ture correlations, in subsequent chapters both recursive and nonrecursive models are treated from a different and more fundamental point of view.

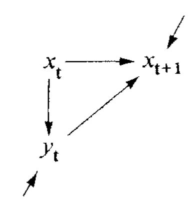
FURTHER READING

The classic exposition of the causal structures that may underlie the correlations among three variables is that of Simon (1954). A didactic presentation of four-variable models much in the spirit of this chapter is given by Blalock (1962–1963). Both papers are reprinted in Blalock (1971).

3

Recursive Models

A model is said to be recursive if all the causal linkages run "one way," that is, if no two variables are reciprocally related in such a way that each affects and depends on the other, and no variable "feeds back" upon itself through any indirect concatenation of causal linkages, however circuitous. However, recursive models do cover the case in which the "same" variable occurs at two distinct points in time, for, in that event, we would regard the two measurements as defining two different variables. For example, a dynamic model like the following:



where t and $t + 1$ are two points in time, is recursive (even though x appears to feed back upon itself). The definition also subsumes the case in which there are two or more ostensibly contemporaneous dependent variables where none of them has a direct or indirect causal linkage to any other. This situation is illustrated by Models II and II′ in Chapter 2 (assuming that ρ_{uv} in Model II′ does not implicitly arise from either a path $y \to z$ or a path $z \to y$). In this case, we have to consider whether or not to specify $\rho_{uv} = 0$. If so, we might term the model "fully recursive"; if not, it is merely "recursive."

With the exception of this last kind of situation—which offers no difficulty in principle, though it requires careful handling in practice—we can state that all the *dependent variables* in a recursive model (those whose causes are explicitly represented in the model) are arrayed in an unambiguous causal ordering. Moreover, all *exogenous variables* (those whose causes are not explicitly represented in the model) are, as a set, causally prior to all the dependent variables. There is, however, no causal ordering of the exogenous variables (if there are two or more) with respect to each other. (In some other model, of course, these variables might be treated as dependent variables.)

It simplifies matters greatly and results in a more powerful model if we can assume there is only one exogenous variable. This may not always be a reasonable assumption. We will consider models of this kind first and then see what modifications are entailed if we have to assume the contrary.

All our exposition of recursive models will rest on illustrations in which there are just four variables. Throughout this book we rely on relatively simple examples, and it is expected that the reader will come to see how the principles pertaining to these examples can be generalized to other models. There is some risk in this procedure, but it is hopefully outweighed by the advantages of making the discussion both concrete and compact—virtues difficult to attain if all principles and theorems must be stated in a perfectly general form.

With four variables, one of them exogenous, the causal ordering will involve a unique arrangement. Any one of the four might be the exogenous variable, any of the remaining three might be the first dependent variable, and either of the remaining two might follow it. Hence, there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible causal orderings of four variables. To build a recursive model means that we must choose one and only one of these 24 as the "true" ordering. That choice, it should be clear from the discussion in the previous chapter, cannot be based on the correlations among the variables, because *any* correlation matrix will be consistent with *any* causal ordering one may propose.

The information in regard to causal ordering is, logically, a priori. Such information is derived from theory, broadly construed, and no amount of study of the formal properties of models can teach one how to come up with a true theory. We can, however, prescribe the task of theory—to provide a causal ordering of the variables. Another way to put it is that the theory must tell us that at least six of the twelve possible causal linkages among four variables are *not* present, and these missing links, moreover, must fall into a triangular pattern. For, if four variables are put into a causal order and then numbered in sequence, we will have a pattern like the following:

	Caused by			
Effect	x_1	x_2	x_3	x_4
(x_1)	⋯	0	0	0
x_2	×	⋯	0	0
x_3	×	×	⋯	0
x_4	×	×	×	⋯

It is immaterial whether one or more of the crosses is replaced by a nought. But all the noughts must be present, to stand for the assumption that x_2 does *not* cause x_1 (though x_1 may cause x_2), and so on. If one enters six noughts in such a matrix (ignoring diagonal cells) before the variables are numbered, it may or may not be possible to triangulate the matrix. If it is, then the matrix defines a recursive causal ordering. If not, one or more nonrecursive relationships is present. Thus, the indispensable contribution of theory is to put noughts into the matrix. It is an odd way to put it, but the decisive criterion of the utility of a theory is that it can tell us definitely what causal relationships do not obtain, not that it can suggest (however evocatively) what relationships may well be present. (This remark will seem even more poignant when, in a later chapter, we discuss the problem of identification for a nonrecursive model.)

There is still another way to describe a recursive model. We may say that all exogenous variables in the model are *predetermined* with respect to all dependent variables. Moreover, each dependent variable is predetermined with respect to any other dependent variable that occurs later in a causal ordering. We take this to mean that all exogenous variables are uncorrelated with the disturbances in all equations of the model. (If one cannot assume this, the remedy is to build a better model.) Moreover, we shall similarly assume that the predetermined variables occurring in any equation are uncorrelated with the disturbance of that equation. (This is virtually a definition of "predetermined." It does leave open the possibility that of two dependent variables in a recursive model, neither is predetermined with respect to the other, while the disturbances of their equations are correlated, as is the case for Model II' in the previous chapter.) Again, this assumption has to be evaluated on its theoretical or substantive merits and, if it must be faulted, the recourse is to propose a better model.

The four-variable model, already described by a matrix of crosses and noughts, is more explicitly represented by this set of equations:

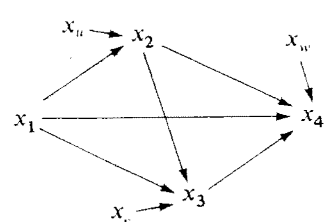
$(x_1 \text{ exogenous})$

$$x_2 = p_{21}x_1 + p_{2u}x_u$$

$$x_3 = p_{32}x_2 + p_{31}x_1 + p_{3v}x_v$$

$$x_4 = p_{43}x_3 + p_{42}x_2 + p_{41}x_1 + p_{4w}x_w$$

or by the path diagram:



We continue to assume that $E(x_h) = 0$ and $E(x_h^2) = 1$, $h = 1,2,3,4,u,v,w$ (the variables are in standard form). Hence, $E(x_h x_j) = \rho_{hj}$, the correlation (in the population) between x_h and x_j. The specifications on the disturbance terms are as follows:

(*a*) The exogenous variable is uncorrelated with the disturbances:

$$E(x_1 x_u) = E(x_1 x_v) = E(x_1 x_w) = 0$$

(*b*) Disturbances are also uncorrelated with any other predetermined variables in an equation:

$$E(x_2 x_v) = E(x_2 x_w) = E(x_3 x_w) = 0$$

A standard manipulation is to "multiply through" one equation of the model by a variable in the model, take expected values, and express in terms of path coefficients (the p's) and correlations (ρ's). Following this procedure with the x_2-equation, making use of (*a*) and (*b*), we deduce that the disturbance in each equation is uncorrelated with the disturbance in any other equation; for example,

$$E(x_2 x_v) = p_{21}E(x_1 x_v) + p_{2u}E(x_u x_v)$$

so that $\rho_{uv} = 0$; similarly, $\rho_{uw} = \rho_{vw} = 0$. But note that the disturbance in each equation has a nonzero correlation with the dependent variable in that equation and (in general) with the dependent variable in each "later" equation. To take another example, if we multiply through the equation for x_3 by x_2, we obtain

$$E(x_2 x_3) = p_{32}E(x_2^2) + p_{31}E(x_1 x_2) + p_{3v}E(x_2 x_v)$$

or

$$\rho_{23} = p_{32} + p_{31}\rho_{12}$$

[since $E(x_2^2) = 1$ and $E(x_2 x_v) = 0$]. Proceeding systematically, we obtain

$$\rho_{12} = p_{21} \qquad \text{(from the } x_2\text{-equation)}$$

$$\left.\begin{aligned} \rho_{13} &= p_{31} + p_{32}\rho_{12} \\ \rho_{23} &= p_{31}\rho_{12} + p_{32} \end{aligned}\right\} \qquad \text{(from the } x_3\text{-equation)}$$

$$\left.\begin{aligned} \rho_{14} &= p_{41} + p_{42}\rho_{12} + p_{43}\rho_{13} \\ \rho_{24} &= p_{41}\rho_{12} + p_{42} + p_{43}\rho_{23} \\ \rho_{34} &= p_{41}\rho_{13} + p_{42}\rho_{23} + p_{43} \end{aligned}\right\} \qquad \text{(from the } x_4\text{-equation)}$$

We may study this set of "normal equations" in two ways:

(*i*) *Solve for the p's in terms of the ρ's*

We obtain, for the first equation of the model,

$$p_{21} = \rho_{12}$$

for the second equation,

$$p_{31} = (\rho_{13} - \rho_{12}\rho_{23})/(1 - \rho_{12}^2)$$

$$p_{32} = (\rho_{23} - \rho_{12}\rho_{13})/(1 - \rho_{12}^2)$$

and for the third equation,

$$p_{41} = \frac{1}{D}\begin{vmatrix} \rho_{14} & \rho_{12} & \rho_{13} \\ \rho_{24} & 1 & \rho_{23} \\ \rho_{34} & \rho_{23} & 1 \end{vmatrix}$$

$$p_{42} = \frac{1}{D}\begin{vmatrix} 1 & \rho_{14} & \rho_{13} \\ \rho_{12} & \rho_{24} & \rho_{23} \\ \rho_{13} & \rho_{34} & 1 \end{vmatrix}$$

$$p_{43} = \frac{1}{D}\begin{vmatrix} 1 & \rho_{12} & \rho_{14} \\ \rho_{12} & 1 & \rho_{24} \\ \rho_{13} & \rho_{23} & \rho_{34} \end{vmatrix}$$

where

$$D = \begin{vmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{vmatrix}$$

Thus, if we knew the correlations we could solve for the coefficients of the model. In practice, we have only estimates (from a sample) of the correlations. If these estimates are inserted into the foregoing formulas (in place of the population correlations), the formulas will yield estimates of the p's. These are, in fact, the same estimates that one obtains from the ordinary least squares (OLS) regression of

x_2 on x_1

x_3 on x_2 and x_1

x_4 on x_3, x_2, and x_1

if the variables are in standard form.

