LECTURE 4: THE GENERAL LISREL MODEL

I. QUICK REVIEW OF A LITTLE MATRIX ALGEBRA.
II. A SIMPLE RECURSIVE MODEL IN LATENT VARIABLES.
III. THE GENERAL LISREL MODEL IN MATRIX FORM.
 A. SPECIFYING STRUCTURAL EQUATIONS.
 B. COMPUTING MOMENTS IN TERMS OF PARAMETERS.

In this lecture, we want to lay out the general LISREL model in matrix form. To do this, we need to review a little elementary matrix algebra. I'll then lay out a simple example in scalar algebra, and show how it is represented in matrix algebra. We'll conclude by computing moments in terms of parameters to get the LISREL model. Here's what we'll end up with: Σ = Σ(θ), where

$$\Sigma(\theta) = \begin{bmatrix} \Lambda_y (\mathrm{I} - \mathrm{B})^{-1} (\Gamma \Phi \Gamma' + \Psi)(\mathrm{I} - \mathrm{B})^{-1\prime} \Lambda_y' + \Theta_\varepsilon & \Lambda_y (\mathrm{I} - \mathrm{B})^{-1} \Gamma \Phi \Lambda_x \\ \Lambda_x \Phi \Gamma' (\mathrm{I} - \mathrm{B})^{-1} \Lambda_y' & \Lambda_x \Phi \Lambda_x' + \Theta_\delta \end{bmatrix}$$

I. QUICK REVIEW OF A LITTLE MATRIX ALGEBRA.

Matrix order:

(**r**ows × **c**olumns)

$$\underset{\substack{(r \times c)\\ 3 \times 2}}{S} = \begin{matrix} row\ 1 \\ row\ 2 \\ row\ 3 \end{matrix} \overset{\begin{matrix}\text{col 1} & \text{col 2}\end{matrix}}{\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \\ s_{31} & s_{32} \end{bmatrix}}$$

A few common matrices

Square matrix contains the same number of rows and columns (r x c), which is the matrix's order.

$$\underset{(3 \times 3)}{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

Triangular matrix contains zeros either above or below the diagonal:

$$\underset{(3 \times 3)}{T} = \begin{bmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

Diagonal matrix is a square matrix with off-diagonal elements all equal to zero:

$$\underset{(3 \times 3)}{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

Identity matrix is a diagonal matrix with ones on the diagonal:

$$\underset{(3 \times 3)}{I} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A symmetric matrix, such as a covariance or correlation matrix, is one in which each row is identical to its corresponding column. The transpose of a symmetric matrix is the matrix: S' = S.

$$\underset{(3\times 3)}{S} = \begin{bmatrix} s_{11} & a & b \\ a & s_{22} & c \\ b & c & s_{33} \end{bmatrix}$$

Transpose (interchanges rows and columns):

$$\underset{(3\times 2)}{\Gamma} = \begin{bmatrix} \gamma_{11} & 0 \\ \gamma_{21} & 0 \\ 0 & \gamma_{32} \end{bmatrix} \qquad \underset{(2\times 3)}{\Gamma'} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ 0 & 0 & \gamma_{23} \end{bmatrix}$$

1. (A')' = A (transpose of the transpose of a matrix is the matrix)

Addition and subtraction (must have *same order* p x q):

$$\underset{(2\times 2)}{B} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \quad \underset{(2\times 2)}{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \underset{(2\times 2)}{I - B} = \begin{bmatrix} 1-\beta_{11} & -\beta_{12} \\ -\beta_{21} & 1-\beta_{22} \end{bmatrix}$$

Properties of addition and subtraction:

