is possible to estimate spatially dependent density variations of the medium.[1] As another application, an inverse solution of the integral transport equation can be used to estimate the spatially dependent temperature in an atmosphere from the Planck radiation source function provided the effects of multiple scattering can be neglected.[2-5] Such inverse attenuated beam or inverse integral equation problems are not the focus here, however, because we consider media that are a mean free path thick or more so that multiple scattering effects are important. Also, because the scattering will be taken to be quite anisotropic, the integrodifferential transport equation is needed to solve inverse and direct problems, rather than the integral equation.

Research on solving inverse problems governed by the linear integrodifferential transport equation dates back at least to the late 1960s (Refs. 6 and 7). Progress accelerated once Siewert[8] was able to show that the simplest inverse problem could be solved to estimate, using only external angular flux measurements and a closed-form expression, the scattering-to-total cross-section ratio for a semi-infinite homogeneous medium that scattered isotropically. Since much of the research on inverse transport theory up to 1985 has been reviewed already,[9-12] the purpose of this overview is to illustrate developments in 1986-1991.

Inverse problems are notorious for generally being poorly conditioned, i.e., unstable to small perturbations in the input data.[13] This more likely occurs when the number of unknowns is large, as when estimating many coefficients of a Legendre polynomial expansion of the angle-dependent scattering cross section or the spatial variation of an inhomogeneous medium, but also occurs when experimental data are insensitive to variations in a single unknown. Sometimes a total of U unknowns $\boldsymbol{u} = \{u_1, u_2, \ldots, u_U\}$ are linked in a coupled set of equations such as $\mathbf{A}\boldsymbol{u} = \boldsymbol{g}$. The solution of this equation formally requires the inversion of matrix $\mathbf{A}$ that depends on directly measured or processed data, as does $\boldsymbol{g}$. The condition number C_K, which can be expressed in terms of norms of matrices as (see, for example, Ref. 14)

$$C_K = \|\mathbf{A}\| \, \|\mathbf{A}^{-1}\| , \tag{1}$$

provides an upper bound on the relative error for matrix inversion,

$$\|\delta \boldsymbol{u}\| / \|\boldsymbol{u}\| \le C_K (\|\delta \boldsymbol{g}\| / \|\boldsymbol{g}\| + \|\delta \mathbf{A}\| / \|\mathbf{A}\|) . \tag{2}$$

The condition number is usually large when there are many unknowns, which means that errors in $\boldsymbol{u}$ are large; this has been encountered, for example, in the estimation of the coefficients of a Legendre polynomial expansion of the angle-dependent scattering cross section.[15]

Inverse radiative transfer problems have been the primary focus to date for research on inverse transport theory. This is partly due to considerations of the energy dependence of the source of particles (i.e., photons or neutrons) and of the detectors used to measure them. For example, lasers can provide convenient sources suitable for "monoenergetic" radiative transfer inverse applications if the energy resolution of the detectors is sufficiently good and scatterings do not appreciably change the wavelength of the radiation; then, the problem involves a scalar transport equation for each interval of the energy spectrum. A scalar equation also is obtained if a detector integrates over the entire energy spectrum; the problem then involves "gray" radiative transfer (or "one-group" neutron transport). Multigroup applications arise, on the other hand, when photons of different energies are tracked separately in problems where the energy change in scattering events is important. Much of the research with scalar equations can be extended to multienergy methods that use matrix equations, but then an independent source for each distinct energy group is needed, for example.[16]

In the field of nuclear engineering, an application of inverse transport methods arises in nuclear geophysics.[17] For nuclear oil well logging, for example, the energy dependence of the return signal from a neutron or gamma source can be compared with data from known configurations to estimate, respectively, the hydrogen density (associated with pore fluids) or the bulk density in the surrounding media.[18] Another possibility is to look for the gamma rays emitted following neutron capture, in which case the detected particles and the source particles are different. Such "active" sensing techniques can be contrasted with "passive" ones in which one might estimate the elemental composition of materials undergoing natural radioactive decay from a gamma-ray spectrum.

Another important consideration in developing a method to solve an inverse problem is that it must be consistent with the directional dependence of the detector(s). For example, for an energy-independent problem, if the angular field of view of a detector is narrow, then the directionally dependent radiance (i.e., energy-independent angular flux) $I(\boldsymbol{r},\boldsymbol{\Omega}_j,t)$ can be measured in different directions $\boldsymbol{\Omega}_j$, or otherwise detectors with different wide-angle response characteristics can be used. All such measurements can be grouped together as

$$I_j(\boldsymbol{r},t) = \int_{4\pi} w_j(\boldsymbol{\Omega}) I(\boldsymbol{r},\boldsymbol{\Omega},t)\, d\boldsymbol{\Omega} \ , \quad j = 1,2,\ldots,M \ , \tag{3}$$

where the angular weight functions $w_j(\boldsymbol{\Omega})$ possibly can be Dirac delta functions. For example, in plane-parallel problems for time-independent radiative transfer applications, with z the optical distance variable, μ the cosine of the polar angle, and ϕ the azimuthal angle, one can measure the moments

$$I_{1,\pm}(z) = \int_0^{2\pi} d\phi \int_0^1 I(z,\pm\mu,\phi)\mu\, d\mu$$

or

$$I_{0,\pm}(z) = \int_0^{2\pi} d\phi \int_0^1 I(z,\pm\mu,\phi)\, d\mu \ , \tag{4}$$

with a flat or 2π-shielded angle-integrating detector, respectively. These are sufficient for an inverse algorithm based on diffusion theory, where only the angle-integrated flux $I_0(z) = I_{0,+}(z) + I_{0,-}(z)$ and net current $I_1(z) = I_{1,+}(z) - I_{1,-}(z)$ are needed.

Another key consideration in developing methods for solving inverse radiative transfer problems is the measurement locations of the detector(s). Are measurements made exterior to the medium (i.e., remotely) or at various locations within the medium? If the measurements are in the medium interior, are they far enough from external boundaries, media discontinuities, and localized sources that diffusion theory can be used instead of transport theory when solving the inverse problem?

So far, we have considered sensing applications in which an overall energy balance is not necessary. There are other situations in which thermodynamic equilibrium constraints may be necessary. One example is thermal radiative transfer in which all absorbed energy is re-emitted with a wavelength dependence governed by the Planck blackbody radiation function for the local temperature; other cases arise in the mixed-mode heat transfer applications discussed by Özişik.[19]

Table I is one possible classification scheme for inverse radiative transfer problems, and Table II gives some constraints that influence the method of solving such problems.