(ii) Solve for the ρ's in terms of the p's

This may be done quite simply, making substitutions in the "normal equations." We obtain

$$\rho_{12} = p_{21}$$

$$\rho_{13} = p_{31} + p_{32}p_{21}$$

$$\rho_{23} = p_{32} + p_{31}p_{21}$$

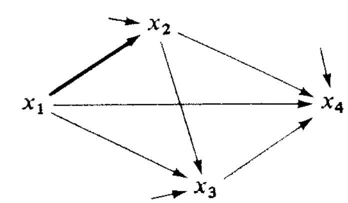
$$\rho_{14} = p_{41} + p_{42}p_{21} + p_{43}(p_{31} + p_{32}p_{21})$$

$$\rho_{24} = p_{42} + p_{43}p_{32} + p_{41}p_{21} + p_{43}p_{31}p_{21}$$

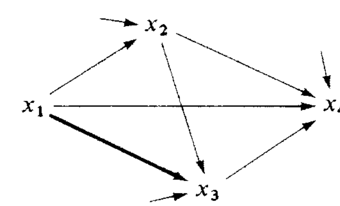
$$\rho_{34} = p_{43} + p_{42}(p_{32} + p_{31}p_{21}) + p_{41}(p_{31} + p_{32}p_{21})$$

These expressions are instructive in that they show how the model generates the (observable) correlations. It is worthwhile to study them with some care.

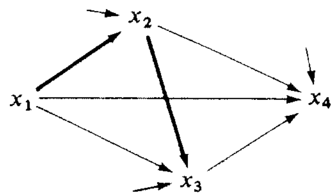
(*1*) We see that the entirety of the correlation between x_1 and x_2 is generated by the direct effect, p_{21}



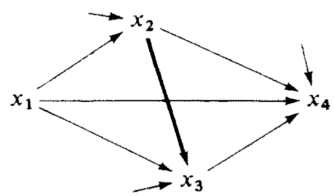
(*2*) The correlation between x_1 and x_3 is generated by two distinct paths, so that ρ_{13} equals the direct effect, p_{31}



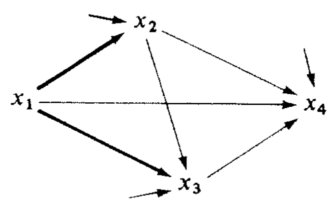
plus the indirect effect, $p_{32}p_{21}$



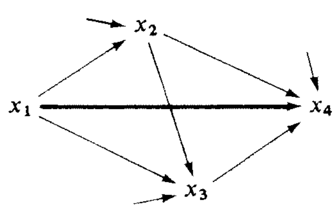
(3) The situation is different in regard to x_2 and x_3, for here we have the total correlation (ρ_{23}) generated as the sum of the direct effect, p_{32}



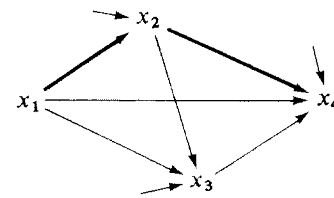
plus correlation due to a common cause, $p_{31}p_{21}$



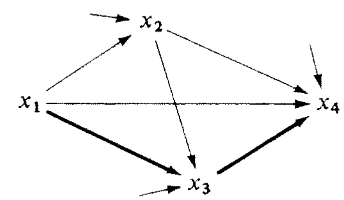
(4) The correlation between x_1 and x_4 is generated by four distinct causal links; ρ_{14} equals the direct effect, p_{41}



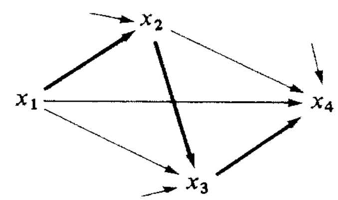
plus indirect effect via x_2, $p_{42}p_{21}$



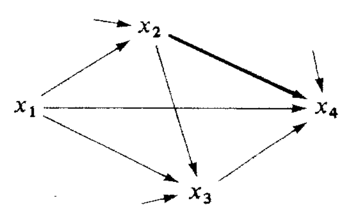
plus indirect effect via x_3, $p_{43}p_{31}$



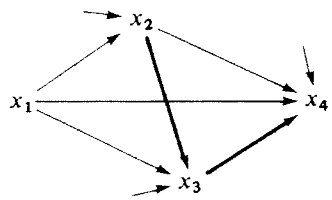
plus indirect effect via x_3 and x_2, $p_{43}p_{32}p_{21}$



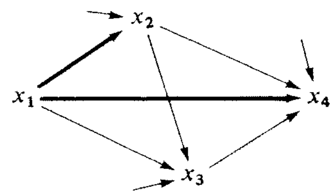
(5) Both an indirect effect and correlation due to common causes are involved in generating the correlation between x_2 and x_4: ρ_{24} equals the direct effect, p_{42}



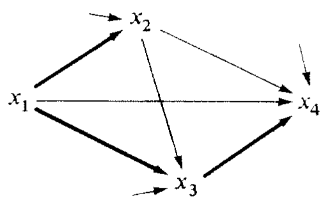
plus indirect effect via x_3, $p_{43}p_{32}$



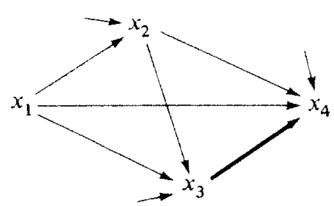
plus correlation due to x_1 operating as a common cause, directly, $p_{41}p_{21}$



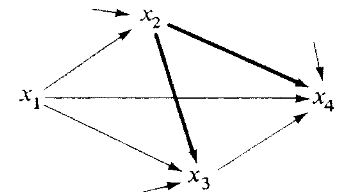
and indirectly (via x_3), $p_{43}p_{31}p_{21}$



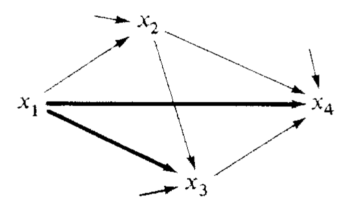
(6) There are no indirect effects in the model producing correlation between x_3 and x_4; but there are two common causes. Hence, ρ_{34} equals the direct effect, p_{43}



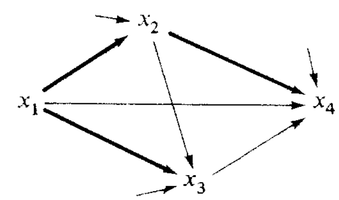
plus correlation due to common causes, working directly, $p_{42}p_{32}$



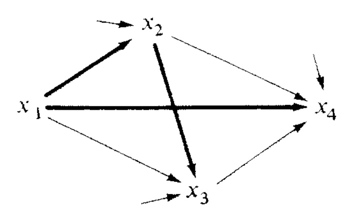
and $p_{41}p_{31}$



or indirectly, $p_{42}p_{31}p_{21}$



and $p_{41}p_{32}p_{21}$



In all six of these cases, the correlation may be read off the path diagram using Wright's (1921) multiplication rule: To find the correlation between x_h and x_j, where x_j appears "later" in the model, begin at x_j and read *back* to x_h along each distinct direct and indirect (compound) path, forming the product of the coefficients along that path. After reading back, read *forward* (if necessary), but only one reversal from back to forward is permitted. Sum the products obtained for all the linkages between x_j and x_h.

We consider next a modification of the model. Suppose both x_1 and x_2 are exogenous, that is, the model cannot explain how they are generated. Since we do not know anything about this, we cannot make any strong assumption about their correlation. Hence, we shall have to suppose, in general, that $\rho_{12} \neq 0$. The model now has but two equations.

$$x_3 = p_{32}x_2 + p_{31}x_1 + p_{3v}x_v$$

$$x_4 = p_{43}x_3 + p_{42}x_2 + p_{41}x_1 + p_{4w}x_w$$

Disturbances are uncorrelated with predetermined (including exogenous) variables. Hence, $\rho_{1v} = \rho_{1w} = \rho_{2v} = \rho_{2w} = \rho_{3w} = 0$; and, as a consequence, $\rho_{vw} = 0$. Normal equations are obtained as before, except, of course, that there is no normal equation with ρ_{12} on the left-hand side. Also, except for the fact that no p_{21} occurs in the model, the solution for the p's is the same as before and has the same interpretation.