1. A + B = B + A Commutative
2. (A + B) + C = A + (B + C) Associative
3. (A + B)' = A' + B' Transpose

Multiplication (must be conformable): $\underset{(pxq)}{A}\ \underset{(qxt)}{B} = \underset{(pxt)}{C}$

$$\underset{(2\times 2)}{c\,B} = \begin{bmatrix} c\,\beta_{11} & c\,\beta_{12} \\ c\,\beta_{21} & c\,\beta_{22} \end{bmatrix}, \text{ where c is a scalar}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad Y' = [y_1 \quad y_2 \quad y_3] \qquad \underset{(3\times 1)}{X}\underset{(1\times 3)}{Y'} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{bmatrix}$$

$$\underset{(2\times 2)}{B} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \qquad \underset{(2\times 3)}{\Gamma} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \end{bmatrix}$$

$$\underset{(2\times 2)}{B}\underset{(2\times 3)}{\Gamma} = \begin{bmatrix} \beta_{11}\gamma_{11}+\beta_{12}\gamma_{21} & \beta_{11}\gamma_{12}+\beta_{12}\gamma_{22} & \beta_{11}\gamma_{13}+\beta_{12}\gamma_{23} \\ \beta_{21}\gamma_{11}+\beta_{22}\gamma_{21} & \beta_{21}\gamma_{12}+\beta_{22}\gamma_{22} & \beta_{21}\gamma_{13}+\beta_{22}\gamma_{23} \end{bmatrix}$$

Properties of multiplication:

1. (A B)' = B' A' A' B' = (B A)' (A B C) = C' B' A' Transpose
2. (A B) C = A (B C) Associative
3. A B ≠ B A Non-communative (except in special cases)
4. A (B + C) = A B + A C Distributive

5. I A = A   A I = A   Identity Matrix (square diagonal matrix with ones on the diagonal, zeros elsewhere)

Trace is the sum of diagonal elements (square matrices only):

$$tr(S) = \sum_{i=1}^{n} S_{ii}$$

$$\underset{(2 \times 2)}{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \quad tr(S) = s_{11} + s_{22}$$

Properties of trace:

1. tr(S) = tr(S')   Trace of a matrix is equal to the trace of its transpose
2. tr(S Σ) = tr(Σ S)   Communative (assuming they're conformable)
3. tr(S + Σ) = tr(S) + tr(Σ)   Distributive (assuming they're conformable)

Determinant (of a square matrix)

$|s_1| = s_1$ (determinant of a scalar is equal to the value of the scalar by definition)

$|S| = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} = s_{11}s_{22} - s_{12}s_{21}$ (subtract cross-products)

$$|S| = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} = s_{11}s_{22}s_{33} - s_{12}s_{21}s_{33} + s_{12}s_{23}s_{31} - s_{13}s_{22}s_{31} + s_{13}s_{21}s_{32} - s_{11}s_{23}s_{32}$$

For larger matrices, it gets complicated:

1. Find the minors, which are the determinants of submatrix S_{ij} when the ith row and jth column have been removed.   In the above 3 x 3 example:

$$|S_{11}| = \begin{vmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{vmatrix} = s_{22}s_{33} - s_{23}s_{32}$$

$$|S_{22}| = \begin{vmatrix} s_{11} & s_{13} \\ s_{31} & s_{33} \end{vmatrix} = s_{11}s_{33} - s_{13}s_{31}$$

   etc, for all nine minors...

2. Transform each minor into a cofactor by changing the sign according to:

   $(-1)^{i+j}\ |S_{ij}|$ when *i* + *j* = even, sign is positive; when *i* + *j* = odd, sign is negative

3. Place these cofactors into a matrix of cofactors:

$$C = \begin{bmatrix} +|S_{11}| & -|S_{12}| & +|S_{13}| \\ -|S_{21}| & +|S_{22}| & -|S_{23}| \\ +|S_{31}| & -|S_{32}| & +|S_{33}| \end{bmatrix}$$

Then take any single row or column and multiply each element of S (the original 3 x 3 matrix) by its corresponding element in C, and sum the resulting products. Let's take row one of C and multiply each element by its corresponding element in S (note the signs in C):

$$s_{11}|S_{11}| - s_{12}|S_{12}| + s_{13}|S_{13}| = s_{11}(s_{22}s_{33} - s_{23}s_{32}) - s_{12}(s_{21}s_{33} - s_{23}s_{31}) + s_{13}(s_{21}s_{32} - s_{22}s_{31})$$

