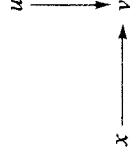

Preliminaries

In this book we will frequently use a notation like the following:



It may be read as “a change in x or u produces a change in y ” or “ y depends on x and u ” or “ x and u are the causes of y .” In all these statements, we need to include a distinction between two sorts of causes, x and u . We intend the former to stand for some definite, explicit factor (this variable has a name) producing variation in y , identified as such in our *model* of the dependence of y on its causes. On the other hand, u stands for all other sources (possibly including many different causes) of variation in y , which are not explicitly identified in the model. It sums up all their effects and serves to account for the fact that no single cause, x , nor even a finite set of causes (a list of specific x 's) is likely to explain all the observable variation in y . (The variable u has no specific name; it is just called “the disturbance.”)

The letters (like x , y , and u), the arrows, and the words (like “depends on”) are elements in a language we use in trying to specify

how we think the world—or, rather, that part of it we have selected for study—works. Once our ideas are sufficiently definite to help us make sense of the observations we have made or intend to make, it may be useful to formalize them in terms of a model. The preceding little arrow diagram, once we understand all the conventions for reading it, is actually a model or, if one prefers, a pictorial representation of a model. Such a representation has been found useful by many investigators as an aid in clarifying and conveying their ideas and in studying the properties of the models they want to entertain.

More broadly useful is the algebraic language of variables, constants, and functions, symbolized by letters and other notations, which are manipulated according to a highly developed grammar. In this language, our little model may be expressed as

$$y = bx + u$$

or, even more explicitly,

$$y = b_{yx}x + u$$

It is convenient, though by no means essential, to follow the rule that y , the “dependent variable” or “effect” is placed on the left-hand side of the equation while x , the “independent variable” or “cause,” goes on the right-hand side. The constant, or coefficient b in the equation tells us by how much x influences y . More precisely, it says that a change of one unit in x (on whatever scale we adopt for the measurement of x) produces a change of b units in y (taking as given some scale on which we measure y). When we label b with subscripts (as in b_{yx}) the order of subscripts is significant: the first named variable (y) is the dependent variable, the second (x) the independent variable.

(Warning: *Although this convention will be followed throughout this book, not all authors employ subscripts to designate the dependent and independent variables.*)

The scale on which u is measured is understood to be the same as that used for y . No coefficient for u is required, for in one sense, u is merely a balancing term, the amount added to the quantity bx to satisfy the equation. (In causal terms, however, we think of y as depending on u , and not vice versa.) This statement may be clearer if we make explicit a feature of our grammar that has been left implicit

up to now. The model is understood to apply (or, to be proposed for application) to the behavior of units in some population, and the variables y , x , and u are variable quantities or “measurements” that describe those units and their behavior. [The units may be individual persons in a population of people. But they could also be groups or collectives in a population of such entities. Or they could even be the occasions in a population of occasions as, for instance, a set of elections, each election being studied as a unit in terms of its outcome (y) and being characterized by properties such as the number of candidates on the ballot (x), for example.] We may make this explicit by supplying a subscript to serve as an identifier of the unit (like the numeral on the sweater of a football player). Then the equation of our model is

$$y_i = b_{yx}x_i + u_i$$

That is, for the i th member of the population we ascertain its score or value on x , to wit x_i , multiply it by b , and add to the product an amount u_i (positive or negative). The sum is equal to y_i , or the score of the i th unit on variable y . Ordinarily we will suppress the observation subscript in the interest of compactness, and the operation of summation, for example, will be understood to apply over all members of a sample of N units drawn from the population.

It is assumed that the reader will have encountered notation quite similar to the foregoing in studying the topic of regression in a statistics course. (Such study is a prerequisite to any serious use of this book.) But what we have been discussing is not statistics. Rather, we have been discussing the form of one kind of model that a scientist might propose to represent his ideas or theory about how things work in the real world. Theory construction, model building, and statistical inference are distinct activities, sufficiently so that there is strong pressure on a scientist to specialize in one of them to the exclusion of the others. We hope in this book to hint at reasons why such specialization should not be carried too far. But we must note immediately some reasons why the last two may come to be intimately associated.

Statistics, in one of its several meanings, is an application of the theory of probability. Whenever, in applied work—and all empirical inquiry is “applied” in this sense—we encounter a problem that probability theory may help to solve, we turn to statistics for guidance.

There are two broad kinds of problems that demand statistical treatment in connection with scientific use of a model like the one we are discussing. One is the problem of inference from samples. Often we do not have information about all units in a population. (The population may be hypothetically infinite, so that one could never know about "all" units; or for economic reasons we do not try to observe all units in a large finite population.) Any empirical estimate we may make of the coefficient(s) in our model will therefore be subject to sampling error. Any inference about the form of our model or the values of coefficients in it that we may wish to base on observational data will be subject to uncertainty. Statistical methods are needed to contrive optimal estimators and proper tests of hypotheses, and to indicate the degree of precision in our results or the size of the risk we are taking in drawing a particular conclusion from them.