In studying the model from standpoint *ii* (solution for ρ's in terms of p's), however, we must reconsider the situation. Algebraic substitutions yield,

$$\rho_{13} = p_{31} + p_{32}\rho_{12}$$

$$\rho_{23} = p_{32} + p_{31}\rho_{12}$$

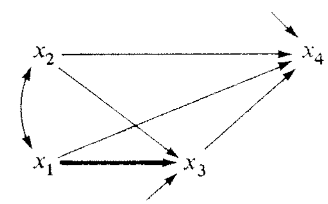
$$\rho_{14} = p_{41} + p_{43}p_{31} + (p_{42} + p_{43}p_{32})\rho_{12}$$

$$\rho_{24} = p_{42} + p_{43}p_{32} + (p_{41} + p_{43}p_{31})\rho_{12}$$

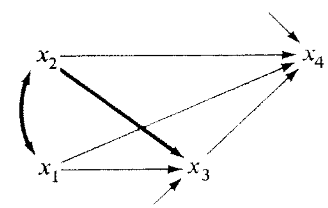
$$\rho_{34} = p_{43} + p_{42}p_{32} + p_{41}p_{31} + (p_{42}p_{31} + p_{41}p_{32})\rho_{12}$$

We cannot eliminate ρ_{12} from the right-hand side of these equations, since the model cannot (by definition) tell us anything about how that

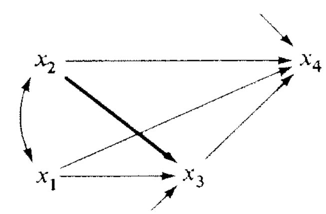
correlation is generated. Moreover, the presence of this correlation means that the remaining correlations are generated in a somewhat ambiguous way. Consider the correlation between x_1 and x_3. We have a direct effect, p_{31}. The other term, $p_{32}\rho_{12}$, consists of the product of the direct effect of x_2 on x_3 and the correlation of x_1 and x_2. It represents a contribution to ρ_{13} by virtue of the fact that *another cause* of x_3 (namely x_2) is correlated (to the extent of ρ_{12}) with the cause we are examining at the moment (namely, x_1). As in Chapter 2 (see Models II′ and III) we use a curved, double-headed arrow to refer to a correlation that cannot be analyzed in terms of causal components within this model. Hence the way to look at this situation is that ρ_{13} equals the direct effect, p_{31}



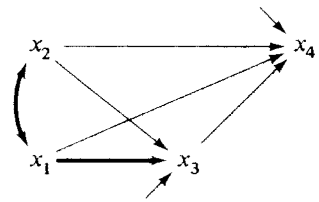
plus correlation due to correlation with another cause, $p_{32}\rho_{12}$



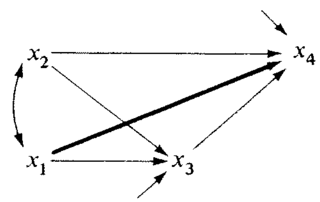
Similarly, we may break down ρ_{23} into the components: direct effect, p_{32}



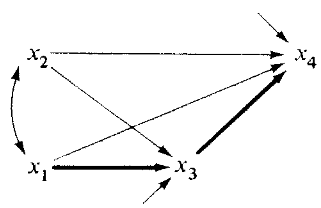
plus correlation due to correlation with another cause, $p_{31}\rho_{12}$



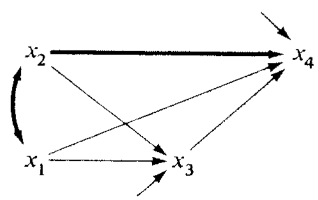
The same kind of reasoning interprets the remaining correlations. For ρ_{14} we obtain the direct effect, p_{41}



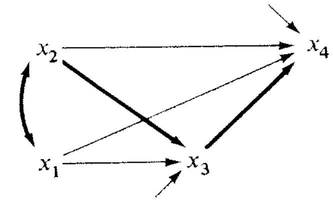
plus the indirect effect, $p_{43}p_{31}$



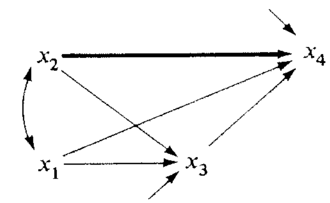
plus correlation due to the correlation of x_1 with another cause (x_2), working both directly, $p_{42}\rho_{12}$



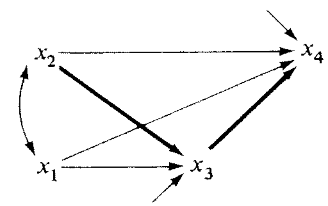
and indirectly, $p_{43}p_{32}\rho_{12}$



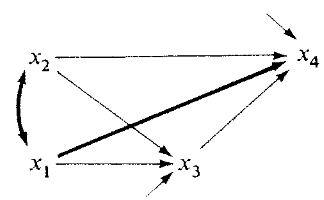
We decompose ρ_{24} into the direct effect, p_{42}



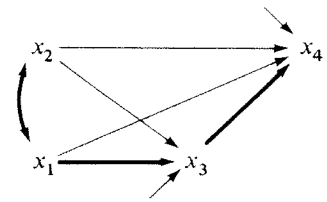
plus the indirect effect, $p_{43}p_{32}$



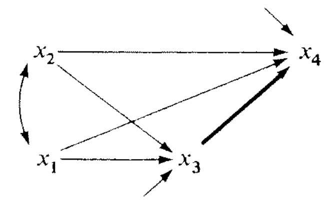
plus correlation due to the correlation of x_2 with another cause (x_1), working both directly, $p_{41}\rho_{12}$



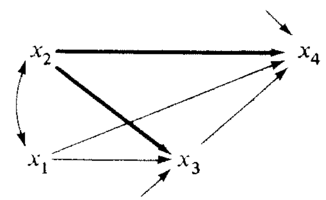
and indirectly, $p_{43}p_{31}\rho_{12}$



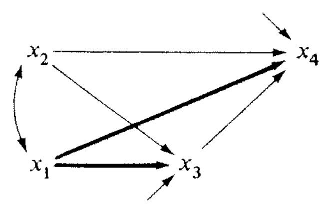
As in the previous version of the model, ρ_{34} involves no indirect effects, but the correlation generated by the common causes (x_1 and x_2) involves both the direct effects of those causes and the correlation due to the fact that they are correlated with each other. Hence, ρ_{34} equals the direct effect, p_{43}



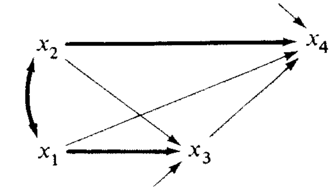
plus correlation due to x_2 as a common cause, $p_{42}p_{32}$



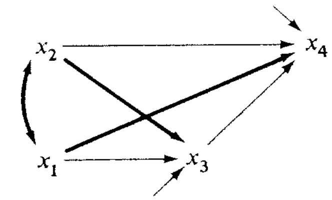
and x_1 as a common cause, $p_{41}p_{31}$



plus correlation due to the correlation of x_1 with another common cause (x_2), $p_{42}p_{31}\rho_{12}$



and correlation due to the correlation of x_2 with another common cause (x_1), $p_{41}p_{32}\rho_{12}$



For some purposes, one might be content to aggregate the last four components, so as to describe the correlation ρ_{34} as being generated by the direct effect (p_{43}) and the correlation due to the influence of the two common causes, x_1 and x_2, on x_3 and x_4.

To bring this discussion within the scope of Sewall Wright's multiplication rule, we stipulate that the curved double-headed arrow is read either back or forward.

It is, of course, an undesirable property of this modified model that we cannot clearly disentangle the effects of its two exogenous variables. But our theory may simply be unable to tell us whether x_1 causes x_2, x_2 causes x_1, each influences the other, both are effects of one or more common or correlated causes, or some combination of these situations holds true. In that event, we cannot know for sure whether a change initiated in (say) x_1 will have indirect effects via x_2 or not, since we do not know whether x_2 depends on x_1. It follows that we cannot, with this model, estimate the *total* effect (defined as *direct* effect plus *indirect* effect) of x_1 on, say, x_4. There may or may not be an indirect causal linkage from x_1 through x_2 to x_4. But since we know nothing about this, we cannot include any such indirect effect in our estimate of total effect.

It should be noted, in both forms of the model, that the zero-order correlation between two variables often is not the correct measure of total effect of one variable on the other, since that correlation may include components other than direct and indirect effects.

In a further modification of the four-variable model, we suppose that x_1, x_2, and x_3 all are exogenous. This gives rise to the degenerate case of a single-equation model:

$$x_4 = p_{43}x_3 + p_{42}x_2 + p_{41}x_1 + p_{4w}x_w$$

The disturbance is uncorrelated with the exogenous variables.

There are three normal equations:

$$\rho_{14} = p_{41} + p_{42}\rho_{12} + p_{43}\rho_{13}$$

$$\rho_{24} = p_{41}\rho_{12} + p_{42} + p_{43}\rho_{23}$$

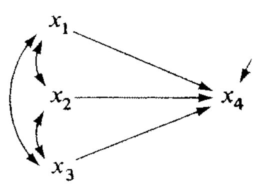
$$\rho_{34} = p_{41}\rho_{13} + p_{42}\rho_{23} + p_{43}$$

As before, if sample correlations are inserted into these equations, the solution for the p's yields least-squares estimates of the parameters.