In this review, mostly monoenergetic, plane-parallel radiative transfer problems are considered for which the time-dependent, azimuthally symmetric radiative transfer equation for the radiance $I(z,\mu,t)$ can be written as[20]

$$\begin{aligned}(c^{-1}\partial_t + \mu\partial_z + 1)I(z,\mu,t) &= \frac{\varpi}{2}\sum_{l=0}^{L}(2l+1)f_l P_l(\mu)\int_{-1}^{1} P_l(\mu')I(z,\mu',t)\,d\mu' \\ &\quad + Q(z,t)/2 \ , \quad 0 \le z \le z_0 \ ,\end{aligned} \tag{5}$$

where z is the optical distance, $\partial_z = \partial/\partial z$, and c^{-1} is the mean time between collisions. Also, ϖ is the albedo of single scattering, and the phase function has been expanded in a series of Legendre polynomials with coefficients f_l, $l = 0$ to L, where $f_0 = 1$. The possibility of an isotropic source $Q(z,t)$ is also considered.

At $z = 0$, the boundary condition for the radiation field is given by the incident radiance $I(0,\mu,t)$, $0 \le \mu \le 1$, while the boundary condition at the other surface

TABLE I

Different Types of Radiative Transfer Inverse Problems

A. Sensing (temperature-independent) applications
   1. Optical property estimation: for the mean number of secondaries and the expansion coefficients of the angular scattering function
   2. Optical thickness estimation: for the thickness of the medium above a surface of known or unknown albedo (e.g., assuming a Lambertian reflection law)
   3. Internal source estimation: for the magnitude and location of sources within the medium
   4. Boundary condition estimation: for the angular distribution of incident or reflected radiation at boundaries of the medium that causes a measured distribution at another spatial location

B. Energy balance (temperature-dependent) applications
   1. Heat flux estimation: for only radiative transfer, or radiative transfer plus conduction or convection or both
   2. Temperature estimation: for only radiative transfer, or radiative transfer plus conduction or convection or both

TABLE II

Considerations When Developing a Solution to an Inverse Radiative Transfer Problem

A. Radiation detectors
   1. Field of view
      a. Narrow (collimated)
      b. Broad (angle integrated)
   2. Location
      a. Outside the medium (remote)
      b. Inside the medium (*in situ*)

B. Source of photons
   1. Type of illumination
      a. Passive (inherent radiation field)
      b. Active (induced radiation field)
   2. Time dependence
      a. Steady state (e.g., a continuous-working laser)
      b. Pulsed (e.g., a pulsed laser)
   3. Location
      a. On the medium boundary (external)
      b. Inside the medium (*in situ*)

located at distance $z = z_0$ often is taken to be that for a vacuum or a purely absorbing underlying medium,

$$I(z_0,-\mu,t) = 0 \ , \quad 0 \le \mu \le 1 \ , \tag{6a}$$

or sometimes that for isotropic (Lambertian) reflection,

$$I(z_0,-\mu,t) = 2\rho \int_0^1 \mu' I(z_0,\mu',t)\,d\mu' \ , \quad 0 \le \mu \le 1 \ , \tag{6b}$$

where ρ, $0 \le \rho \le 1$, is the albedo of the surface at z_0.

Although solutions of plane-parallel problems generally are only idealized applications for neutron and gamma-ray applications, inverse radiative transfer solutions have practical utility for geophysical applications that involve atmospheres or oceans. A second reason for initially developing and numerically testing inverse methods for such a geometry is that if a method does not work for the simplest of configurations, why bother trying to make it work for more general ones? For these reasons, most of the past work on inverse transport has been been for plane-parallel geometries.

A general discussion of the methods for solving inverse radiative transfer problems is given in Sec. II, followed in Sec. III by a summary of recent research on the estimation of optical properties in a plane-parallel medium using a variety of detectors. Then, in Sec. IV, solutions to other sensing problems for a plane-parallel medium are addressed. Section V contains an introduction to solutions in nonplane-parallel media that do not involve energy balance considerations, while Sec. VI is directed at inverse problems that do involve the medium temperature. Then, Sec. VII gives a few concluding comments.

II. GENERAL METHODS OF SOLUTION

There are two dramatically different approaches to solving an inverse problem governed by the radiative transfer equation: explicit and implicit methods. Explicit methods are based on manipulations of the transport equation itself or of analytic solutions to direct problems. The goal of an inversion algorithm for explicitly estimating unknowns is to use only measured data as in Eq. (3) or (4) and known parameters for the problem, which makes the algorithm easy to implement if no radiative transfer calculations are required.

Explicit algorithms for solving the inverse problem are difficult to develop because one has an integral (scattering) operator plus a differential operator for each independent spatial variable. If the medium is optically thin enough that multiple scatterings can be neglected, then Larsen's general geometry solution can be used.[21] Otherwise, explicit algorithms traditionally have been developed for a plane-geometry medium for the case of plane-parallel illumination so as to deal with only one spatial variable. In other cases such as a point illumination incident on a slab medium, a Fourier transformation of the data over the two transverse spatial variables can be used to reduce the problem to a quasi-one-dimensional one.[22]

An estimate of the unknowns obtained from an explicit algorithm can be used either as the only solution to a problem or as a starting estimate for an implicit method.

Implicit methods for solving an inverse radiative transfer problem, on the other hand, combine measurements with repeated solutions of the direct problem obtained with different values of the unknowns. Iteration proceeds until a functional that represents the goodness of fit is reduced to an acceptably small value. An iterative approach works best when estimates of the unknowns are changed in directions that maximize the rate of approach to the minimum value of the functional.