The second, not unrelated, kind of problem that raises statistical issues is the supposition that some parts of the world (not excluding the behavior of scientists themselves, when making fallible measurements) may be realistically described as behaving in a stochastic (chance, probabilistic, random) manner. If we decide to build into our models some assumption of this kind, then we shall need the aid of statistics to formulate appropriate descriptions of the probability distributions.

This last point is especially relevant at this stage in the presentation of our little model, for there is one important stipulation about it that we have not yet stated. We think of the values of u as being drawn from a probability distribution. We said before that, for the i th unit of observation, u_i is the amount added to bX_i to produce y_i . Now we are saying that u_i itself is produced by a process that can be likened to that of drawing from a large set of well-mixed chips in a bowl, each chip bearing some value of u . The description of our model is not complete until we have presented the "specification on the disturbance term," calling u the "disturbance" in the equation (for reasons best known to the econometricians who devised the nomenclature), and meaning by the "specification" of the model a statement of the assumptions made about its mathematical form and the essential stochastic properties of its disturbance.

Throughout this book, we will assume that the values of the disturbance are drawn from the same probability distribution for all units in

the population. This subsumes, in particular, the assumption of "homoskedasticity." It can easily be wrong in an empirical situation, and tests for departures from homoskedasticity are available. When the assumption is too wide of the mark, special methods (for example, transformation of variables, or weighting of regression estimators) are needed to replace the methods sketched in this book. No special attention is drawn to this assumption in the remainder of the text: but the reader must not forget it, nonetheless. Another assumption made throughout is that the mean value of the disturbance in the population is zero. The implications of assumptions about the disturbance are discussed in Chapter 11.

Frequently we shall assume explicitly that the disturbance is uncorrelated with the causal variable(s) in a model, although this assumption will be modified when the logic of the situation requires. Thus for the little model under study now, we specify that $E(xu) = 0$. (E is the sign for the expectation operator. If the reader is not familiar with its use in statistical arguments, he should look up the properties of the operator in an intermediate statistics text such as Hays, 1963, Appendix B.) The assumption that an explanatory or causal variable is uncorrelated with the disturbance must always be weighed carefully. It may be negated by the very logic of the model, as already hinted. If it is supposed, not only that y depends on x , but also that x simultaneously depends on y , it is contradictory to assume that the disturbance in the equation explaining y is uncorrelated with x . The specification $E(xu) = 0$ may also be contrary to fact, even when it is not inherently illogical. The difficulty is that we will never know enough about the facts of the case to be sure that the assumption is true—that would be tantamount to knowing everything about the causes of y . Lacking omniscience, we rely on theory to tell us if there are substantial reasons for faulting the assumption. If so, we shall have to eschew it—however convenient it may be—and consider how, if at all, we may modify our model or our observational procedures to remedy the difficulty. For we must have *this assumption in the model in some form*—though not necessarily in regard to all causal variables—if *any statistical procedures* (estimation, hypothesis testing) *are to be justified*. Here we distinguish sharply between (1) statistical description, involving summary measures of the joint distributions of observed variables, which may serve the useful purpose of data reduction, and (2) statistical methods

applied to the problem of estimating coefficients in a *structural model* (as distinct from a "statistical model") and testing hypotheses about that model. One can do a passably good job of the former without knowing much about the subject matter (witness the large number of specialists in "multivariate data analysis" who have no particular interest in any substantive field). But one cannot even get started on the latter task without a firm grasp of the relevant scientific theory, because the starting point is, precisely, the model and not the statistical methods.

In summary, we have proposed a model,

$$y = b_{yx}x + u$$

and stated a specification on its disturbance term, $E(xu) = 0$. Without mentioning it before, we have also been assuming that $E(x) = 0$, which is simply a convention as to the location of the origin on the scale of the independent variable. It follows at once that

$$E(y) = b_{yx}E(x) + E(u) = 0.$$

Now, each of the variables in our model has a variance, and it is convenient to adopt the notation,

$$\begin{aligned}\sigma_{yy} &= E(y^2) \\ \sigma_{xx} &= E(x^2) \\ \sigma_{uu} &= E(u^2)\end{aligned}$$

for the variances (writing σ_{yx} , for example, in place of the usual σ_y^2). There are also three covariances,

$$\begin{aligned}\sigma_{yx} &= E(yx) \\ \sigma_{yu} &= E(yu) \\ \sigma_{xu} &= E(xu) = 0\end{aligned}$$

The disappearance of the last of these covariances is merely a restatement of the original specification on the disturbance term. To evaluate σ_{yu} we multiply the equation by u and take expectations, finding

$$E(yu) = b_{yx}E(xu) + E(uu)$$

so that $\sigma_{yu} = \sigma_{uu}$, in view of the fact that $E(xu) = 0$. Let us multiply through the equation of our model by y , obtaining,

$$y^2 = b_{yx}xy + yu$$

We take expectations

$$E(y^2) = b_{yx}E(xy) + E(yu)$$

and thereby find that we can write the variance of y as

$$\sigma_{yy} = b_{yx}\sigma_{xy} + \sigma_{uu}$$

since $\sigma_{yu} = \sigma_{uu}$ as already noted. Let us next multiply through by x and take expectations. We find

$$E(xy) = b_{yx}E(x^2) + E(xu)$$

or

$$\sigma_{xy} = b_{yx}\sigma_{xx}$$

Substituting this result into the expression for the variance of y , we obtain

$$\sigma_{yy} = b_{yx}^2\sigma_{xx} + \sigma_{uu}$$

(The same result is obtained upon squaring both sides of $y = b_{yx}x + u$ and taking expectations.)