Multiply this out and obtain:

$$= s_{11}\, s_{22}\, s_{33} - s_{12}\, s_{21}\, s_{33} + s_{12}\, s_{23}\, s_{31} - s_{13}\, s_{22}\, s_{31} + s_{13}\, s_{21}\, s_{32} - s_{11}\, s_{23}\, s_{32}$$

Properties of Determinants:

1. $|S'| = |S|$ The determinant of the transpose of a square matrix is the matrix.
2. If S has two identical rows or columns, $|S| = 0$
3. If S has a zero row or column, $|S| = 0$
4. $|S\,\Sigma| = |S|\ |\Sigma|$ The determinant of the product of two square matrices is equal to the product of their determinants.
5. For a diagonal matrix, D, determinant is the product of the diagonal elements: $\prod_{i=1}^{n} d_{ii}$
6. For a lower triangular matrix (zeros above the diagonal), T, determinant is the product of the diagonal elements: $\prod_{i=1}^{n} t_{ii}$
7. If the elements of a single row or column of a square matrix *S* are multiplied by a scalar *c*, the determinant is equal to $c\ |S|$. If every element is multiplied by the constant, then $|c\ S| = c^n\ |S|$.

Inverse of a square matrix:

The inverse of a matrix is the matrix analogue of scalar division.

The inverse of a square matrix S is that matrix S^{-1} that yields an identity matrix when multiplied by S:

$S\ S^{-1} = S^{-1}\ S = I$ where I = identity matrix (ones on diagonals; zeros elsewhere)

The inverse matrix can be expressed as:

$S^{-1} = (1/|S|)(\text{adj } S)$ where adj S = C': the adjoint matrix is the transpose of the matrix of cofactors

The inverse of a matrix can be obtained in four steps: **M**inor **C**ofactor **T**ranspose **D**eterminant. For a 2 x 2 matrix it is easy:

$$\underset{(2\times 2)}{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} s_{22} & s_{21} \\ s_{12} & s_{11} \end{bmatrix} \rightarrow \begin{bmatrix} s_{22} & -s_{21} \\ -s_{12} & s_{11} \end{bmatrix} \rightarrow \begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix} \rightarrow \frac{1}{s_{11}s_{22} - s_{12}s_{21}} \begin{bmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{bmatrix}$$

(1) **M**inor (2) **C**ofactor (3) **T**ranspose (4) **D**eterminant

For our example,

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} \rightarrow \frac{1}{(3)(2)-(4)(1)} \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -½ & 1½ \end{bmatrix}$$

(1) **M**inor (2) **C**ofactor (3) **T**ranspose (4) **D**eterminant (5) **I**nverse

To check these results, premultiply the result by the original matrix and we should get an identity matrix:

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -½ & 1½ \end{bmatrix} = \begin{bmatrix} (3)(1)+(4)(-½) & (3)(-2)+(4)(1½) \\ (1)(1)+(2)(-½) & (1)(-2)+(2)(1½) \end{bmatrix} = \begin{bmatrix} 3+2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Properties of Inverses:

1. $(S')^{-1} = (S^{-1})'$ inverse of transpose of S is equal to the transpose of the inverse

2. $(S\ \Sigma)^{-1} = \Sigma^{-1}\ S^{-1}$ Distributive property

   $(A\ B\ C)^{-1} = C^{-1}\ B^{-1}\ A^{-1}$

3. $(c\ S)^{-1} = 1/c\ S^{-1}$ where c is a scalar

4. For a diagonal matrix with d_{ii} on the diagonal, the inverse is a diagonal matrix with $1/d_{ii}$ on the diagonal.

$$\underset{(3 \times 3)}{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \qquad \underset{(3 \times 3)}{D^{-1}} = \begin{bmatrix} 1/d_{11} & 0 & 0 \\ 0 & 1/d_{22} & 0 \\ 0 & 0 & 1/d_{33} \end{bmatrix}$$