To illustrate an implicit method for the case of M data points, we can define a least-squares functional F depending on the I_j of Eq. (3) that is a function of vector $\boldsymbol{u}$ of unknowns $\{u_k\}$, $k = 1$ to U,

$$F(\boldsymbol{u},\boldsymbol{p},\boldsymbol{I}^m) = \frac{1}{2}\sum_{j=1}^{M} w_j^*[I_j^c(\boldsymbol{u},\boldsymbol{p}) - I_j^m]^2$$

$$= \frac{1}{2}\boldsymbol{f}^T\mathbf{W}\boldsymbol{f} \ , \tag{7}$$

where $\boldsymbol{f}(\boldsymbol{u},\boldsymbol{p},\boldsymbol{I}^m) = \boldsymbol{I}^c(\boldsymbol{u},\boldsymbol{p}) - \boldsymbol{I}^m$ is the vector of differences of the calculated and measured values of the functions I_j of the radiance and $\boldsymbol{f}^T$ is its transpose, and vector $\boldsymbol{p}$ contains parameters $\{p_i\}$, $i = 1$ to P, that are assumed known. The diagonal weight matrix $\mathbf{W} = \mathrm{diag}\{w_j^*\}$ contains the weights w_j^* that express the confidence we have in the precision of the measurements and should be selected to be consistent with the accuracy of the weight functions $w_j(\boldsymbol{\Omega})$ of Eq. (3). For example, if the data consist of the set of radiances $I(0,-\mu_j)$ and $I(z_0,\mu_j)$ emerging from a slab with given boundary conditions $I(0,\mu)$ and $I(z_0,-\mu)$, $\mu \geq 0$, and if the emerging radiances can be measured just as precisely at both surfaces, then the simplest form of Eq. (7) is

$$F = \frac{1}{2}\sum_{j=1}^{M}\{[I^c(0,-\mu_j) - I^m(0,-\mu_j)]^2 + [I^c(z_0,\mu_j) - I^m(z_0,\mu_j)]^2\} \ , \quad \mu_j \geq 0 \ , \tag{8}$$

and the unknowns might be the optical thickness z_0 and ϖ, while the known parameters could be the f_l coefficients.

An iterative solution of the inverse problem is obtained by minimizing the value of the functional. The iterations can be assumed to have converged if all the measured values are obtained within a given relative error ϵ_1^*,

$$\epsilon_1 = \max_j \left| \frac{I_j^c - I_j^m}{I_j^m} \right|^2 < \epsilon_1^* \ ,$$

or if for two consecutive iterations,

$$\epsilon_2 = \left| \frac{F^{(n)} - F^{(n-1)}}{F^{(n-1)}} \right| < \epsilon_2^* \ ,$$

which means that the functional F has become stationary.

One way to minimize the value of the functional F with respect to the unknown parameters in $\boldsymbol{u}$ is to use the system of nonlinear equations

$$\nabla_u F = 0 \ . \tag{9}$$

This system is solved by an iteration procedure. At the start of the n'th iteration, we solve the radiative transfer equation using the presumed values $\boldsymbol{u}^{(n)}$ and then compute the corresponding value for the functional $F^{(n)}$. If the convergence criteria are not satisfied, then the new value $\boldsymbol{u}^{(n+1)}$ can be estimated by a second-order local approximation to the functional at $\boldsymbol{u} = \boldsymbol{u}^{(n)}$,

$$F(\boldsymbol{u} + \Delta\boldsymbol{u}) \approx F(\boldsymbol{u}) + \Delta\boldsymbol{u}\cdot\nabla_u F + \tfrac{1}{2}(\Delta\boldsymbol{u}\cdot\nabla_u)^2 F \ .$$

The minimization yields[23,24] an algebraic system of equations for $\Delta\boldsymbol{u}$,

$$\mathbf{H}\Delta\boldsymbol{u} = -\boldsymbol{h} \ , \tag{10}$$

where

$$\boldsymbol{h} = \nabla_u F = \mathbf{GW}\boldsymbol{f} \ , \tag{11a}$$

while $\mathbf{H}$ is the Hessian matrix whose elements are $\partial^2 F/\partial u_i \partial u_j$,

$$\mathbf{H} = \nabla_u\nabla_u F \approx \mathbf{GWG}^T \ . \tag{11b}$$

Here, $\mathbf{G}$ is the $U \times M$ matrix

$$\mathbf{G} = \nabla_u \boldsymbol{f} = \nabla_u(\boldsymbol{I}^c - \boldsymbol{I}^m) = \nabla_u \boldsymbol{I}^c \ ,$$

and $\mathbf{G}^T$ is its transpose. Also, the second-order derivative terms have been neglected in Eq. (11b) in the calculation of $\mathbf{H}$.

In the usual iteration procedure, one numerically evaluates the derivatives $\nabla_u \boldsymbol{I}^c$ from two consecutive iterations and uses the results with the latest $\boldsymbol{I}^c$ values to estimate $\boldsymbol{u}$, but such evaluations often are very sensitive to inaccuracies in the calculations of the derivatives. A better approach, when possible, is to develop an analytical procedure for computing the derivatives, and this has been done for some inverse radiative transfer problems.[25]

Another way to estimate the unknowns in $\boldsymbol{u}$ is to use the conjugate gradient method[24,26,27] or a combination of a steepest descent method for the first few iterations followed by a conjugate gradient method for subsequent iterations. (The conjugate gradient method is better as the objective function becomes smaller and closer to the optimal point.)

The searching direction is very important in an optimization algorithm used to minimize a functional. With the conjugate gradient method, the searching direction $\boldsymbol{d}_k$ is determined from

$$\boldsymbol{d}_k = -\boldsymbol{g}_k + \beta_k \boldsymbol{d}_{k-1} \ . \tag{12a}$$

Here, $g_k = \nabla_u F(u_k)$ and

$$\beta_k = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}} \ , \qquad (12b)$$

and the initial search direction is selected as $d_0 = -g_0$. The magnitude of the displacement along the searching direction is α_k, which is solved by the linear minimization rule,

$$F(u_k + \alpha_k d_k) = \min_{\alpha} F(u_k + \alpha d_k) \ . \qquad (13)$$

The new solution vector then is found by

$$u_{k+1} = u_k + \alpha_k d_k \ , \qquad (14)$$

and this value is used to obtain $F(u_{k+1})$ so that the procedure can be repeated until an optimal point of convergence is reached.