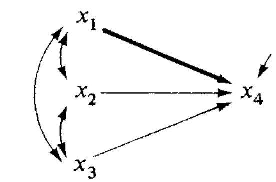
None of the correlations on the right-hand side of the normal equations can be expressed in terms of path coefficients. Therefore, we cannot separate indirect effects from correlation due to common or correlated causes. Thus, the only decomposition we can provide is the following:

Causal variable	*Total correlation*	=	*Direct effect on x_4*	+	*Correlation due to common and/or correlated causes*
x_1	ρ_{14}		p_{41}		$p_{42}\rho_{12} + p_{43}\rho_{13}$
x_2	ρ_{24}		p_{42}		$p_{41}\rho_{12} + p_{43}\rho_{23}$
x_3	ρ_{34}		p_{43}		$p_{41}\rho_{13} + p_{42}\rho_{23}$

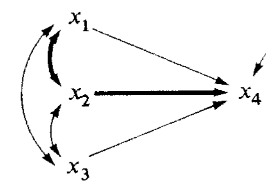
The path diagram for this single-equation model is shown below:



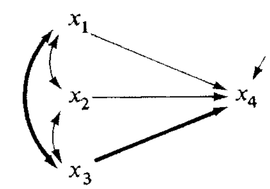
Correlations between exogenous variables are represented by curved, double-headed arrows. The normal equations can be written using Sewall Wright's rule. The curved arrow can be read either forward or backward, but *only one* curved arrow can be included in a given trajectory. Thus, to find ρ_{14} (for example), we read p_{41}



plus $p_{42}\rho_{12}$



plus $p_{43}\rho_{13}$



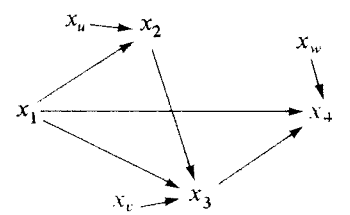
The single-equation model correctly represents the *direct effects* of the exogenous variables. But, since the causal structure of relationships among the exogenous variables is unknown, this model cannot tell us anything about how the indirect effects (not to mention total effects) are generated. In this respect, it is even less satisfactory than the two-equation model.

A major goal of theory, therefore, should be to supply a model that will make some of the exogenous variables endogenous.

This discussion has illustrated a significant theorem: The *direct* effects of predetermined variables in one equation of a model on the dependent variable of that equation are the same, irrespective of the causal relationships holding among the predetermined variables. Thus, returning to the complete model discussed at the beginning, the values of p_{41}, p_{42}, and p_{43} in the third equation do not depend on whether x_2 causes x_1 or vice versa in the first equation or on whether x_1, x_2, or x_3 is the dependent variable in the second equation.

Thus far we have only considered models in which all direct paths allowed by the causal ordering are, in fact, present in the model. We must now consider procedures suited to recursive models in which one or more such paths are (or may be) missing. These procedures have to do with the distinct (though related) problems of *estimation* and *testing*. We discuss them in that order.

We have seen that in the fully recursive model with direct paths from each "earlier" variable to each "later" variable, the path coefficients may be estimated by OLS regression. Suppose, however, that our model, while recursive, explicitly specifies that one or more coefficients are zero. To take a concrete example, consider this path diagram



The equations of the model are

$$(x_1 \text{ exogenous})$$

$$x_2 = p_{21}x_1 + p_{2u}x_u$$

$$x_3 = p_{32}x_2 + p_{31}x_1 + p_{3v}x_v$$

$$x_4 = p_{43}x_3 + p_{41}x_1 + p_{4w}x_w$$

The only change from the model on page 28 is that $p_{42} = 0$. We continue to assume that all variables (including disturbances) are in standard form. In each equation of the model, the disturbance is uncorrelated with the predetermined variables. Moreover, we take the model to be fully recursive, so that the disturbance in each equation is uncorrelated with predetermined variables in all "earlier" equations. (Here, as elsewhere in this chapter, we allow the zero correlation of two disturbances to appear as a consequence of the zero correlation of disturbances with prior predetermined variables.) The force of this specification, with special reference to the present example, is that $\rho_{2w} = 0$, even though x_2 does not appear (explicitly) in the x_4-equation. We have, then, the following specification on the disturbances: $\rho_{1u} = \rho_{1v} = \rho_{1w} = \rho_{2v} = \rho_{2w} = \rho_{3w} = 0$. As a consequence of this specification, we find that it is also true that $\rho_{uv} = \rho_{uw} = \rho_{vw} = 0$.

The normal equations for the x_2-equation and the x_3-equation are the same as before, and OLS estimates of their path coefficients are obtained by formulas given earlier.

Multiplying through the x_4-equation by each predetermined variable, we find

$$\rho_{14} = p_{41} + p_{43}\rho_{13}$$

$$\rho_{24} = p_{41}\rho_{12} + p_{43}\rho_{23}$$

$$\rho_{34} = p_{41}\rho_{13} + p_{43}$$

Assuming the ρ's are known, we have three equations in the two unknown path coefficients. In mathematical terms, the solution for the p's is overdetermined. In the language of structural equation models, the x_4-equation is *overidentified*. In the event that an equation in the model is overidentified, we may deduce that one or more *overidentifying restrictions* must hold if the model is true. Here, we can ascertain the overidentifying restriction by writing out each of the solutions for the p's obtained upon solving a pair of the normal equations. There are three distinct solutions. If the model holds, the values obtained in all three must be equal. Thus,

$$\begin{array}{cccc} & (i) & (ii) & (iii) \\ p_{41} = & \dfrac{\rho_{14} - \rho_{13}\rho_{34}}{1 - \rho_{13}^2} = & \dfrac{\rho_{14}\rho_{23} - \rho_{13}\rho_{24}}{\rho_{23} - \rho_{12}\rho_{13}} = & \dfrac{\rho_{24} - \rho_{23}\rho_{34}}{\rho_{12} - \rho_{13}\rho_{23}} \\ p_{43} = & \dfrac{\rho_{34} - \rho_{13}\rho_{14}}{1 - \rho_{13}^2} = & \dfrac{\rho_{24} - \rho_{12}\rho_{14}}{\rho_{23} - \rho_{12}\rho_{13}} = & \dfrac{\rho_{12}\rho_{34} - \rho_{13}\rho_{24}}{\rho_{12} - \rho_{13}\rho_{23}} \end{array}$$

where solution number (*i*) makes use of the first and third normal equations, number (*ii*) is from the first and second normal equations, and number (*iii*) is from the last two normal equations. If $p_{41}^{(i)} = p_{41}^{(ii)}$ it follows that

$$\rho_{24} + \rho_{13}\rho_{14}\rho_{23} + \rho_{12}\rho_{13}\rho_{34} - \rho_{24}\rho_{13}^2 - \rho_{23}\rho_{34} - \rho_{12}\rho_{14} = 0$$

We note that this expression is just the expansion of the determinant given as the numerator of p_{42} on page 30; and that determinant must be zero if $p_{42} = 0$. We reach the same conclusion from any of the other equalities of solutions for p_{41} or p_{43}. These several equalities are not independent. There is actually only one overidentifying restriction on this model.

Now, the overidentifying restriction must hold in any *population* in which the model applies. But if we have only *sample* values of the correlations we cannot expect it to hold exactly, nor can we expect the three solutions for each path coefficient to be exactly equal. In that event, to estimate the path coefficients we must choose one of the solutions, or perhaps some average of them. Since each solution makes use of only two of the normal equations, it would appear that averaging the solutions would be advisable, since we would then be making use of all the sample correlations rather than only some of them. *This intuition, however, is wrong.* It turns out that the preferred estimate is obtained upon inserting sample correlations into solution (*i*). It will be noted that the estimates of p_{41} and p_{43} obtained in this way are just the OLS regression coefficients of x_4 on x_1 and x_3. The general rule, then, is this: In a fully recursive model (where the correlation between each pair of disturbances is zero), estimate the coefficients in each equation by OLS regression of the dependent variable on the predetermined variables included in that equation.

The basis for this rule is a proof that the sampling variance of a $\hat{p}$ estimated by OLS is smaller than the variance of any other unbiased estimate of the same coefficient, even if such an estimate appears to use more information in the sense of combining correlations involving the included variables with correlations involving the excluded predetermined variable(s). Some of the earlier literature on path analysis was in error on this point; it was called to the attention of sociologists by A. S. Goldberger (1970).

We now turn to the problem of *testing*. In the preceding example, we discussed estimation on the assumption that the model and, in particular, the overidentifying restriction on the model are known in advance

to be true. But the investigator may not feel confident of this specification. Indeed, he may be undertaking a study precisely to test that aspect of his theory which says that a particular coefficient should be zero. Much of the literature on causal models in the 1960s—particularly papers discussing or using the so-called "Simon–Blalock technique"—focused on this very question. Sometimes the problem was described as that of "making causal inferences from correlational data," but that ambiguous phrase seems to promise far too much. In the light of our preceding discussion, it would be more accurate to describe the problem as that of *testing the overidentifying restriction(s)* of a model.

We first take note of two plausible and conceptually correct procedures for making such tests on recursive models. But, since these procedures are not convenient from the standpoint of the standard methods of statistical inference, we conclude with an alternative recommendation.

Continuing with the example already described, suppose the analyst computes estimates of the path coefficients, p_{41} and p_{43}, by some method (not necessarily the OLS estimates recommended above). If these estimates—call them $\tilde{p}_{41}$ and $\tilde{p}_{43}$—are combined with sample correlations according to the normal equations, we have

$$r_{14}^* = \tilde{p}_{41} + \tilde{p}_{43} r_{13}$$
$$r_{24}^* = \tilde{p}_{41} r_{12} + \tilde{p}_{43} r_{23}$$
$$r_{34}^* = \tilde{p}_{41} r_{13} + \tilde{p}_{43}$$

where r_{hj} is an observed sample correlation and r_{hj}^* is the "implied" ("predicted" or "reproduced") correlation that would be observed if the overidentifying restriction(s) held exactly in the sample. Because of sampling error, implied and observed correlations will ordinarily not all be equal. Thus, we have a set of discrepancies.

$$d_{14} = r_{14} - r_{14}^*$$
$$d_{24} = r_{24} - r_{24}^*$$
$$d_{34} = r_{34} - r_{34}^*$$

If the OLS method of estimation were used, we would have

$d_{14} = d_{34} = 0$ but $d_{24} \neq 0$. If some other method were used we would still find one or more d's differing from zero. However, if the model holds true in the population, any such difference(s) should be "small," that is, no larger than one might reasonably expect as a consequence of sampling error alone.