5. If $|S| = 0$, S^{-1} does not exist -- because we cannot divide by zero -- and S is said to be a *singular* matrix. A *nonsingular* matrix has an inverse. E.g., $(X'X)^{-1}$ does not exist if $|X'X| = 0$, such as when it has a zero row or column or has two columns or rows that are identical. The latter occurs in the case of perfect collinearity in multiple regression, and $\hat{\beta} = (X'X)^{-1} X'Y$ cannot be defined. And, empirically, as $|X'X|$ gets closer and closer to zero, $(X'X)^{-1}$ gets smaller and smaller, and $\sigma^2 (X'X)^{-1}$ gets bigger and bigger, causing standard errors to get bigger and bigger.

6. B B = I where B is an idempotent matrix

II. A SIMPLE RECURSIVE MODEL IN LATENT VARIABLES.

A. REPRESENTATION IN SCALAR ALGEBRA.

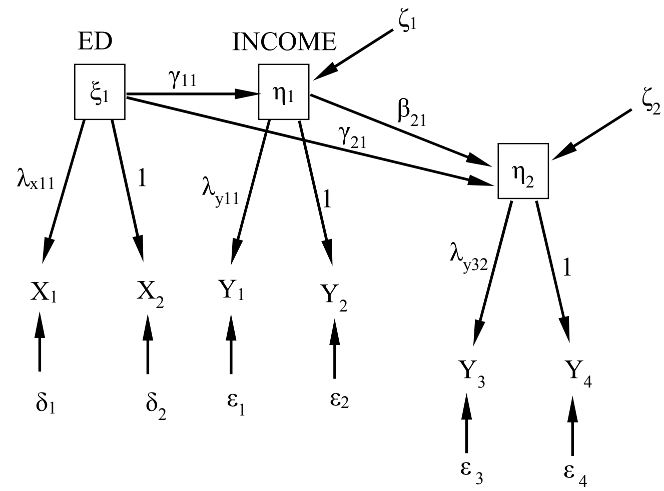
We have two substantive equations:

$\eta_1 = \gamma_{11}\ \xi_1 + \zeta_1$
$\eta_2 = \gamma_{21}\ \xi_1 + \beta_{21}\ \eta_1 + \zeta_2$

and six measurement equations:

$x_1 = \lambda_{11}\ \xi_1 + \delta_1$
$x_2 = \xi_1 + \delta_2$

$y_1 = \lambda_{11}\ \eta_1 + \varepsilon_1$
$y_2 = \eta_1 + \varepsilon_2$
$y_3 = \lambda_{32}\ \eta_2 + \varepsilon_3$
$y_4 = \eta_2 + \varepsilon_4$



Assume we're in deviation scores, so all observed variables (and latent variables) have expected values of zero, and make the usual assumptions of disturbances (zero expected values, mutually uncorrelated, and uncorrelated with independent variables in its equations). Note we have six observed variables, and therefore, 21 observed moments -- [k (k + 1)]/2 = (6) (7)/2 = 21.

To keep track of the number of variables, let's let

p = number of y variables (observed measures of endogenous latent variables ηs)
q = number of x variables (observed measures of exogenous latent variables ξs)
n = number of ξ variables (unobserved exogenous variables)
m = number of η variables (unobserved endogenous variables)

In our model, we have p = 4, q = 2, n = 1, m = 2. Furthermore, our model has 15 parameters:

$\sigma_{\xi 1}^2$, γ_{11}, γ_{21}, β_{21}, $\sigma_{\zeta 1}^2$, $\sigma_{\zeta 2}^2$, λ_{x11}, λ_{y11}, λ_{y32}, $\sigma_{\delta 1}^2$, $\sigma_{\delta 2}^2$, $\sigma_{\varepsilon 1}^2$, $\sigma_{\varepsilon 2}^2$, $\sigma_{\varepsilon 3}^2$, $\sigma_{\varepsilon 4}^2$.