The three symbols on the right-hand side stand for the basic parameters of this model as it applies in a well-defined *population*:

- the structural coefficient b_{yx} ,
- the variance of the exogenous variable x ,
- the variance of the disturbance u .

The variance in the dependent variable is traceable to these three distinct sources.

The expression just obtained for the covariance of the two observable variables is a suggestive one, for we can immediately rewrite it as

$$b_{yx} = \frac{\sigma_{xy}}{\sigma_{xx}}$$

We see that if we knew σ_{xy} and σ_{xx} we could calculate the value of the structural coefficient. We do not and, in general, cannot know these quantities exactly. But we can estimate them, or their ratio, from data

pertaining to a *sample* of the population to which the model applies. How to use this sample information in a correct and efficient manner is a topic studied in the statistical theory of estimation. In this book, we will draw upon a few important results from that theory, but will not try to demonstrate those results.

Exercise. Our model

$$y = bx + u$$

could be solved for x , to read

$$x = \frac{1}{b}y - \frac{1}{b}u$$

Let $1/b$ be renamed c and $-u/b$ be called v . Someone could, therefore, assert that our model is equally well written

$$x = cy + v$$

But on the assumption that our original model is true (including the specification on the disturbance term) show that the disturbance is not uncorrelated with the variable on the right-hand side, that is, $E(vy) \neq 0$. Show also that we cannot solve for c using the same kind of formula developed for the original model, that is, $c \neq \sigma_{xy}/\sigma_{yy}$. How do you square this result with the well-known fact in statistics, that there are two regressions, Y on X and X on Y ?

FURTHER READING

The statistical methods of simple and multiple regression are well presented in Snedecor and Cochran (1967, Chaps. 6, 7, and 13). A judicious discussion of issues raised in the use of the regression model to represent a causal relationship is given by Rao and Miller (1971, Chap. 1).

2

Correlation and Causation

Partly for historical reasons, the topic of structural equation models has often been approached by considering the implications of causal relationships for observable correlations or, inversely, the problem of rendering a causal interpretation of observed correlations. Taking the correlation coefficient as a point of departure has been particularly characteristic of psychometrics. But sociology, as well, has depended heavily on this measure of the degree of linear association between two quantitative variables. We suggest in Chapter 4 that a more fundamental view of structural equation models is secured by foregoing the algebra of correlation. Nevertheless, some aspects of our topic are quite convenient to develop in terms of correlation, so that is the way we shall begin.

Let us reconsider briefly the illustrative model studied in Chapter 1:

$$y = b_{yx}x + u$$

We already specified that $E(x) = E(u) = 0$ [whence it follows that $E(y) = 0$], and that $E(xu) = 0$. Let us now suppose that each variable (y , x , and u) is to be reexpressed in units of its own standard deviation. We can make the equality hold, while retaining the form of the model,

by introducing ratios of standard deviations in the following fashion:

$$\frac{y}{\sigma_y} = b_{yx} \frac{x}{\sigma_x} + \frac{\sigma_u}{\sigma_y} \frac{u}{\sigma_u}$$

The expressions

$$b_{yx} \frac{\sigma_x}{\sigma_y} = \rho_{yx}$$

$$\frac{\sigma_u}{\sigma_y} = \rho_{yu}$$

were termed *path coefficients* by Sewall Wright (1921, 1960, 1968), the great pioneer in the development of structural equation models (Li, 1956, 1968; Goldberger, 1972a). Let us now revise our notation, so that y , x , and u are the *standardized values* of the dependent variable, the independent variable, and the disturbance. Henceforth, in Chapters 2 and 3 only, we shall suppose that $E(y^2) = E(x^2) = E(u^2) = 1$, since the variance of a standardized variable is unity.

Writing our model in terms of path coefficients and standardized variables, we have

$$y = \rho_{yx}x + \rho_{yu}u$$

with the specification $E(xu) = 0$, which is not affected by standardization. (*Why?*) We may study the properties of this model, as before, by multiplying through the equation by one or another of the variables, taking expectations, and simplifying. The last step calls upon the theorem that the covariance of two standardized variables is the coefficient of correlation. Hence, using the letter *r* to designate the correlation between two variables in the population under study (reserving the letter *r* for the corresponding correlation in a sample), we have

$$E(xy) = \rho_{xy}$$

We see at once that, since

$$E(xy) = \rho_{yx}E(x^2) + \rho_{yu}E(xu)$$

we have an equality of the path coefficient and the correlation,

$$\rho_{xy} = \rho_{yx}$$

recalling that $E(xu) = \rho_{xu} = 0$. (We hasten to add that the path coefficient has the same value as the correlation because this model is one with only a single explanatory variable; this will not hold true in general.) In writing path coefficients, as for structural coefficients, the first subscript refers to the variable affected, the second to the causal variable. In writing simple correlations, the order of subscripts is immaterial, since $\rho_{xy} = \rho_{yx}$.