It would seem plausible to use the set of d's in a formal statistical test against the null hypothesis which asserts the truth of the overidentifying restriction(s) of the model. However, this procedure is less convenient than the standard test described later, in the event that this test is available. Under some circumstances—though not in the case of the model used as an example here—the method of implied correlations may be recommended cautiously as a heuristic expedient, if an appropriate standard statistical test is not available. It may also be of use in the initial stage of specifying a model, where the investigator wishes to make an informal test of his ideas.

A similar approach to testing of overidentifying restrictions is the Simon–Blalock procedure. Consider any fully recursive model in which one or more paths are taken to be missing, that is, to have the value zero. It is then possible to deduce that certain simple and/or partial correlations will be zero. Blalock (1962–1963) has actually provided an exhaustive enumeration of the "predictions" for all possible four-variable models. The example given here appears in Blalock's enumeration as "Model E," and we find that in this model $\rho_{24.13} = 0$. The corresponding sample partial correlation ($r_{24.13}$) should, therefore, be close to zero. If it is not—if the difference from zero is too great to attribute to sampling error—we should be obliged to call into question the overidentifying restriction of this model. Blalock does not develop formal procedures of statistical inference for this kind of test. Again, we conclude that the proposed test is conceptually valid and is useful to the investigator who wants to be sure he understands the properties of his model. The research worker should not, however, rely on mere inspection of sample partial correlations. We recommend instead the standard statistical test described hereafter.

In our example, the issue as to the specification of the model is whether $p_{42} = 0$ or $p_{42} \neq 0$. In other words, we must decide as between the competing specifications of the x_4-equation:

$$x_4 = p_{43}x_3 + p_{41}x_1 + p_{4w}x_w$$

and

$$x_4 = p_{43}x_3 + p_{42}x_2 + p_{41}x_1 + p_{4w}x_w$$

We proceed on the latter specification and estimate by OLS the equation which includes p_{42}. In the usual routine for multiple regression we obtain as a by-product of our calculations the quantities necessary to compute the standard errors of our estimated coefficients (see, for example, the chapter on multiple regression in Walker and Lev, 1953, where the procedures are described for standardized variables). We may then form the ratio,

$$t = \hat{p}_{42}/\text{S.E.}(\hat{p}_{42})$$

and refer it to the t-distribution with the appropriate degrees of freedom. Roughly speaking, if one is working with a reasonably "large" sample, when $|t| \geq 2.0$, we may conclude with no more than 5% risk of error that the null hypothesis is false. In this event, we would reject the overidentifying restriction of the model and, presumably, respecify it to include a nonzero value of p_{42}.

In the case of failure to reject the null hypothesis—that is, if the t-ratio is not statistically significant—the situation is intrinsically ambiguous. Clearly, one is not obliged to *accept* the null hypothesis unless there is sufficient a priori reason to do so. It could happen, for example, that the true value of p_{42} is positive but small, so that our sample is just not large enough to detect the effect reliably. If our theory strongly suggests this is the case, we would do well to keep p_{42} in the equation despite the outcome of the test. In any event, it is good practice to publish standard errors of all coefficients, so that the reader of the research report may draw his own conclusion as well as have some idea of the precision of the estimates of coefficients. A good discussion of the issues raised by tests of this kind is given by Rao and Miller (1971); their discussion is presented in the context of a single-equation model, but it carries over to the problem of testing an overidentifying restriction on any equation of a recursive model.

This is not the place to develop the theory and techniques of statistical inference. What has been said can be reduced to a simple rule: If OLS regression is the appropriate method of estimation, the theory and techniques of statistical inference, as presented in the literature on the multiple regression model, should be drawn upon when making tests of overidentifying restrictions. (Perhaps we should note that such a test will not always involve the t-statistic for a single coefficient but may lead to the F-test for a whole set of coefficients.) This rule does not cover all cases, as will become apparent when we consider models for which OLS is not the appropriate method of estimation.

It is vital to keep the matter of tests of overidentifying restrictions in perspective. Valuable as such tests may be, they do not really bear upon what may be the most problematical issue in the specification of a recursive model, that is, the causal ordering of the variables. It is the gravest kind of fallacy to suppose that, from a number of competing models involving different causal orderings, one can select the true model by finding the one that comes closest to satisfying a particular test of overidentifying restrictions. (Examples of such a gross misunderstanding of the Simon–Blalock technique can be found, among other places, in the political science literature of the mid-1960s.) In fact, a test of the causal ordering of variables is beyond the capacity of any statistical method; or, in the words of Sir Ronald Fisher (1946), "if . . . we choose a group of social phenomena with no antecedent knowledge of the causation or absence of causation among them, then the calculation of correlation coefficients, total or partial, will not advance us a step toward evaluating the importance of the causes at work [p. 191]."

Exercise. *Show how to express all the correlations in terms of path coefficients in the model on page 44, using Sewall Wright's multiplication rule for reading a path diagram.*

Exercise. *Change the model on page 28 so that p_{43} is specified to be zero. Draw the revised path diagram. Obtain an expression for the overidentifying restriction. Indicate how one would estimate the coefficients in the revised model, supposing the overidentifying restriction was not called seriously into question.*

FURTHER READING

See Walker and Lev (1953, Chapter 13) for a treatment of multiple regression with standardized variables. Sociological studies using recursive models are numerous; for examples, see Blalock (1971, Chapter 7) and Duncan, Featherman, and Duncan (1972).

4

Structural Coefficients in Recursive Models

In Chapter 3 a four-variable recursive model was formulated in terms of standardized variables. That procedure has some advantages.

(*1*) Certain algebraic steps are simplified.
(*2*) Sewall Wright's rule for expressing correlations in terms of path coefficients can be applied without modification.
(*3*) Continuity is maintained with the earlier literature on path analysis and causal models in sociology.
(*4*) It shows how an investigator whose data are available only in the form of a correlation matrix can, nevertheless, make use of a clearly specified model in interpreting those correlations.

Despite these advantages (see also Wright, 1960), it would probably be salutary if research workers relinquished the habit of expressing variables in standard form. The main reason for this recommendation is that standardization tends to obscure the distinction between the structural coefficients of the model and the several variances and covariances that describe the joint distribution of the variables in a certain population.

Although it will involve some repetition of ideas, we will present the four-variable recursive model again, this time avoiding the stipulation that all variables have unit variance. The model is

$$(x_1 \text{ exogenous})$$

$$x_2 = b_{21}x_1 + u$$

$$x_3 = b_{32}x_2 + b_{31}x_1 + v$$

$$x_4 = b_{43}x_3 + b_{42}x_2 + b_{41}x_1 + w$$

We continue to assume that all variables have zero expectation; unlike standardization, this achieves a useful simplification without significant loss of generality or confusion of issues. (Only in special cases, where "regression through the origin" is involved, does this stipulation require modification.)

The specification on the disturbances is that the disturbance in each equation has zero covariance with the predetermined variables in that equation and all "earlier" equations. The one strictly exogenous variable x_1 is predetermined in each equation. In addition, x_2 is a predetermined variable in the x_3-equation and the x_4-equation, while x_3 is a predetermined variable in the x_4-equation. Thus we specify $E(x_1 u) = E(x_1 v) = E(x_1 w) = E(x_2 v) = E(x_2 w) = E(x_3 w) = 0$. We find, as a consequence of this specification, that it also is the case that $E(uv) = E(uw) = E(vw) = 0$.

(We note, for future reference, that in both recursive and nonrecursive models, the usual specification is zero covariance of predetermined variables in an equation with the disturbance of that equation. In the case of recursive models this generally implies zero covariances among the disturbances of different equations. This does not, however, hold true for nonrecursive models.)

To deduce properties of the model from the specifications regarding its functional form and its disturbances, we multiply through equations of the model by variables in the model, and take expectations. For convenience, we denote the variance $E(x_j^2)$ by σ_{jj}, using the double subscript in place of the exponent of the usual notation, σ_j^2. Similarly, a covariance is denoted by $\sigma_{hj} = E(x_h x_j)$, $h \neq j$.