That implies six tetrad difference overidentifying restrictions, of the following form:

$\sigma_{y2x2}\,\sigma_{y1x1} = \sigma_{y2x1}\,\sigma_{y1x2}$.

III. THE GENERAL LISREL MODEL IN MATRIX FORM.

A. SPECIFYING STRUCTURAL EQUATIONS.

Let's express our model in matrix form. Start with the substantive equations:

$$\underset{(m \times 1)}{\eta} = \underset{(m \times m)}{B}\;\underset{(m \times 1)}{\eta} + \underset{(m \times n)}{\Gamma}\;\underset{(n \times 1)}{\xi} + \underset{(m \times 1)}{\zeta}$$

$$\underset{(m \times 1)}{\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_m \end{bmatrix}} = \underset{(m \times m)}{\begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \cdots & \beta_{1m} \\ \beta_{21} & 0 & \beta_{13} & \cdots & \beta_{2m} \\ \beta_{31} & \beta_{32} & 0 & \cdots & \beta_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \beta_{m3} & \cdots & 0 \end{bmatrix}} \underset{(m \times 1)}{\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_m \end{bmatrix}} + \underset{(m \times n)}{\begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m1} & \gamma_{m2} & \cdots & \gamma_{mn} \end{bmatrix}} \underset{(n \times 1)}{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{bmatrix}} + \underset{(m \times 1)}{\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \vdots \\ \zeta_n \end{bmatrix}}$$

In our model above, we have:

$\eta_1 = \gamma_{11}\,\xi_1 + \zeta_1$

$\eta_2 = \gamma_{21}\,\xi_1 + \beta_{21}\,\eta_1 + \zeta_2$

Therefore, in matrix form we have:

$$\underset{(2\times 1)}{\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}} = \underset{(2\times 2)}{\begin{bmatrix}0 & 0\\ \beta_{21} & 0\end{bmatrix}}\underset{(2\times 1)}{\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}} + \underset{(2\times 1)}{\begin{bmatrix}\gamma_{11}\\ \gamma_{21}\end{bmatrix}}\underset{(1\times 1)}{[\,\xi_1]} + \underset{(2\times 1)}{\begin{bmatrix}\zeta_1\\ \zeta_2\end{bmatrix}}$$

Aside: if we included β_{12} in the model, it would be a non-recursive model.

It will be useful later to express our substantive model in a way that isolates η on the left-hand side. We can do that with a little matrix manipulation:

$$\underset{(m\times 1)}{\eta} = \underset{(m\times m)}{B}\underset{(m\times 1)}{\eta} + \underset{(m\times n)}{\Gamma}\underset{(n\times 1)}{\xi} + \underset{(m\times 1)}{\zeta}$$

$$\underset{(m\times 1)}{\eta} - \underset{(m\times 1)}{B\eta} = \underset{(m\times 1)}{\Gamma\xi} + \underset{(m\times 1)}{\zeta}$$

$$\underset{(m\times m)(m\times 1)}{(I - B)\eta} = \underset{(m\times 1)}{\Gamma\xi} + \underset{(m\times 1)}{\zeta}$$

$$\underset{(m\times 1)}{\eta} = \underset{(m\times m)}{(I - B)^{-1}}(\underset{(m\times 1)}{\Gamma\xi} + \underset{(m\times 1)}{\zeta})$$

We can take the transpose of η, which will be useful later:

$$\eta' = (\Gamma\xi + \zeta)'(I - B)^{-1\prime} \qquad (A B)' = B' A'$$
$$\eta' = [(\Gamma\xi)' + \zeta')](I - B)^{-1\prime} \qquad (A + B)' = A' + B'$$
$$\eta' = (\xi'\Gamma' + \zeta')(I - B)^{-1\prime} \qquad (A B)' = B' A'$$