Because of the poorly conditioned nature of inverse problems, it is important to investigate the effects of errors induced in the unknowns caused by errors in the measurements I_j^m and the "given" parameters p_i. From a linear perturbation analysis, the relative error in the unknown u_k can be expressed in terms of the relative errors $|\Delta x_n / x_n|$ for $x_n = I_j^m, p_1, p_2$, etc., as

$$\left| \frac{\Delta u_k}{u_k} \right| \leq \sum_n s_{x_n}^{(k)} \left| \frac{\Delta x_n}{x_n} \right| \ . \qquad (15)$$

The sensitivity coefficients $s_{x_n}^{(k)}$ have the form

$$s_{x_n}^{(k)} = \left| \frac{\partial u_k}{\partial x_n} \frac{x_n}{u_k} \right| \qquad (16)$$

and represent the ratio of the relative error in the unknown u_k to the relative error in x_n. Sensitivity coefficients can be used to estimate the confidence in a solution to an inverse problem: the smaller the coefficients, the better the inversion procedure. An advantage of an explicit inverse radiative transfer algorithm is that sometimes these sensitivity coefficients can be derived analytically without the need for radiative transfer computations.

One way to test the efficacy of a method for solving an inverse problem is with experimental data from a known specimen, in which case the specimen properties must be very well characterized. A second way is to assume all "unknown" parameters are available and perform direct problem calculations to obtain "exact" values of the simulated radiance quantities I_j^m. Then, to incorporate the effects of imprecision in the measurements, one can sample from an assumed probability distribution to obtain perturbations to impose on the direct problem results. Once obtained, the numerically generated "experimental" data can be used in the inverse solution procedure to obtain the sensitivity coefficients.

III. ESTIMATION OF OPTICAL PROPERTIES

The solutions of inverse problems in a plane-parallel medium that are reviewed in this section are categorized according to whether (a) the problem is time independent or time dependent, (b) the medium is known to be spatially uniform or not, and (c) the detectors are exterior or interior to the medium.

III.A. Time-Independent Problems for Spatially Uniform Media and Exterior Detectors

Prior knowledge that the medium is spatially uniform simplifies a problem since the optical properties for only one medium need to be determined. Explicit solutions for this class of problems received considerable attention prior to 1986. More recently, Kamiuto and Seki[28] developed an explicit algorithm for estimating ϖ and f_1 for an isotropic illumination of a slab using the transmitted angle-integrated radiance $I_{0,+}(z_0)$ of Eq. (4). The method was based on P_1 theory that used one measurement for a single thickness. If $z_0 \gg 1$, then

$$\ln I_{0,+}(z_0) = K_1 - K_2 z_0 \ ,$$

and the unknowns can be estimated from

$$\varpi = 1 - K_3^2 \quad \text{and} \quad f_1 = [3 - (K_2/K_3)^2]/3\varpi \ , \qquad (17)$$

where

$$K_3 = [2K_2 \exp(-K_1)]^{1/2} \{1 - [1 - \exp(K_1)]^{1/2}\}/2 \ .$$

The authors also used an implicit method that assumed that measurements could be made on specimens of different optical thicknesses.

An explicit inversion approach was investigated by Agarwal and Mengüç[29] for recovering ϖ and f_1 for very thin targets in which multiple scattering beyond second order can be neglected. They found that by limiting their analysis to second-order scattering they could get good agreement between the solution of the direct problem and experiments on mono- and polydispersions of particles provided $z_0 < 1$. When solving the inverse problem, they were able to recover the unknowns to within 10% if $z_0 < 0.5$.

Dressler[30] tried to estimate ϖ and all the f_l coefficients by an explicit method that requires measurement of the radiances $I(0,-\mu)$ and $I(z_0,-\mu)$. He concluded that his inverse solution is inherently poorly conditioned. No numerical tests have yet been done to see if the procedure is better than one proposed earlier[15] that also is poorly conditioned.

A method of estimating optical properties for use in two-stream (forward/backward direction) models of deep ocean water has been developed by Gordon.[31] The explicit method relies on correlations developed by Monte Carlo calculations for the variation of the ratio $I_{1,-}(0)/I_{1,+}(0)$ with different values of the incident radiation $I(0,\mu)$, $0 \leq \mu \leq 1$.

Yi[32] has devised a way to estimate the f_l coefficients from emerging radiances $I(0,-\mu)$ and $I(z_0,\mu)$ using an explicit algorithm based on invariant imbedding equations. Preliminary numerical experiments with the algorithm indicate that he has been able to circumvent some of the poor conditioning encountered with an algorithm that requires azimuthal measurements be made.[15]

An implicit method, on the other hand, has been used to estimate ϖ and the scattering asymmetry factor g for a medium having a Henyey-Greenstein scattering function defined by the expansion coefficients $f_n = g^n$. A normally incident radiance at $z = 0$ and the boundary condition of Eq. (6a) were used by Kamiuto,[33,34] who first estimated the optical thickness by an approximate procedure in which the exponentially attenuated uncollided beam was separated out from the normally directed, diffusely transmitted beam. This estimated optical thickness then was used to compute the $I(0,-\mu_j)$ and $I(z_0,\mu_j)$ in the F of Eq. (8), and then ϖ and g were estimated using the equations

$$\partial F/\partial\varpi = 0 \quad \text{and} \quad \partial F/\partial g = 0 \ .$$

The method was applied to experimental data taken on packed beds of large spheres of stainless steel, alumina, and glass, and it was found that the albedo could be estimated more accurately than the scattering asymmetry factor[35]; similar experiments on nickel-chromium porous plates also have been done.[36] In addition, the macroscopic cross section Σ (i.e., the optical attenuation coefficient) has been estimated for packed-sphere systems using estimated values of the optical thickness z_0 and the equation $\Sigma = z_0/\tilde{z}_0$, where $\tilde{z}_0$ is the measured thickness (e.g., in metres) of the system.[37]

III.B. Time-Independent Problems for Spatially Nonuniform Media and Exterior Detectors

Since this class of problems is inherently more difficult to deal with than those with prior information that a medium is homogeneous, it is not surprising that implicit methods are needed. To test such methods, a variety of idealized models of the spatially varying properties has been utilized. The spatially varying single scattering albedo

$$\varpi(z) = \varpi_0 \exp(-z/s) \ , \quad s \text{ known} \ ,$$

has been considered, for example, where ϖ_0 was estimated.[38] For a somewhat different albedo,

$$\varpi(z) = \sum_{j=0}^{J} \varpi_j z^j \ ,$$

Ho and Özişik[39] used an implicit method to estimate the optical thickness z_0 and the ϖ_j, $j = 0$ to J, from the emerging radiances $I(0,-\mu)$ and $I(z_0,\mu)$; for a problem with $J = 2$, the authors claim that only two-digit accuracy in the measurements is needed to obtain acceptable accuracy for most engineering applications.