The variance of the dependent variable is given by

$$E(y^2) = \rho_{yx}E(yx) + \rho_{yu}E(yu)$$

But, since $E(y^2) = 1$, we have

$$1 = \rho_{yx}\rho_{yx} + \rho_{yu}\rho_{yu}$$

Multiplying the equation of the model through by u , we find

$$E(yu) = \rho_{yx}E(xu) + \rho_{yu}E(uu)$$

so that

$$\rho_{yu} = \rho_{yu}$$

Hence, with one dependent variable, one explanatory variable, and the disturbance, we find that the variance (set at unity) of the dependent variable is partitioned into

$$1 = \rho_{yx}^2 + \rho_{yu}^2$$

the "explained" (by x) and "unexplained" portions.

As the reader must have guessed, this little model has been used primarily to illustrate notation, nomenclature, and the basic techniques to be used henceforth in studying properties of a model. It is time to complicate the discussion, and we do so by introducing a third explicit variable in addition to x and y . Three-variable models are interesting mainly for didactic purposes. But they illustrate all the essential principles that apply in more elaborate models.

Let us call the three variables x , y , and z and assume that each is in standard form. For the time being, we will suppose that nothing is known about the causal relationships—which (if any) of these variables cause(s) any of the others. It has sometimes been thought that correlations among variables can be employed in a quasi-deductive logic: If A and B are positively correlated, and if B and C are positively

correlated, then it is quite likely that A and C are positively correlated. Whatever its plausibility in any particular concrete instance, this mode of reasoning clearly is not rigorous in general. If ρ_{xy} , ρ_{xz} , and ρ_{yz} are the correlations among three variables, then it can be shown that a certain determinant must be nonnegative; that is, that

$$\begin{vmatrix} 1 & \rho_{xy} & \rho_{xz} \\ \rho_{xy} & 1 & \rho_{yz} \\ \rho_{xz} & \rho_{yz} & 1 \end{vmatrix} = 1 + 2\rho_{xy}\rho_{yz}\rho_{xz} - \rho_{xy}^2 - \rho_{yz}^2 - \rho_{xz}^2 \geq 0$$

This theorem places only very broad constraints on the possible range of values of any one of the correlations. To see this, put in hypothetical values for two of the correlations, evaluate the determinant, and analyze the result. For example, if $\rho_{xz} = \rho_{yz} = .5$, we have

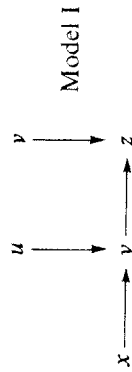
$$1 - 2\rho_{xy}^2 + \rho_{xy} \geq 0$$

so that ρ_{xy} can have any value between $-.5$ and $+1.0$. If $\rho_{xz} = \rho_{yz} = \sqrt{.5}$ (about .707), we have

$$\rho_{xy} - \rho_{xy}^2 \geq 0$$

so that ρ_{xy} is constrained to fall between zero and unity. Thus, one must assume rather high values for two of the correlations in order to decide with certainty even the sign of the third correlation.

If, on the other hand, we are in a position to assume causal relationships among the variables, rather strong deductions may be possible. To illustrate, consider the diagram



We interpret this as a representation of a two-equation model,

$$\begin{aligned} y &= \rho_{yx}x + \rho_{yu}u \\ z &= \rho_{zy}y + \rho_{zt}t \end{aligned} \quad \text{Model I}$$

where the p 's are path coefficients and the specification on the disturbances is $\rho_{ux} = \rho_{zx} = \rho_{ty} = 0$. That is, each disturbance is uncorrelated with all "prior" causal variables. (But note that neither ρ_{uy} ,

ρ_{uz} , nor ρ_{tz} is zero.) Let us substitute the value of y , as given by the y -equation, into the z -equation:

$$z = \rho_{zy}\rho_{yx}x + \rho_{zy}\rho_{yu}u + \rho_{zt}t$$

Multiply through by x and take expectations:

$$\rho_{zx} = \rho_{zy}\rho_{yx}$$

since $\rho_{xu} = \rho_{xt} = 0$ and $E(x^2) = 1.0$. But both the y -equation and the z -equation are just like the model studied previously. So we already know that $\rho_{zy} = \rho_{yz}$ and $\rho_{yx} = \rho_{xy}$. Hence, we conclude that for this "simple causal chain" model

$$\rho_{zx} = \rho_{zy}\rho_{yx}$$

whatever the values of the two correlations on the right-hand side.