Let's express our measurement models in matrix form. First our model for the Xs:

$$\underset{(q\times 1)}{x} = \underset{(q\times n)}{\Lambda_x}\underset{(n\times 1)}{\xi} + \underset{(q\times 1)}{\delta}$$

$$\underset{(q\times 1)}{\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_q\end{bmatrix}} = \underset{(q\times n)}{\begin{bmatrix}\lambda_{11}^{(x)} & \lambda_{12}^{(x)} & \dots & \lambda_{1n}^{(x)}\\ \lambda_{21}^{(x)} & \lambda_{22}^{(x)} & \dots & \lambda_{2n}^{(x)}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{q1}^{(x)} & \lambda_{q2}^{(x)} & \dots & \lambda_{qn}^{(x)}\end{bmatrix}}\underset{(n\times 1)}{\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \vdots\\ \xi_n\end{bmatrix}} + \underset{(q\times 1)}{\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \vdots\\ \delta_n\end{bmatrix}}$$

In our example, we have:

$x_1 = \lambda_{11}\,\xi_1 + \delta_1$

$x_2 = \xi_1 + \delta_2$

$$\underset{(2\times 1)}{\begin{bmatrix}x_1\\ x_2\end{bmatrix}} = \underset{(2\times 1)}{\begin{bmatrix}\lambda_{11}^{(x)}\\ 1\end{bmatrix}}\underset{(1\times 1)}{[\,\xi_1]} + \underset{(2\times 1)}{\begin{bmatrix}\delta_1\\ \delta_2\end{bmatrix}}$$

For our measurement model of ys, we have the following:

$$\underset{(p \times 1)}{y} = \underset{(p \times m)}{\Lambda_y} \underset{(m \times 1)}{\eta} + \underset{(p \times 1)}{\varepsilon}$$

$$\underset{(p \times 1)}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_q \end{bmatrix}} = \underset{(p \times m)}{\begin{bmatrix} \lambda_{11}^{(y)} & \lambda_{12}^{(y)} & \cdots & \lambda_{1m}^{(y)} \\ \lambda_{21}^{(y)} & \lambda_{22}^{(y)} & \cdots & \lambda_{2m}^{(y)} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p1}^{(y)} & \lambda_{p2}^{(y)} & \cdots & \lambda_{pm}^{(y)} \end{bmatrix}} \underset{(m \times 1)}{\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_m \end{bmatrix}} + \underset{(p \times 1)}{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_p \end{bmatrix}}$$

In our example, we have:

$$y_1 = \lambda_{11} \ \eta_1 + \varepsilon_1$$
$$y_2 = \eta_1 + \varepsilon_2$$
$$y_3 = \lambda_{32} \ \eta_2 + \varepsilon_3$$
$$y_4 = \eta_2 + \varepsilon_4$$

In matrix form we get:

$$\underset{(4 \times 1)}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}} = \underset{(4 \times 2)}{\begin{bmatrix} \lambda_{11}^{(y)} & 0 \\ 1 & 0 \\ 0 & \lambda_{32}^{(y)} \\ 0 & 1 \end{bmatrix}} \underset{(2 \times 1)}{\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}} + \underset{(4 \times 1)}{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}}$$

This gives us matrix representations for slope parameters (λ, γ, and β). But we need representations of the error variances for ξ, ζ, δ, and ε as well.

$$\underset{(n \times 1)(1 \times n)}{E(\xi \ \xi')} = \underset{n \times n}{\Phi} = \begin{bmatrix} \phi_{11} & & & \\ \phi_{21} & \phi_{22} & & \\ \vdots & \vdots & \ddots & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}$$ where $E(\xi_n \xi_n) = \phi_{nn}$

$$\underset{(m \times 1)(1 \times m)}{E(\zeta\zeta')} = \underset{m \times m}{\psi} = \begin{bmatrix} \psi_{11} & & & \\ \psi_{21} & \psi_{22} & & \\ \vdots & \vdots & \ddots & \\ \psi_{m1} & \psi_{m2} & \cdots & \psi_{mm} \end{bmatrix}$$ where $E(\zeta_m \zeta_m) = \psi_{mm}$