Subramaniam and Mengüç[40] solved several inverse problems for an exponentially varying or spatially uniform single scattering albedo. They used an implicit method in which the direct problem was solved by a Monte Carlo method. To minimize the number of unknowns in the inverse solution when the scattering was anisotropic, they approximated the phase function with an isotropic portion plus a constant "step" value for the near-forward direction. The use of a Monte Carlo method to solve the direct problem has the advantage of being easily modified to nonplane-parallel geometries, but the method does introduce statistical errors from the finite number of particle histories used in the direct problem simulation.

A formal method to estimate $\varpi(z)$, based on an existence and uniqueness proof for the solution of the integral equation, also has been reported for an isotropic scattering problem.[41]

The absorption and scattering coefficients for each layer of a two-layer medium have been estimated from an implicit method with calculations based on the two-stream model for radiation transport.[42] The scheme utilizes the diffusely backscattered and transmitted moments $I_{1,-}(0)$ and $I_{1,+}(z_0)$ of Eq. (4) plus the collimated (uncollided) radiance, measured when illuminating the top layer, and the corresponding three measured values when illuminating the bottom layer. The method has the inherent limitation that the fractional thickness of each layer must be known, however.

III.C. Time-Independent Problems and Interior Detectors

The use of detectors within a medium permits an easier assessment of spatially varying properties than with only external detectors. The disadvantage, however, is that it may be more difficult to obtain a sufficient number of linearly independent measurements I_j^m to evaluate very many unknowns. The radiation deep in the interior of a homogeneous, source-free region is in the asymptotic regime, for example, and exhibits little dependence on any azimuthal dependence introduced by boundary conditions; hence, in plane-parallel geometries, use of 2π field-of-view detectors to measure only the moments in Eq. (4) may be most appropriate.[43] By introducing a spatially varying angle-dependent source (e.g., a submerged searchlight), however, one could take more advantage of detectors with a small field of view, but the inverse problem then would be multidimensional and more difficult to solve.

The probability per unit distance of travel that a photon will be absorbed [i.e., $(1-\omega)$ times the macroscopic cross section Σ] is just the absorption cross section. That cross section can be estimated for a source-free plane-parallel medium by integrating transport equation (5) over all directions to obtain the conservation equation

$$d_z I_1(z) + (1-\varpi) I_0(z) = 0 \ ,$$

which involves the differentiation operator $d_z = d/dz$ that can be treated as a finite difference operator. Thus, a simple explicit algorithm for $(1-\omega)\Sigma$ is

$$(1-\varpi)\Sigma = -\frac{2}{\Delta\tilde{z}}\left[\frac{I_1(\tilde{z}_{n+1}) - I_1(\tilde{z}_n)}{I_0(\tilde{z}_{n+1}) + I_0(\tilde{z}_n)}\right] , \quad (18)$$

where $\tilde{z} = z/\Sigma$ is the spatial position (e.g., in metres). This conservation relation, known to oceanographers as the Gershun[44] equation, has been used (but not with complete success) to estimate the absorption coefficient in seawater, for example.

For an application with interior detectors located away from the surfaces, there is an explicit algorithm for estimating the similarity parameter s, defined by

$$s^2 = (1-\varpi)/(1-\varpi f_1) .$$

(The value of s^2 is just the ratio of the absorption-to-transport cross sections.) The algorithm[45]

$$s^2 = [\nu_0^2(1-\varpi)(1-\varpi f_1)]^{-1}\frac{d_z[I_1(z)]^2}{d_z[I_0(z)]^2} \quad (19)$$

contains the factor $[\nu_0^2(1-\varpi)(1-\varpi f_1)]^{-1}$, which depends on ν_0, the asymptotic eigenvalue of the transport equation[46] that can be expressed in terms of s. Numerical tests with simulated measurements for seawater showed that, as expected, the algorithm does not work well near the surface, especially when the incident radiation is monodirectional.[45]

III.D. Time-Dependent Problems

An initial impression is that it should be easier to estimate optical properties for such problems because there is an additional variable from which to extract information. If a pulsed laser is used to initiate the time dependence, however, to obtain any spatial resolution, one needs a complicated detection system to extract the resulting pulse shape: The time constants for laboratory-scale experiments are on the order of picoseconds.

A time-dependent, explicit inverse radiative transfer algorithm for estimating ϖ and the f_n coefficients of the phase function has been numerically tested.[47] The algorithm is based on the use of Fourier azimuthal moments $I_m(0,-\mu,t)$ of the radiance backscattered from a plane-parallel semi-infinite medium illuminated by a pulsed laser. These moments are calculated from the backscattered radiance $I(0,-\mu,\phi,t)$ using

$$I_m(0,-\mu,t) = [\pi(1+\delta_{m0})]^{-1} \times \int_0^{2\pi} I(0,-\mu,\phi,t)\cos m\phi \, d\phi ,$$

where δ_{mn} is unity if $m = n$ and zero otherwise. Long after the incident pulse centered about time $t = 0$, the backscattered moments asymptotically decay according to[47]

$$I_m(0,-\mu,t) \cong C_m(\mu)t^{-3/2}\exp[-(1-\varpi f_m)ct] , \quad (20)$$

where $C_m(\mu)$ need not be evaluated as long as measurements are made for a fixed μ. The algorithm has been tested with experimental measurements for light with a pulse width of order 50 ps incident on a thick medium of polystyrene spheres, but only ϖ and f_1 could be successfully reproduced because of the difficulty of getting accurate higher order Fourier moments of the radiance.[48,49] Extensions of the algorithm for the measurement of azimuthally symmetric components of polarized light have been proposed as a possible way to extract information about the size distribution of particles[50]; such a technique would provide unique answers, however, only if the particle diameter is considerably smaller than the wavelength of the laser radiation.

A formally exact, explicit expression for evaluating ϖ and the f_l coefficients as angular moments of the Laplace transform of the backscattered return from a semi-infinite medium also has been derived,[30] but the need to make measurements at early times following the onset of a pulse will make the method impractical to implement. Another reason is that the finite width of any incident pulse means that a convolution of the incident and emerging radiance is needed, but incident laser pulse shapes are difficult to control and measure.