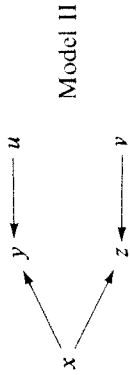
It is a plausible conjecture that when we attempt to reason from the values of two correlations to the value of a third, we must actually be working with an implicit causal model. With the right kind of causal model, the reasoning is valid. Without one, the reasoning is loose. Would it not be advantageous in such cases to make the causal model explicit, so as to be able to check our reasoning carefully?

In general, the advantages of having the model explicit are seen to lie in (1) making our arguments consistent (so that we are not altering our premises surreptitiously in the course of a discussion), (2) making our conclusions precise (so that it is easier to see what evidence is, and what is not, compatible with them), and thereby (3) rendering our conclusions susceptible of empirical refutation.

In the case of our causal chain, if $\rho_{zy} = .7$ and $\rho_{yx} = .6$, we infer that $\rho_{zx} = .42$. The conclusion is, indeed, precise. If it is belied by the facts, we must question one or more of the premises, for the argument itself is explicit and unexceptionable. There are two difficulties, one material, the other logical. The material difficulty is that we can never know ρ_{zx} exactly but can only estimate it from sample data. Hence the statement $\rho_{zx} \neq .42$ is based on a statistical inference and is, therefore, subject to uncertainty (the degree of uncertainty, however, we may hope to estimate by statistical methods). The logical difficulty is that, having rejected the conclusion of our argument because it is contradicted by the facts, we do not know which one(s) of the premises is (are) in error. We may have entered into the argument with erroneous values of

either ρ_{xy} or ρ_{yx} or both. Moreover, we may have taken the wrong model as the basis for the argument. If, however, we are confident of the facts as to the two correlations and also reasonably confident that the conclusion is unsupported by evidence, we must reject the model, realizing the third advantage claimed for making the model explicit—albeit an advantage the full enjoyment of which calls for a certain taste for irony, if not masochism.

Consider another diagram:



Again, we have two equations, but this time they have the same causal variable:

$$\begin{aligned} y &= \rho_{yx}x + \rho_{yu}u \\ z &= \rho_{zx}x + \rho_{zv}v \end{aligned} \quad \text{Model II}$$

In each equation we specify a zero correlation between the causal variable and the disturbance, so that $\rho_{xu} = \rho_{xv} = 0$. We further specify a zero correlation between the disturbances of the two equations, $\rho_{uv} = 0$.

Exercise. Show that in the case of Model I it was unnecessary to make this condition explicit, since it was implied by the specification, $\rho_{ux} = \rho_{vx} = \rho_{xy} = 0$.

We now multiply through each equation by each variable in the model, take expectations, and express the result in terms of correlations.

Exercise. Verify the following results.

$$\begin{aligned} \rho_{yx} &= \rho_{yx} \\ \rho_{zx} &= \rho_{zx} \\ \rho_{yz} &= \rho_{yx}\rho_{zx} \\ \rho_{uz} &= \rho_{vy} = 0 \\ \rho_{uy} &= \rho_{yu} \\ \rho_{vz} &= \rho_{zv} \end{aligned}$$

We also find:

$$\begin{aligned} E(y^2) &= 1 = \rho_{yx}^2 + \rho_{yu}^2 \\ E(z^2) &= 1 = \rho_{zx}^2 + \rho_{zv}^2 \end{aligned}$$

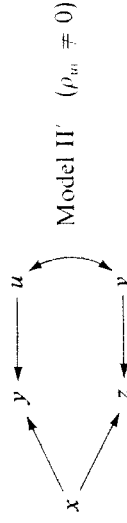
The key result for Model II is that $\rho_{yz} = \rho_{yx}\rho_{zx}$. Again we see the possibility of rejecting the model, upon discovering that the value of one of the correlations is inconsistent with the values of the other two, assuming Model II is true.

It is worth noting that this test of Model II is equivalent to a test of the null hypothesis that the partial correlation $\rho_{yz|x} = 0$. This is interesting because the tradition of empirical sociology has placed strong emphasis upon partial correlation, partial association, and “partializing” in general as a routine analytical procedure. By contrast, the approach taken in this book leaves rather little scope for partial correlation as such, although it does attempt to exploit the insights into causal relationships that often guide the capable analyst working with partial correlations.

To show the connection of partial correlation with Model II, recall the definition of a first-order partial correlation in terms of simple correlations:

$$\rho_{yz|x} = \frac{\rho_{yz} - \rho_{yx}\rho_{zx}}{\sqrt{1 - \rho_{yx}^2} \sqrt{1 - \rho_{zx}^2}}$$

We see at once that, if Model II holds, the numerator of this partial correlation is zero, since $\rho_{yz} = \rho_{yx}\rho_{zx}$, and thus $\rho_{yz|x} = 0$. Let us suppose that Model II has been rejected since $\rho_{yz|x}$ in the population under study clearly is not zero. We can represent the situation by relinquishing the specification $\rho_{uv} = 0$ which was originally part of the model. We require a new path diagram.