The normal equations are obtained by multiplying through each equation by its predetermined variables:

$$\sigma_{12} = b_{21}\sigma_{11} \qquad \text{(from the } x_2\text{-equation)}$$

$$\left.\begin{aligned} \sigma_{13} &= b_{31}\sigma_{11} + b_{32}\sigma_{12} \\ \sigma_{23} &= b_{31}\sigma_{12} + b_{32}\sigma_{22} \end{aligned}\right\} \qquad \text{(from the } x_3\text{-equation)}$$

$$\left.\begin{aligned} \sigma_{14} &= b_{41}\sigma_{11} + b_{42}\sigma_{12} + b_{43}\sigma_{13} \\ \sigma_{24} &= b_{41}\sigma_{12} + b_{42}\sigma_{22} + b_{43}\sigma_{23} \\ \sigma_{34} &= b_{41}\sigma_{13} + b_{42}\sigma_{23} + b_{43}\sigma_{33} \end{aligned}\right\} \qquad \text{(from the } x_4\text{-equation)}$$

Clearly, it is possible to solve uniquely for the b's, which we shall term the *structural coefficients*, in terms of the population variances and covariances. In practice, of course, the latter are unknown. Hence, we can only estimate the structural coefficients. If, in the normal equations, the σ's are replaced by sample moments, the estimates obtained are equivalent to those of ordinary least-squares (OLS) regression of x_2 on x_1; x_3 on x_2 and x_1; and x_4 on x_3, x_2, and x_1. By sample moments we mean the quantities

$$m_{jj} = \sum x_j^2$$

$$m_{hj} = \sum x_h x_j \qquad (h \neq j)$$

where the summation is over all the observations in the sample and each observation on x_j is expressed as a deviation from the mean of x_j in the sample.

Along with the preceding normal equations, we will find it useful to obtain expressions for the variances of the dependent variables by multiplying through each equation of the model by its dependent variable:

$$\sigma_{22} = b_{21}\sigma_{12} + \sigma_{2u}$$

$$\sigma_{33} = b_{32}\sigma_{23} + b_{31}\sigma_{13} + \sigma_{3v}$$

$$\sigma_{44} = b_{43}\sigma_{34} + b_{42}\sigma_{24} + b_{41}\sigma_{14} + \sigma_{4w}$$

These may be simplified slightly by noting that $\sigma_{2u} = \sigma_{uu}$, $\sigma_{3v} = \sigma_{vv}$, and $\sigma_{4w} = \sigma_{ww}$, which facts are deduced by multiplying through each equation of the model by its disturbance.

It is instructive to rewrite the expressions for the variances and covariances in the form given below (the algebra involves only straightforward, though tedious, substitutions in the equations already given). The variances may be written:

σ_{11} exogenous

$$\sigma_{22} = b_{21}^2\sigma_{11} + \sigma_{uu}$$

$$\sigma_{33} = A_6\sigma_{11} + b_{32}^2\sigma_{uu} + \sigma_{vv}$$

$$\sigma_{44} = A_9\sigma_{11} + A_0\sigma_{uu} + b_{43}^2\sigma_{vv} + \sigma_{ww}$$

and the covariances:

$$\sigma_{12} = b_{21}\sigma_{11}$$

$$\sigma_{13} = A_1\sigma_{11}$$

$$\sigma_{23} = A_3\sigma_{11} + b_{32}\sigma_{uu}$$

$$\sigma_{14} = A_2\sigma_{11}$$

$$\sigma_{24} = A_4\sigma_{11} + A_5\sigma_{uu}$$

$$\sigma_{34} = A_7\sigma_{11} + A_8\sigma_{uu} + b_{43}\sigma_{vv}$$

where

$$A_1 = b_{31} + b_{32}b_{21}$$

$$A_2 = b_{41} + b_{42}b_{21} + b_{43}A_1$$

$$A_3 = b_{21}(b_{31} + b_{32}b_{21})$$

$$A_4 = b_{41}b_{21} + b_{42}b_{21}^2 + b_{43}A_3$$

$$A_5 = b_{42} + b_{43}b_{32}$$

$$A_6 = b_{32}A_3 + b_{31}A_1$$

$$A_7 = b_{41}A_1 + b_{42}A_3 + b_{43}A_6$$

$$A_8 = b_{32}(b_{42} + b_{43}b_{32})$$

$$A_9 = b_{43}A_7 + b_{42}A_4 + b_{41}A_2$$

$$A_0 = b_{42}A_5 + b_{43}A_8$$

The A's have been introduced merely to abbreviate the presentation and have no particular interpretation in themselves. It is important to note, however, that all the A's are nonlinear combinations of the structural coefficients (the b's) and *involve no other terms*. Thus, we can draw an important conclusion. The variances and covariances are all functions of (at most) three kinds of quantities: (*1*) the variance of the exogenous variable; (*2*) the variance(s) of one or more disturbances; and (*3*) a nonlinear combination of structural coefficients. Table 4.1

Table 4.1 Sources of Observable Variances and Covariances

Variance or	*Is a function of*									
covariance	σ_{11}	σ_{uu}	σ_{vv}	σ_{ww}	b_{21}	b_{31}	b_{32}	b_{41}	b_{42}	b_{43}
σ_{11}	×	...	...	...	...	...	...	...	...	...
σ_{22}	×	×	...	...	×	...	...	...	...	...
σ_{33}	×	×	×	...	×	×	×	...	...	...
σ_{44}	×	×	×	×	×	×	×	×	×	×
σ_{12}	×	...	...	...	×	...	...	...	...	...
σ_{13}	×	...	...	...	×	×	×	...	...	...
σ_{14}	×	...	...	...	×	×	×	×	×	×
σ_{23}	×	×	...	...	×	×	×	...	...	...
σ_{24}	×	×	...	...	×	×	×	×	×	×
σ_{34}	×	×	×	...	×	×	×	×	×	×

makes this explicit in each instance. The first component (σ_{11}) is involved in all the variances and covariances. One or more of the disturbance variances (σ_{uu}, σ_{vv}, σ_{ww}) are involved in the variances of all the dependent variables in the model and in the covariances of these variables with each other. Some combination of structural coefficients (the b's) is involved in all variances (except σ_{11}) and covariances. Thus, it is possible to regard the variances and covariances of the observed variables as having arisen entirely from these three sources. Moreover, we can describe these components as separable, in the following sense. We can suppose without contradiction (that is, without violating any other property of the model) that one of these components may change without either of the others having to change. If any of them changes, however, the observable variances and covariances will, in general, change.

This remarkable property of the model should be considered carefully by the investigator, for it has some far-reaching implications.

Suppose we had two populations under study and we specified our three-equation model as holding in each. It could happen that the structural coefficients are the same in the two populations and the variances of the several disturbances are likewise the same. But if only σ_{11} differs between the two populations, we will observe differences in all the other variances and all the covariances. Incidentally, we will also observe differences (in general) in all the correlations in the two populations and also (in general) in all the standardized path coefficients, not to mention other purported measures of "relative importance" or "unique contribution" of variables. Thus the observable facts about the two populations (as reflected in sample estimates of variances, covariances, and correlations) will suggest that they differ in many ways. But the premise of the illustration is that they differ in only one way: with respect to the variance of the exogenous variable. The model is invariant across populations with respect to the structural coefficients and the variances of the disturbances.

Another possibility is that both σ_{11} and the disturbance variances differ as between two populations, but the structural coefficients are the same. Again we would observe entirely different variance–covariance (or correlation) matrices in the two populations, even though only four of the ten quantities in the column headings of the table actually differ.

The possibilities just described are only hypothetical. But there would not be much purpose in devising a model to use in interpreting data if we did not have some hope that at least some features of our model would be invariant with respect to some changes in the circumstances under which it is applied. If all the model is good for is to describe a particular set of data—if with any new set of data we will be obliged to change all the quantities listed in the column headings of the table, even though we continue to specify the same mathematical form of the model—then we might as well forego the effort of devising and estimating the model. It offers no economy of description, since there are as many parameters across the top of the table (neglecting the possibility that some b's may be zero and therefore may be omitted) as there are variances and covariances listed in the stub. We can transform the variances and covariances into the parameters, or vice versa, by mathematical operations already described in these pages. Hence, from a purely descriptive standpoint, we might as well let stand the first set of estimates we compute—the variances and covariances, or correlations—and not bother with the structural coefficients.

Another line of reflection is suggested by this analysis. It could happen that a macrosociological model proposed for a given society gave rise to (nearly) invariant estimates of structural coefficients over a period of time (for example, for several successive birth cohorts), even though the variances of the exogenous variable and the disturbances were changing. One would then observe "social change" with respect to variances and covariances of the dependent variables. But in another sense, no "social change" would be occurring, since the structural coefficients were staying constant. If, on the other hand, the latter should change, one would really be dealing with social change, in a deeper sense of the term. The first kind of social change would, in a sense, be "explained" by the model (though, of course, the model does not speak to the sources of change in the exogenous variable and disturbances themselves). The second kind of social change—modification of structural coefficients (or "structural change," if one likes)—cannot be explained in any sense by the model. Even so, one might argue, the model—if one held to it with good reason, despite changes in structural coefficients—would at least make clear what it is about the social changes occurring that requires explanation. But surely our scientific aspirations and efforts should be directed toward the construction of models which are themselves "explanatory" in a proper scientific sense of the word, and not merely in the sense of providing some parametrization of the descriptive statistics which serves merely as a clue to the task of scientific explanation.