In an interesting application of the invariant imbedding method of radiative transfer, Ayoubi and Nelson[51] have developed an explicit technique to estimate the concentration profile of a spatially nonuniform medium from the time-dependent backscattered radiance $I(0,-1,t)$, $0 \le t \le T$. The method, which was based on the two-stream approximation of transport theory, was used to estimate the concentration of particles for $0 \le z \le cT/2$, where c^{-1} is the mean time between collisions. The algorithm relies on a "layer-peeling" solution, a technique in which the incremental change in the backscattered radiance from one time step to the next gives information about the density in the next corresponding spatial increment. The method was stable when subjected to simulated data that had discontinuities or prominent spatial variations in the scattering coefficient. For this reason, the algorithm could be important for the light detection and ranging (i.e., lidar sensing) of both the location and concentration of clouds or pollutants in the atmosphere because the inversion method traditionally used for this purpose does not account for multiple scattering.

A second time-dependent method for estimating the spatially dependent density in a plane-parallel medium also has been developed.[52] It too is an explicit method based on principles of invariant imbedding, but it is not restricted to a two-stream approximation nor to radiation of a single wavelength. The method relies on unfolding the density by using different values of the Laplace transform of the emerging radiance. For a small Laplace transform variable λ, the transformed radiance depends more on multiple scatterings from deeper depths, while for large values of λ, only the near-surface scatterings contribute. As yet, however, the approach has been tested only for a simple case and has not been shown to be stable to simulated measurement errors, for example.

IV. ESTIMATION OF OPTICAL THICKNESS OR SPATIALLY DISTRIBUTED SOURCE

In this section, we consider the estimation of an optical thickness z_0 or a spatially distributed source $Q(z)$ in a plane-parallel medium. Knowledge of the optical thickness, for example, can be used in two ways. If all the optical properties and the albedo ρ of an underlying surface are known, then one can estimate the spatial location $\tilde{z}$ of the obscured surface or the energy transmitted through the medium. On the other hand, if some of the optical properties are unknown but the spatial thickness $\tilde{z}_0$ is available (e.g., in metres), the ratio $z_0/\tilde{z}_0$ can be used to estimate Σ, the probability of a photon interaction per unit distance of travel (i.e., the total macroscopic cross section).

IV.A. Optical Thickness Problems

The optical thickness z_0 of a medium has been estimated by an approximate method that assumes it is possible to separate out the transmission of the attenuated uncollided beam from the diffusely transmitted radiation[33-37]; such a method becomes more prone to errors as the optical thickness increases. On the other hand, King[53] developed an explicit method to estimate a large optical thickness z_0 from the backscattered radiance $I(0,-\mu,\phi)$ due to a monodirectional illumination of a medium; the method was based on the angle-dependent backscattered radiance derived from asymptotic (thick-medium) transport theory with the assumption that the boundary condition of Eq. (6b) was valid.

In a paper related to King's, two sets of explicit algorithms were developed[54] to estimate z_0 from measurements of the irradiance $I_{1,-}(0)$ or $I_{1,+}(z_0)$, assuming that $I_{1,+}(0)$ is known. The closely related sets of equations differed somewhat according to whether an asymptotic radiative transfer or transport-corrected diffusion (tcd) approximation was implemented. An estimate of z_0 using measurements of the reflectance ratio $r(z_0) = I_{1,-}(0)/I_{1,+}(0)$ follows from either approximation as[54]

$$z_0 = \frac{\nu_0}{2} \ln\left\{ G\left[\gamma + \frac{\alpha}{r(\infty) - r(z_0)}\right]\right\} . \qquad (21)$$

Here, ν_0 again is the asymptotic eigenvalue of the transport equation,[46] where $r(\infty)$ is the limiting value as $z_0 \to \infty$. For the tcd approximation, for example,

$$\alpha = 1 - r^2(\infty) ,$$

$$G = [r(\infty) - \rho]/[1 - \rho r(\infty)] ,$$

where ρ is the albedo of the surface at z_0. The corresponding equation for either approximation for estimating z_0 from measurements of the transmittance ratio $t(z_0) = I_{1,+}(z_0)/I_{1,+}(0)$ has the form

$$z_0 = \nu_0 \ln\left[\frac{\beta}{2t(z_0)}\left(1 + \left\{1 + \gamma G\left[\frac{2t(z_0)}{\beta}\right]^2\right\}^{1/2}\right)\right] , \qquad (22)$$

where for the tcd approximation,

$$\gamma = r(\infty) ,$$

$$\beta = [1 - r^2(\infty)]/[1 - \rho r(\infty)] .$$

Numerical tests and sensitivity tests have been performed[54] on algorithms (21) and (22) and confirmed that z_0 cannot be accurately estimated from the former equation, for example, when the medium becomes so thick that $r(\infty) \approx r(0)$ or when $r(\infty) \approx \rho$, which corresponds to $G \approx 0$.

Explicit methods to estimate the optical thickness have been put to practical use for clouds. King's backscattered radiance algorithm has been used[55-57] with measurements at two different wavelengths to estimate the optical thickness and mean radius of water droplets.

Implicit algorithms also have been used to estimate optical thicknesses. Ho and Özişik[58] solved a problem in which the angular scattering function was known, and they estimated z_0 and ϖ by minimizing either of two functionals similar to that of Eq. (8),

$$F_1 = \frac{1}{2}\sum_{j=1}^{M} [I^c(0,-\mu_j) - I^m(0,-\mu_j)]^2 ,$$

$$F_2 = \frac{1}{2}\sum_{j=1}^{M} [I^c(z_0,\mu_j) - I^m(z_0,\mu_j)]^2 , \quad \mu_j \geq 0 . \qquad (23)$$

In their numerical tests, simulated measurements at 19 angles were used, as obtained by dividing 0 deg $\leq \cos^{-1}\mu \leq$ 90 deg into 18 equal intervals. When they tested the method for isotropic scattering and $z_0 \leq 2$, with radiances having 5% normally distributed random errors at 99% confidence, they found that using either reflected or transmitted radiances worked for $\varpi \geq 0.5$, but for smaller values of ϖ, better estimates of z_0 came from transmitted exit radiances, and better estimates of ϖ came from the reflected radiances. It would be interesting to see how sensitive these results are to the number of measurement directions and to test this implicit scheme at larger values of z_0.