in which the curved, double-headed arrow linking u and v stands for the correlation (ρ_{uv}) between these two variables (leaving unresolved the issue of whether that correlation has any simple causal interpreta-

tion). We note that the equations of Model II' can be solved, respectively, for u and v ,

$$u = \frac{v - \rho_{yx}x}{\rho_{yu}}$$

$$v = \frac{z - \rho_{zx}x}{\rho_{zv}}$$

from which it is but a short step to

$$E(uv) = \rho_{uv} = \frac{\rho_{yz} - \rho_{yx}\rho_{zx}}{\rho_{yu}\rho_{zv}}$$

in view of results already obtained. We now note that

$$\rho_{yu}^2 = 1 - \rho_{yx}^2 = 1 - \rho_{xy}^2$$

and

$$\rho_{zv}^2 = 1 - \rho_{zx}^2 = 1 - \rho_{xz}^2$$

so that we may write

$$\rho_{uv} = \frac{\rho_{yz} - \rho_{yx}\rho_{zx}}{\sqrt{1 - \rho_{yx}^2}\sqrt{1 - \rho_{zx}^2}}$$

But this is precisely the formula for $\rho_{yz,x}$ given earlier.

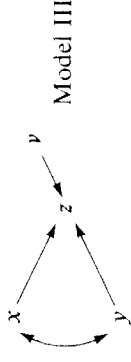
We see that Model II is a way of expressing the hypothesis that a single common cause (here called x) exactly accounts for the entirety of the correlation between two other variables (here, y and z). Model II' asserts that y and z do indeed share x as a common cause, but that some other factor or factors, unrelated to x , also serve to induce a correlation between them. Note that the hypothesis underlying Model II is subject to a direct test, once we have sufficiently reliable estimates of the three observable correlations ρ_{yz} , ρ_{yx} , and ρ_{zx} . Model II' is not so easily rejected. Indeed, one can (if one wishes) defend the truth of Model II' in the face of any conceivable set of three correlations (that is, correlations that satisfy the condition stated for the determinant on page 12). To call Model II' seriously into question really requires that we develop and justify a different model such that, if the new model is correct, Model II' cannot be true. (When you think you have learned enough about the technique of working with structural equation

models to try your hand at it, you should undertake the exercise of posing a countermodel to Model II', such that a set of observable correlations *could* show decisively that, if the new model is true, Model II' cannot be.)

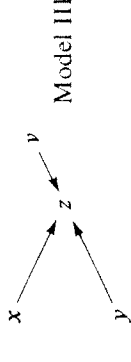
We consider next a model in which one variable has the other two as causes:

$$z = \rho_{zy}y + \rho_{zx}x + t \quad \text{Model III}$$

It takes only one equation to write this model. Since the equation has two variables that are causally prior to z , we specify that both are uncorrelated with the disturbance term: $\rho_{yt} = \rho_{xt} = 0$. The model as such says nothing about the correlation of y and x . In the absence of definite information to the contrary, we would do best to assume that this correlation may not be zero. Hence, the path diagram will be drawn as



In some previous literature (for example, Blalock, 1962-1963), the contrary assumption led to Model III', as below, with $\rho_{xy} = 0$.



Model III' has the ostensible advantage that one can test and possibly reject it. The "test" would consist merely in ascertaining whether a set of sample data are consistent with the hypothesis, $\rho_{xy} = 0$. But this "test" does not get at the *causal* properties of the model in any way. Indeed, one should think of the value of ρ_{xy} as being entirely exogenous in this model. The model might well hold good in different populations with varying values (not excluding zero) of ρ_{xy} . We will have to work harder to get a cogent test of Model III.

Let us multiply through the equation of Model III by each of the causal variables, take expectations, and express the result in terms of

correlations. (By now, this technique should be quite familiar.) We find

$$\begin{aligned} \rho_{yz} &= p_{zy} + p_{zx}p_{xy} \\ \rho_{xz} &= p_{zy}p_{xy} + p_{zx} \end{aligned} \quad (\text{from Model III})$$

This set of equations may be looked at from two viewpoints. First, it shows how the correlations involving the dependent variable are generated by the path coefficients and the exogenous correlation, ρ_{xy} . Second, if we know the values of the three correlations in a population to which Model III applies, we may solve these equations uniquely for the path coefficients:

$$\begin{aligned} p_{zy} &= \frac{\rho_{zy} - \rho_{zx}p_{xy}}{1 - \rho_{xy}^2} \\ p_{zx} &= \frac{\rho_{zx} - \rho_{zy}p_{xy}}{1 - \rho_{xy}^2} \end{aligned}$$

We note that any conceivable set of three correlations (that is, correlations meeting the condition on the determinant on page 12) will be consistent with the truth of Model III. (It may happen, incidentally, that a path coefficient will have a value greater than unity or less than -1.0 , for this coefficient is not constrained in the same way that a correlation is.) Thus to call Model III into question—other than by questioning the plausibility of estimated values of the path coefficients—really requires one to pit it against some alternative model, supported by equally strong or stronger theoretical considerations, such that if the new model is true Model III is necessarily faulty. (We have not yet developed a broad enough grasp of the possibilities to exemplify this strategy here; but upon completing your study of this book, you should be able to employ it with confidence.)