We gain still another perspective on the concept of structural coefficients in learning how to transform the model into a different set of equations. We continue with the model presented at the beginning of this chapter. By straightforward substitution we eliminate x_2 from the x_3-equation and both x_2 and x_3 from the x_4-equation; the x_2-equation is repeated as it stands. These manipulations yield the following three equations as the *reduced form* of the model:

$$
\begin{aligned}
&(x_1 \text{ exogenous})\\
&x_2 = b_{21}x_1 + u\\
&x_3 = (b_{31} + b_{32}b_{21})x_1 + b_{32}u + v\\
&x_4 = [b_{41} + b_{42}b_{21} + b_{43}(b_{31} + b_{32}b_{21})]x_1 + (b_{42} + b_{43}b_{32})u\\
&\qquad + b_{43}v + w
\end{aligned}
$$

To obtain a compact notation for the coefficients and disturbances of the reduced form, we rewrite the foregoing equations, making use of the following definitions:

$$
\begin{aligned}
a_{21} &= b_{21}\\
a_{31} &= b_{31} + b_{32}b_{21}\\
a_{41} &= b_{41} + b_{42}b_{21} + b_{43}(b_{31} + b_{32}b_{21})\\
u' &= u\\
v' &= b_{32}u + v\\
w' &= (b_{42} + b_{43}b_{32})u + b_{43}v + w
\end{aligned}
$$

This yields

$$
\begin{aligned}
x_2 &= a_{21}x_1 + u'\\
x_3 &= a_{31}x_1 + v'\\
x_4 &= a_{41}x_1 + w'
\end{aligned}
$$

We find that the exogenous variable is uncorrelated with the reduced-form disturbances, since

$$
\begin{aligned}
E(x_1u') &= E(x_1u) = 0\\
E(x_1v') &= b_{32}E(x_1u) + E(x_1v) = 0\\
E(x_1w') &= (b_{42} + b_{43}b_{32})E(x_1u) + b_{43}E(x_1v) + E(x_1w) = 0
\end{aligned}
$$

making use of the initial specification on the disturbances in the model. However, in the reduced form (unlike the *structural form* in which the model was originally specified), it is no longer true that disturbances

are uncorrelated among themselves. In fact, we can derive explicit expressions for the covariances among reduced-form disturbances (recalling that the covariances among structural-form disturbances are zero):

$$
\begin{aligned}
\sigma_{u'v'} &= E(u'v') = b_{32}E(u^2) + E(uv) = b_{32}\sigma_{uu}\\
\sigma_{u'w'} &= (b_{42} + b_{43}b_{32})\sigma_{uu}\\
\sigma_{v'w'} &= b_{32}(b_{42} + b_{43}b_{32})\sigma_{uu} + b_{43}\sigma_{vv}
\end{aligned}
$$

The reduced-form disturbance variances likewise are functions of structural coefficients and variances of structural-form disturbances:

$$
\begin{aligned}
\sigma_{u'u'} &= \sigma_{uu}\\
\sigma_{v'v'} &= b_{32}^2\sigma_{uu} + \sigma_{vv}\\
\sigma_{w'w'} &= (b_{42} + b_{43}b_{32})^2\sigma_{uu} + b_{43}^2\sigma_{vv} + \sigma_{ww}
\end{aligned}
$$

Suppose we regard the expressions for the variances and covariances of the reduced-form disturbances, taking these quantities as known, as six equations in the six unknowns, σ_{uu}, σ_{vv}, σ_{ww}, b_{43}, b_{42}, and b_{32}. Although the equations involve nonlinear combinations of the unknowns, they are readily solved by a sequence of simple substitutions. Having solved for these parameters, we could return to the three equations defining the a's (taking them as known) and solve for the remaining unknown structural coefficients, b_{41}, b_{31}, and b_{21}.

Thus, if the reduced-form coefficients (the a's) and the variance covariance matrix of the reduced-form disturbances were known, we could solve for structural coefficients (the b's) and the variances of structural disturbances. From a computational point of view, this result is of no great practical value. Because of its conceptual interest, however, we indicate how one might proceed. Multiplying each reduced-form equation through by the exogenous variable and taking expectations, we obtain:

$$
\begin{aligned}
\sigma_{12} &= a_{21}\sigma_{11}\\
\sigma_{13} &= a_{31}\sigma_{11}\\
\sigma_{14} &= a_{41}\sigma_{11}
\end{aligned}
$$

Least squares estimates of the a's are, therefore,

$$\hat{a}_{21} = \frac{m_{12}}{m_{11}}$$

$$\hat{a}_{31} = \frac{m_{13}}{m_{11}}$$

$$\hat{a}_{41} = \frac{m_{14}}{m_{11}}$$

Multiplying through each reduced-form equation by each dependent variable and each reduced-form disturbance yields (after a little algebraic manipulation, which may serve as an exercise for the interested reader):

$$\sigma_{u'u'} = \sigma_{22} - a_{21}\sigma_{12}$$

$$\sigma_{v'v'} = \sigma_{33} - a_{31}\sigma_{13}$$

$$\sigma_{w'w'} = \sigma_{44} - a_{41}\sigma_{14}$$

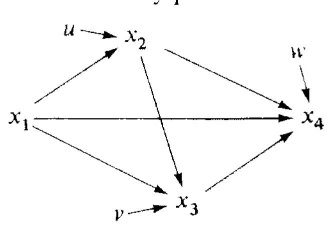
$$\sigma_{u'v'} = \sigma_{23} - a_{21}\sigma_{13}$$

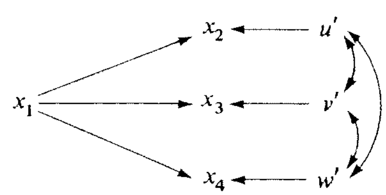
$$\sigma_{u'w'} = \sigma_{24} - a_{21}\sigma_{14}$$

$$\sigma_{v'w'} = \sigma_{34} - a_{31}\sigma_{14}$$

Thus, if we combine sample estimates of the variances and covariances of our observed variables with the least-squares estimates of the a's, we will generate estimates of the reduced-form disturbance variances and covariances. Putting these through the solution routine outlined earlier will yield estimates of structural coefficients and structural-form disturbance variances precisely the same as the estimates obtained by direct least-squares estimation of the structural equations themselves.

Comparing the path diagrams of the structural and reduced forms of the model may put some of these results in perspective:





The diagrams are equivalent in one sense for, as we have shown, given the parameters (coefficients, variances, and covariances of disturbances) of one form, we may solve for the parameters of the other. Each diagram depicts a model with ten parameters. In the structural form we have:

one variance of the exogenous variable
six structural coefficients
three variances of disturbances

In the reduced form we have:

one variance of the exogenous variable
three reduced-form coefficients
three variances of reduced-form disturbances
three covariances among reduced form disturbances

The paths in the reduced-form diagram represent (typically) some combination of compound paths in the structural-form diagram. This fact is an instructive implication of our definitions of the a's (see Figure 4.1).

Thus, the reduced-form coefficients sum up the several direct and indirect paths through which the exogenous variable exerts its effects on each dependent variable. If one cared to know only the *total* effect of the exogenous variable on a dependent variable, the reduced-form coefficient tells the whole story. But if one is interested in how that effect comes about, the greater detail of the structural model is informative. After all, in the reduced form, a great deal of the "structure" is buried in the rather uninformative variances and covariances of the reduced-form disturbances.

Exercise. *Derive the reduced form for a two-equation model consisting only of the x_3-equation and the x_4-equation in the model just discussed. Both x_1 and x_2 are exogenous, so that the variances of both and their covariance as well must be assumed to arise from exogenous sources. Express the variances and covariances of the reduced-form disturbances in terms of structural coefficients and structural-form disturbance variances. Show how the direct and indirect effects of exogenous variables (insofar as these are explicit in the model) are summed up in reduced-form coefficients. Compare the number of parameters in the structural and reduced forms.*

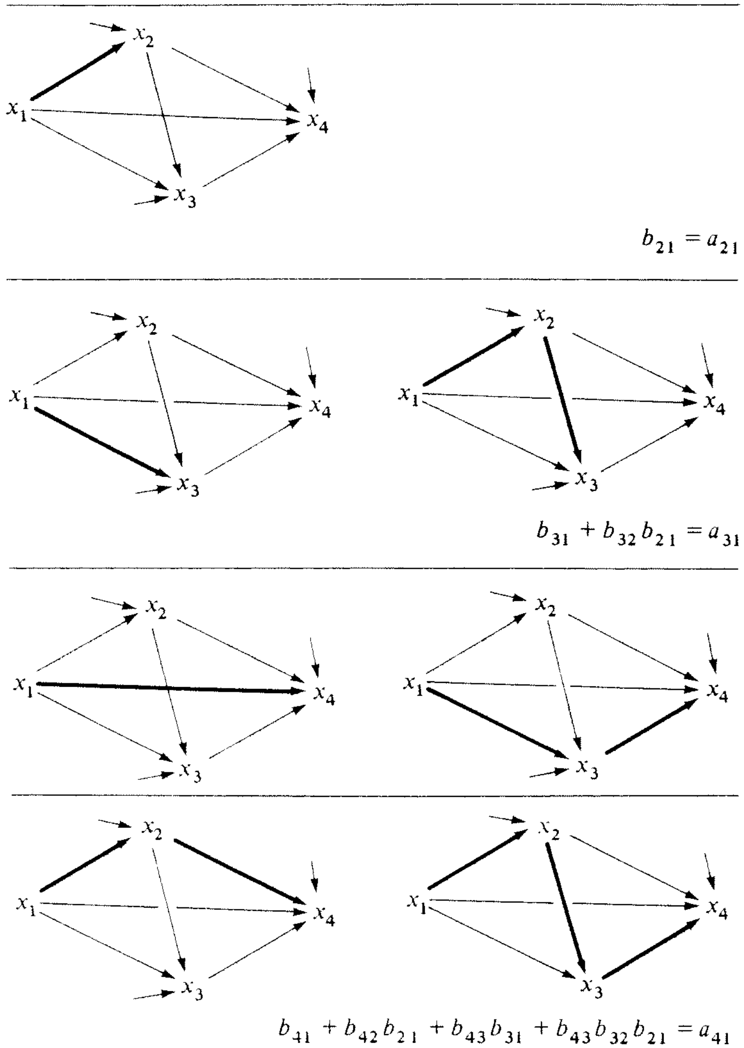


FIG. 4.1. How structural coefficients combine into reduced form coefficients in a fully recursive model

We have managed to postpone to this point a matter that many sociologists consider—erroneously, we believe—to be the single most important feature of a model, the proportion of variation in each dependent variable that is "explained." Although this emphasis upon the misnamed "explanatory power" of a model is mistaken, there is a limited utility in the multiple-correlation statistic. We proceed to indicate how it fits into the account of recursive models offered here.