Sanchez, McCormick, and Yi[25] used an implicit approach to solve a different problem in which the optical thickness z_0 or the surface albedo ρ of Eq. (6b) or both were unknown. Various combinations of detectors of three types were considered: $I(0,-1)$, $I_{0,-}(0)$, or $I_{1,-}(0)$ and $I(z_0,1)$, $I_{0,+}(z_0)$, or $I_{1,+}(z_0)$. If either z_0 or ρ was known and only one detector was required, the method rapidly converged to the correct value in a few iterations. For simulated measurements at the illuminated surface, the region in the (z_0, ρ) phase-space that gave good estimates for z_0 increased with the albedo of single scattering ϖ, although there was no significant improvement with ϖ for estimates of ρ. When simultaneously estimating z_0 and ρ with two detectors, there was a large increase in the sensitivity coefficients compared with those when estimating only one unknown.

IV.B. Spatially Distributed Source Problems

Applications can arise in which there is an inhomogeneous isotropic source such as the term $Q(z)/2$ on the right-hand side of Eq. (5). Such a source can occur because of bioluminescence in seawater or as re-emission at the wavelength of interest of radiation absorbed at a different wavelength.

Two explicit methods have been proposed and tested to estimate the mean value of $Q(z)$ between two measurement positions in the interior of a medium. One consists of an extension[59] of the ideas that led to Eq. (19) and the other to an algorithm based on particle conservation,[60]

$$Q(z) = d_z I_1(z) + (1 - \varpi) I_0(z) \ . \qquad (24)$$

If ϖ was known, this equation worked well except near the surface for cases with strong external illumination. For situations in which ϖ was unknown, the source magnitude sometimes could be accurately estimated using an approximation for $(1 - \varpi)$; this was done by first scanning the ratio $-d_z I_1(z)/I_0(z)$ versus z in order to estimate the minimum value of $(1 - \varpi)(z)$ corresponding to source-free seawater from

$$(1 - \varpi)_{min}(z) = \max[0, -d_z I_1(z)/I_0(z)] \ .$$

This gave a value of $(1 - \varpi)$ that equaled the actual value in regions where there is no source and was less than the actual value in the presence of a source. Once ϖ had been estimated, $Q(z)$ could be estimated using Eq. (24).

V. NONPLANE-PARALLEL SENSING PROBLEMS

V.A. Time-Independent Problems

Explicit inverse radiative transfer solutions are difficult to obtain for nonplane configurations unless the medium is optically thin enough that Larsen's approach can be used.[16] Occasionally, one is able to obtain approximate answers, however, for optically thick media. One idealized example is the estimation of the macroscopic cross section Σ of a medium from external detector measurements of radiation arising from an isotropic point source that is located at depth z_s in a homogeneous half-space medium that scatters isotropically. This Milne problem for a point source requires measurements as a function of distance $\tilde{r}$ (e.g., in metres) in the xy plane away from a point on the surface directly above the source. A rarely referenced solution by Elliott[61] for $r = \Sigma\tilde{r}$ shows that for a weakly absorbing medium,

$$I_{0,-}(z=0,r) = \int_0^{2\pi} d\phi \int_0^1 d\mu \, I(z=0,r,-\mu,\phi) \ ,$$
$$\propto r^{-3}(1 + r/\nu_0)\exp(-r/\nu_0) + O(z_s^3/r^5) \ . \qquad (25)$$

Thus, by measuring $I_{0,-}(z=0,\tilde{r})$ versus $\tilde{r}$, one can estimate Σ if ϖ is known so that ν_0 can be calculated. To overcome the highly restrictive assumption of isotropic scattering, one could develop an iterative method by proceeding along the lines of Gordon[62] who investigated $I_{0,-}(z=0,r)$ for seawater (which scatters in a highly anisotropic manner) with a Monte Carlo solution of the direct problem.

Optical properties of a multilayer medium have been estimated using an explicit inversion method based on an approximate diffusion theory solution of the direct problem.[63] The scheme was developed for measuring the angular moment $I_{0,-}(z=0,\tilde{r})$ versus $\tilde{r}$ that results from an incident illumination $I_{0,+}(z = 0,0)$. Numerically simulated measurements for two layers of human skin were used to test the method, and it was concluded that it would be difficult to distinguish the profile of $I_{0,-}(z=0,\tilde{r})$ from that for homogeneous tissue unless the optical properties of the layers differ markedly.

The optical imaging of tissue also has been investigated by Aronson et al.[64] using a Monte Carlo solution of the adjoint transport equation to develop a detector response function. A back-projection method was used to infer a three-dimensional image of the absorption profile within the medium.

Implicit methods ultimately should become the preferred approach for problems with general geometries. Wang and Ueno,[65] for example, have estimated the two-dimensional (xy) variation of the albedo ρ at $z = z_0$ from measurements of the radiance $I(x,y,\mathbf{\Omega})$ emerging from the illuminated surface at $z = 0$.

An interesting implicit approach has been developed by Grünbaum[66] and Singer et al.,[67] who considered particles undergoing discrete random walks in an optically thick body that was partitioned into little cubes. Particles that enter cube n through one surface can undergo absorption in the cube with probability a_n, or they are forward scattered out through the opposite surface with probability $(1 - a_n)f_n$ or a side surface with probability $(1 - a_n)s_n$, or are backscattered through the entrant surface with probability $(1 - a_n)b_n$. The f_n, s_n, and b_n are three conditional scattering transition probabilities that obey the constraint $f_n + 4s_n + b_n = 1$. The inverse problem consists of estimating the a_n and two of the three conditional transition probabilities for each cube in the body from measurements at different locations on its external surface. The spatial resolution of the imaging is governed by the number of cubes in the body; measurements at a large number of locations are necessary if the cube size is small.