We have now surveyed the whole field of possibilities for the class of three-variable models involving just two causal paths and no “feedback” or “reciprocal” relationships. Other models of the kinds already studied can be developed, however, merely by interchanging the positions of the variables in Model I, II, or III. It is instructive to study the entire set of models that can be formulated in this way.

In the diagrams in Table 2.1 the disturbances are not given letter names (as they would be in the algebraic representation of these models), but are symbolized by the arrows that originate outside the

Table 2.1 Three-Variable Recursive Models with 2 Causal Arrows

Form of model	Case A	Case B	Case C
Model I			
(i)			
(ii)			
	$(\rho_{xz} = \rho_{xy}\rho_{yz})$	$(\rho_{xy} = \rho_{xz}\rho_{zy})$	$(\rho_{yz} = \rho_{yx}\rho_{xz})$
Model II			
	$(\rho_{yz} = \rho_{xy}\rho_{xz})$	$(\rho_{xz} = \rho_{xy}\rho_{yz})$	$(\rho_{xy} = \rho_{xz}\rho_{yz})$
Model III			

diagram. The specifications on the disturbances are the same, in form, as those already stated for Models I, II, and III.

Exercise. Derive the conditions written below Models IB, IC, IIB, and IIC. For each of Models IIIB and IIIC, derive a pair of equations expressing correlations in terms of path coefficients.

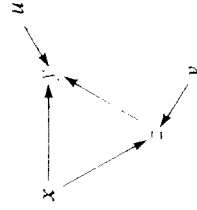
With the aid of this figure, it is worth pondering the problem of "causal inference." Suppose that one does not know which variables cause which other variables, but does know the values of the correlations among the three variables in some population. Imagine that these correlations satisfy the condition given for Model IA(i) ($\rho_{xz} = \rho_{xy}\rho_{yz}$). Should one report his "discovery" that z depends on y and y in turn depends on x ? No, for this same condition is also consistent with either Model IA(ii) or Model IIB, where the causal relationship is quite different.

You should immediately leap to the conclusion that one can *never* infer the causal ordering of two or more variables knowing only the values of the correlations (or even the partial correlations!).

We can reason in the other direction, however. Knowing the causal ordering, or, more precisely, the causal model linking the variables, we can sometimes infer something about the correlations. Or, assuming a model for sake of argument, we can express its properties in terms of correlations and (sometimes) find one or more conditions that must hold if the model is true but that are potentially subject to refutation by empirical evidence. Thus any one of Models IA, IB, IC, IIA, IIB, IIC can be rejected on the basis of reliable estimates of the three correlations, if their estimated values do not come close to satisfying the condition noted in the figure. Failure to reject a model, however, does not require one to accept it, for the reason, already noted, that some other model(s) will always be consistent with the same set of data. Only if one's theory is comprehensive and robust enough to rule out all these alternatives would the inference be justified.

We turn more briefly to three-variable models with one causal arrow connecting each pair of variables in one direction or the other.

There are actually only two kinds of models. One is the recursive model of the form

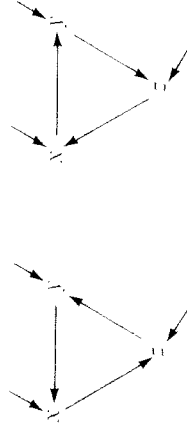


in which each variable depends on all "prior" or "predetermined" variables.

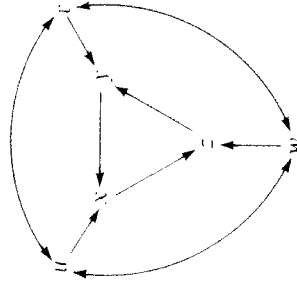
Exercise. Enumerate the six possible models of this general form, interchanging variables.

The three-variable recursive model is included within the four-variable recursive model treated in the next chapter, so we shall not investigate it further here.

The second kind of model involving three causal arrows is represented by either of the diagrams below:



As we shall discover in the next chapter, neither of these is a recursive model. For reasons that are probably not clear now, but should become so upon study of nonrecursive models, we are no longer entitled to assume that the disturbance in a particular equation is uncorrelated with the causal variable in that equation. Indeed, that assumption is not merely contrary to fact, it gives rise to logical contradictions. The kinds of deductions made for Models I, II, and III are no longer legitimate. Moreover, with nonrecursive models it is often not plausible to assume that the disturbances in the several equations are uncorrelated among themselves (although this assumption can be made if there are adequate grounds for doing so). In that event, we should confront a model like the following:



But this model is "underidentified," the meaning and implications of which condition we shall consider in Chapter 6.