Suppose the structural coefficients of the model (page 52) have been estimated by OLS. This method minimizes the sum of squares of the sample residuals and thereby insures that the residual from an estimated regression equation is uncorrelated with the regressors in that equation. Hence we have $\sum x_1 \hat{u} = \sum x_2 \hat{v} = \sum x_1 \hat{v} = \sum x_3 \hat{w} = \sum x_2 \hat{w} = \sum x_1 \hat{w} = 0$, where the summation is over the entire sample and $\hat{u}$, $\hat{v}$, and $\hat{w}$ are the sample residuals that estimate the corresponding disturbances. This allows us to operate on the equations of the model, when the structural coefficients therein are replaced by their OLS estimates, in much the same way that we have hitherto worked with the model itself. Thus, *in the sample*, we have

$$x_2 = \hat{b}_{21}x_1 + \hat{u}$$
$$x_3 = \hat{b}_{32}x_2 + \hat{b}_{31}x_1 + \hat{v}$$
$$x_4 = \hat{b}_{43}x_3 + \hat{b}_{42}x_2 + \hat{b}_{41}x_1 + \hat{w}$$

(These are, in effect, the formulas for computing $\hat{u}$, $\hat{v}$, and $\hat{w}$, respectively.) Each variable will have been expressed as a deviation from its sample mean. Multiplying each equation through by the residual and the dependent variable, we find

$$\sum x_2\hat{u} = m_{\hat{u}\hat{u}}$$
$$\sum x_3\hat{v} = m_{\hat{v}\hat{v}}$$
$$\sum x_4\hat{w} = m_{\hat{w}\hat{w}}$$
$$m_{22} = \hat{b}_{21}m_{12} + m_{\hat{u}\hat{u}}$$
$$m_{33} = \hat{b}_{32}m_{23} + \hat{b}_{31}m_{13} + m_{\hat{v}\hat{v}}$$
$$m_{44} = \hat{b}_{43}m_{34} + \hat{b}_{42}m_{24} + \hat{b}_{41}m_{14} + m_{\hat{w}\hat{w}}$$

The respective *coefficients of determination* for the three equations are

$$R^2_{2(1)} = 1 - \frac{m_{\hat{u}\hat{u}}}{m_{22}}$$

$$R^2_{3(21)} = 1 - \frac{m_{\hat{v}\hat{v}}}{m_{33}}$$

$$R^2_{4(321)} = 1 - \frac{m_{\hat{w}\hat{w}}}{m_{44}}$$

(The *multiple correlation* is the square root of R^2.) If we wished to put our results in the framework of standardized variables and path coefficients, we would proceed to note that

$$\hat{p}_{2u} = \sqrt{1 - R^2_{2(1)}}$$
$$\hat{p}_{3v} = \sqrt{1 - R^2_{3(21)}}$$
$$\hat{p}_{4w} = \sqrt{1 - R^2_{4(321)}}$$

which are the so-called residual paths.

It will be seen that the definition of R^2 rests on the distinction between variation "explained" by an equation in the model and the "unexplained" variation of that equation's dependent variable. It is sometimes suggested that the formula defining R^2 be used to effect a partitioning of the explained variation into portions due uniquely to the several determining causes. Thus, according to this suggestion, $R^2_{3(21)}$ (for example) would be allocated between x_1 and x_2 in proportion to $\hat{b}_{31} m_{13}$ and $\hat{b}_{32} m_{23}$, respectively (or, in the framework of path coefficients, $\hat{p}_{31} r_{13}$ and $\hat{p}_{32} r_{23}$). But the suggestion is mistaken. It is true that $\hat{b}_{31}$ estimates the direct effect of x_1 on x_3, but m_{13} reflects a mixture of effects arising from diverse sources. If we multiply through our estimate of the x_3-equation by x_1, we find

$$m_{13} = \hat{b}_{32} m_{12} + \hat{b}_{31} m_{11}$$

Hence

$$\hat{b}_{31} m_{13} = \hat{b}_{32} \hat{b}_{31} m_{12} + \hat{b}^2_{31} m_{11}$$

and the first term on the right is certainly not an unalloyed indicator of the role of x_1 alone in producing variation in x_3.

Indeed the "problem" of partitioning R^2 bears no essential relationship to estimating or testing a model, and it really does not add anything to our understanding of how a model works. The simplest recommendation—one which saves both work and worry—is to eschew altogether the task of dividing up R^2 into unique causal components. In a strict sense, it just cannot be done, even though many sociologists, psychologists, and other quixotic persons cannot be persuaded to forego the attempt.

Indeed the whole issue of what to make of a multiple correlation is clarified by noting that R^2 does not estimate any parameter of the model as such, but rather a parameter that depends upon both the model and the population in which the model applies. We have no reason to expect R^2 to be invariant across populations. If either σ_{ww} or σ_{11} (and, therefore, σ_{44}) changes in going to a new population, while the structural coefficients remain fixed, R^2 will be changed. This is a matter of special concern in the event that the value of σ_{11} is essentially under the investigator's control, according to whether he (for example) puts greater or lesser variance into the distribution of his experimental stimulus x_1, or samples disproportionately from different parts of the "natural" range of x_1. As we have seen, such a modification of σ_{11} will change all variances and covariances, as well as R^2, even though there is no change in structural coefficients—and there is no reason to expect them to be affected by either of these aspects of study design.

The real utility of R^2 is that it tells us something about the precision of our estimates of coefficients, since the standard error of a coefficient is a function, among other things, of R^2.

It is a mistake—the kind of mistake easily made by the novice—to focus too much attention on the magnitude of R^2. *Other things being equal*, it is, of course, true that one prefers a model yielding a high R^2 to one yielding a lower value. But the *ceteris paribus* clause is terribly important. Merely increasing R^2 by lengthening the list of regressors is no great achievement unless the role of those variables in an extended causal model is properly understood and correctly represented.

Suppose, for example, that in using the four-variable recursive model we have been studying, the investigator became dissatisfied with the low value of $R^2_{3(21)}$. It would be quite easy to get a higher value of R^2, for example, by running the regression of x_3 on x_4, x_2, and x_1, and reporting the value of $R^2_{3(421)}$. But this regression does not correspond to the causal ordering of the variables, which was required to be specified at the outset. It is, therefore, an exceedingly misleading statistic. (One does not often see the mistake in quite this crude a form. But naïvely regressing causes on effects is far from being unknown in the literature.)

Another way to raise $R^2_{3(21)}$ would be to introduce another variable, say x'_3, that is essentially an alternative measure of x_3, though giving slightly different results. The regression of x_3 on x'_3, x_2, and x_1 is then guaranteed to yield a high value of R^2.

Indeed, the best-known examples of very high correlations are those selected to convey the notion of "spurious correlation," "nonsense correlation in time series," or other kinds of artifact. This shows us that high values of R^2, in themselves, are not sufficient to evaluate a model as successful.

Before worrying too much about his R^2, therefore, the investigator does well to reconsider the entire specification of the model. If that specification cannot be faulted on other grounds, the R^2 as such is not sufficient reason to call it into question.

Exercise. *To conclude, for the time being, your study of fully recursive models, review the material on estimation and testing in Chapter 3 and restate the essential points so that they apply to the recursive model as expressed without standardization of variables (page 52).*

FURTHER READING

An example of a recursive sociological model presented in terms of both standardized and nonstandardized coefficients appears in Duncan (1969). Note that the more interesting conclusions were developed on the basis of the latter. On the questionable value of commonly used measures of "relative importance" or "unique contribution" of the several variables in an equation, see Ward (1969), Cain and Watts (1970), and Duncan (1970).

5

A Just-Identified Nonrecursive Model

The model considered throughout this chapter is

$$x_3 = b_{31}x_1 + b_{34}x_4 + u$$

$$x_4 = b_{42}x_2 + b_{43}x_3 + v$$

For convenience, $E(x_j) = 0$, $j = 1, \ldots, 4$, and $E(u) = E(v) = 0$. However, we do *not* put the variables in standard form. Variables x_1 and x_2 are *exogenous*; their variances and their covariance are not explained within the model. Variables x_3 and x_4 are *jointly dependent* or *endogenous*; the purpose of the model is to explain the behavior of these variables. Variables u and v are, respectively, the *disturbances* in the x_3-equation and the x_4-equation. Their presence accounts for the fact that x_3 and x_4 are not fully explained by their explicit determining factors. The model will be operational only if we can assume that disturbances are uncorrelated with exogenous variables: hence the specification $E(x_1 u) = E(x_1 v) = E(x_2 u) = E(x_2 v) = 0$. *This is a serious assumption.* The research worker must carefully consider what circumstances would violate it and whether his theoretical understanding of the situation under study permits him to rule out such violations.

In contrast to the case of a fully recursive model, in the nonrecursive model the specification of zero covariances between disturbances and