To solve the direct problem for use in this iterative procedure, one must select a method to analyze the particle diffusion within a cube and also select a method to describe the transfer between cubes. One possibility would be to analyze the diffusion within a cube with a collision probability method, for example, while the interface current method could be used to analyze the transfer between cubes.[68]

V.B. Time-Dependent Problems

Patterson, Chance, and Wilson[69] used diffusion theory to develop an explicit inversion algorithm for estimating optical properties of a medium using the backscattered return $I_{1,-}(z=0,\tilde{r},t)$ arising from a laser pulse illumination at $\tilde{r}=0$. For example, for a large separation distance $r=\Sigma\tilde{r}$ between the incident illumination and the detector,

$$I_{1,-}(z=0,r,t) \propto t^{-5/2}\exp[-(1-\varpi)ct] \times \exp[-r^2/(4Dct)] \ , \quad (26)$$

where D is the dimensionless diffusion coefficient $\nu_0^2(1-\varpi) \approx [3(1-\varpi f_1)]^{-1}$. This result, analogous to Eq. (20) with $m=0$, also has been tested against experimental measurements; good agreement of the time-dependent pulse shape with measurements on human tissue was obtained.[69]

A closely related time-dependent explicit inverse algorithm can be obtained by recognizing the connection between problems with pulsed sources and those with oscillatory sources. To implement an algorithm with a frequency-dependent source, the observable quantities are the modulation $M_{1,-}(z=0,\tilde{r},f)$ and phase $\psi(\tilde{r},f)$ of the detected signal at distance $\tilde{r}$ and frequency f. Patterson et al.[70] have shown that a large phase shift in the detected signal occurs for frequencies in the range 0 to 250 MHz, and this has been observed experimentally.[71,72]

At low frequencies, there is a linear dependence of the phase function ψ on the modulation frequency. This linear relationship is characteristic of the "crow-flight" distance traveled by a photon from the source to the detector. Thus, measurement of the phase function in this range gives a parameter that is a function of the properties of the medium.

VI. TEMPERATURE-DEPENDENT PROBLEMS

The basic difficulty of treating energy balance applications involving radiative transfer (as categorized in Table I) is that the temperature dependence arising from the source and boundary conditions causes a problem to become nonlinear.[19] Thus, even the solution of the direct problem involves iteration,[73,74] so it is clear that only an iterative method can be considered for the solution of an inverse problem.

We first consider gray radiative transfer with reemission and radiative properties that do not depend on radiation wavelength. Then, we can use Eq. (5) for the radiance, but with the source term replaced by $(1-\varpi)(n^2\sigma/\pi)T^4(z)$, where n is the index of refraction and σ is the Stefan-Boltzmann constant.

Li and Özişik[75] recently solved such an inverse problem to estimate the temperature profile $T(z)$ and the diffuse reflectivity ρ in Eq. (6b) while assuming the surface at $z=0$ was transparent. They were able to estimate quite accurately a fourth-degree-polynomial spatial dependence of the temperature for slab thicknesses $z_0 \leq 5$, for example, although the estimated value of ρ tended to be somewhat in error. The results were more sensitive to increases in simulated measurement errors as the optical thickness increased, as would be expected.

Let us now consider a radiation-conduction problem with the same source term for the radiative transfer equation, but the temperature $T(z,t)$ is to satisfy the energy equation

$$\zeta c_p \partial_t T(z,t) = \partial_z[k\partial_z T(z,t)] - 2\pi\int_{-1}^{1}\mu\partial_z I(z,\mu,t)\,d\mu \ , \quad (27)$$

subject to the boundary conditions

$$T(0,t) = T_1(t) \quad \text{and} \quad T(z_0,t) = T_2(t) \ , \quad (28)$$

where

ζ = density

c_p = specific heat

k = thermal conductivity of the medium.

A general boundary condition for a reflecting surface that replaces that of Eq. (6a) might be

$$I(z_0,-\mu,t) = \epsilon_2\frac{n^2\sigma}{\pi}T_2^4(t) + \rho_2^s I(z_0,\mu,t) + 2\rho_2^d\int_0^1\mu' I(z_0,\mu',t)\,d\mu' \ , \quad 0 \leq \mu \leq 1 \ , \quad (29a)$$

and a corresponding boundary condition for the other surface is

$$I(0,\mu,t) = \epsilon_1\frac{n^2\sigma}{\pi}T_1^4(t) + \rho_1^s I(0,-\mu,t) + 2\rho_1^d\int_0^1\mu' I(0,-\mu',t)\,d\mu' \ , \quad 0 \leq \mu \leq 1 \ , \quad (29b)$$

where

ϵ_i = emissivity of surface i

ρ_i^s = specular reflection coefficient

ρ_i^d = diffuse reflection coefficient.

For this problem, we note that only the time-independent form of Eq. (5) need be solved, not the time-dependent one, since transients associated with the propagation of radiation can be neglected; the dependence of $I(z, \mu, t)$ on time arises because of $T(z, t)$.

It appears that an implicit method for estimating the temperature profile for an inverse problem defined by Eq. (5), with the temperature-dependent source, and Eqs. (27), (28), and (29) has not yet been implemented with a general method for solving the direct problem. Matthews, Viskanta, and Incropera,[76] however, have utilized a two-stream radiative transfer method to solve the inverse problem. Their solution is also interesting because they assumed the optical properties were such that there were two spectral bands: one in which the medium was semitransparent to radiation (for $\lambda < \lambda_c$) and the other in which it was opaque. For $\lambda > \lambda_c$, they replaced boundary condition (27) by one involving convection.

Matthews, Viskanta, and Incropera[76] obtained the measurements for their inverse problem using a combination of thermocouples located within the medium and incident and transmitted heat flux detectors. Their analysis showed that the transmittance and reflectance were more sensitive to changes in ϖ than to other medium properties.

The use of thermocouples interior to a medium to estimate the thermal conductivity and surface heat fluxes, for example, often occurs in inverse problems for which only conduction need be analyzed. Indeed, inverse heat conduction problems have been much more extensively investigated[77–81] than inverse radiative transfer problems.

VII. FUTURE WORK

This paper has concentrated on research published in the last six years on explicit and implicit solutions of inverse radiative transfer problems, with an emphasis on applications. During this time, a much greater variety of problems has been considered than earlier.[9–11]

There has been a definite evolution away from estimating unknowns with only explicit methods, which are simple to implement if no radiative transfer computations are required. Problems now are more likely to be solved with implicit methods that require iterative transport calculations. Also, more care is now taken to numerically test the sensitivity of the estimated parameters to errors arising from measurements and from parameters that are assumed known.

More realistic inverse radiative transfer problems undoubtedly will be solved in the future using implicit methods. These include general geometry problems and those previously formulated with only explicit algorithms, such as optical property estimation for multigroup[16] or polarized light applications.[82] Also, there likely will be more developments in inverse radiative transfer methods for potential biomedical imaging applications.

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