Note on Partial Correlation

In the foregoing we paid no attention to partial correlations except when comparing Model II with Model II', in which the partial correlation $\rho_{y:z,x} = \rho_{zy}$ is actually a parameter of the model. This omission may seem odd to readers acquainted with the literature on "causal inference" in sociology, wherein partial correlations seem to play a central role. It is true, of course, that each of the conditions tabulated on page 19 could be restated in terms of a certain partial correlation. Thus, if, as for Model IA,

$$\rho_{xz} = \rho_{xy}\rho_{yz}$$

it will also be true that

$$\rho_{xz,y} = \frac{\rho_{xz} - \rho_{xy}\rho_{yz}}{\sqrt{1 - \rho_{xy}^2} \sqrt{1 - \rho_{yz}^2}} = 0.$$

Accordingly, we might treat the sample partial correlation $r_{xz,y}$ as a test statistic for the null hypothesis, $\rho_{xz} = \rho_{xy}\rho_{yz}$. But we shall find in Chapter 3 that it is not necessary to invoke the concept of partial correlation in order to carry out an equivalent test.

We also note that the partial correlation, unless it has a well-defined role as a parameter of the model or as a test statistic, may actually be misleading. Continuing with the example of Model IA, suppose it occurred to an investigator to estimate the partial correlation $\rho_{z,y,x}$ —not for any clearly defined purpose, but simply because it is a "good idea" to look at the partial correlations when one is working with three or more variables. We know that, by definition,

$$\rho_{z,y,x} = \frac{\rho_{zy} - \rho_{zx}\rho_{yx}}{\sqrt{1 - \rho_{zx}^2} \sqrt{1 - \rho_{yx}^2}}$$

If Model IA is true, we have seen that $\rho_{zx} = \rho_{zy}\rho_{yx}$, so we may substitute that value into the foregoing formula to see what $\rho_{z,y,x}$ would be under this model:

$$\begin{aligned} \rho_{z,y,x} &= \frac{\rho_{zy} - \rho_{zy}\rho_{yx}}{\sqrt{1 - \rho_{zx}^2} \sqrt{1 - \rho_{yx}^2}} \\ &= \rho_{zy} \sqrt{\frac{1 - \rho_{yx}^2}{1 - \rho_{zx}^2}} \end{aligned}$$

Now, if all the correlations are less than unity and $\rho_{zx} = \rho_{zy}\rho_{yx}$ (as it does when the model is true), we must have $\rho_{zx}^2 < \rho_{zy}^2$, and also $\rho_{zx}^2 < \rho_{yx}^2$. It follows that the fraction on the right-hand side is less than unity, so that

$$|\rho_{z,y,x}| < |\rho_{zy}|$$

or, if ρ_{zy} is positive,

$$\rho_{z,y,x} < \rho_{zy}$$

Now, it is not always clear what one does after "partialing," but it seems likely that the investigator who goes so far as to estimate $\rho_{z,y,x}$ will not leave the result unreported. Most likely he will announce something like, "The relationship between z and y is reduced when x is held constant," and perhaps offer reasons why this might happen. But, if Model IA is true, this result is artifactual, and the investigator's report is erroneous.

We might also note, in light of our work with Model III, that if the investigator had computed, not the partial correlation $\rho_{z,y,x}$, but the corresponding partial *regression* coefficient, the outcome would not be misleading. For the value of that regression coefficient is

$$\frac{\rho_{zy} - \rho_{zx}\rho_{yx}}{1 - \rho_{yx}^2}$$

or, upon substituting $\rho_{zy}\rho_{yx}$ for ρ_{zx} on the assumption that Model I is true,

$$\frac{\rho_{zy} - \rho_{zy}\rho_{yx}^2}{1 - \rho_{yx}^2} = \rho_{zy} = \rho_{zy}$$

That is, the partial regression coefficient is the same as the path coefficient, as it should be for this kind of model, and the numerical result is intelligible in terms of the model, while the one for the partial correlation is misleading.

We find, therefore, that partial correlations may only be a distraction when studying some of the models treated in this chapter (see also Duncan, 1970). Indeed the very concept of correlation itself has a subordinate role in the development of structural equation models. Although in Chapter 3 the presentation of recursive models does fea-

ture correlations, in subsequent chapters both recursive and nonrecursive models are treated from a different and more fundamental point of view.

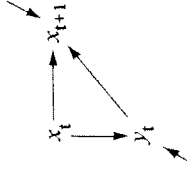
FURTHER READING

The classic exposition of the causal structures that may underlie the correlations among three variables is that of Simon (1954). A didactic presentation of four-variable models much in the spirit of this chapter is given by Blalock (1962-1963). Both papers are reprinted in Blalock (1971).

3

Recursive Models

A model is said to be recursive if all the causal linkages run "one way," that is, if no two variables are reciprocally related in such a way that each affects and depends on the other, and no variable "feeds back" upon itself through any indirect concatenation of causal linkages, however circuitous. However, recursive models do cover the case in which the "same" variable occurs at two distinct points in time, for, in that event, we would regard the two measurements as defining two different variables. For example, a dynamic model like the following:



where t and $t + 1$ are two points in time, is recursive (even though x appears to feed back upon itself). The definition also subsumes the case in which there are two or more ostensibly contemporaneous dependent variables where none of them has a direct or indirect causal