$$\underset{(q\times 1)(1\times q)}{E(\delta\delta')} = \underset{q\times q}{\Theta_\delta} = \begin{bmatrix} \theta^\delta_{11} & & & \\ \theta^\delta_{21} & \theta^\delta_{22} & & \\ \vdots & \vdots & \ddots & \\ \theta^\delta_{q1} & \theta^\delta_{q2} & \dots & \theta^\delta_{qq} \end{bmatrix} \quad \text{where } E(\delta_q \delta_q) = \theta^\delta_{qq}$$

$$\underset{(p\times 1)(1\times p)}{E(\varepsilon\varepsilon')} = \underset{p\times p}{\Theta_\varepsilon} = \begin{bmatrix} \theta^\varepsilon_{11} & & & \\ \theta^\varepsilon_{21} & \theta^\varepsilon_{22} & & \\ \vdots & \vdots & \ddots & \\ \theta^\varepsilon_{p1} & \theta^\varepsilon_{p2} & \dots & \theta^\varepsilon_{pp} \end{bmatrix} \quad \text{where } E(\varepsilon_p \varepsilon_p) = \theta^\delta_{qq}$$

For the model above, these matrices will be

$$\underset{1\times 1}{\Phi} = [\phi_{11}]$$

$$\underset{m\times m}{\psi} = \begin{bmatrix} \psi_{11} & \\ 0 & \psi_{22} \end{bmatrix} \qquad \underset{2\times 2}{\Theta_\delta} = \begin{bmatrix} \theta^\delta_{11} & \\ 0 & \theta^\delta_{22} \end{bmatrix}$$

$$\underset{p\times p}{\Theta_\varepsilon} = \begin{bmatrix} \theta^\varepsilon_{11} & & & \\ 0 & \theta^\varepsilon_{22} & & \\ 0 & 0 & \theta^\varepsilon_{33} & \\ 0 & 0 & 0 & \theta^\varepsilon_{44} \end{bmatrix}$$

B. COMPUTING MOMENTS IN TERMS OF PARAMETERS.

Let's find expressions of moments in terms of parameters. Begin with the input covariance matrix. We have a vector of Xs and Ys:

$$\underset{q\times 1}{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_q \end{bmatrix} \quad \underset{p\times 1}{Y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_q \end{bmatrix} \qquad \underset{1\times q}{X'} = [x'_1 \quad x'_2 \quad \dots \quad x'_q] \quad \underset{1\times p}{Y'} = [y'_1 \quad y'_2 \quad \dots \quad y'_p]$$

Now, compute covariances:

$$E(XX') = \underset{q\times q}{\Sigma_{XX}} = \begin{bmatrix} \sigma^2_{x_1} & & & \\ \sigma_{x_2x_1} & \sigma^2_{x_2} & & \\ \vdots & \vdots & \ddots & \\ \sigma_{x_qx_1} & \sigma_{x_qx_2} & \dots & \sigma^2_{x_q} \end{bmatrix} \qquad E(YY') = \underset{p\times p}{\Sigma_{YY}} = \begin{bmatrix} \sigma^2_{y_1} & & & \\ \sigma_{y_2y_1} & \sigma^2_{y_2} & & \\ \vdots & \vdots & \ddots & \\ \sigma_{y_py_1} & \sigma_{y_py_2} & \dots & \sigma^2_{y_p} \end{bmatrix}$$

$$E(XY') = \underset{q \times p}{\Sigma_{XY'}} = \begin{bmatrix} \sigma_{x_1y_1} & \sigma_{x_1y_2} & \cdots & \sigma_{x_1y_p} \\ \sigma_{x_2y_1} & \sigma_{x_2y_2} & \cdots & \sigma_{x_2y_p} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{x_qy_1} & \sigma_{x_qy_2} & \cdots & \sigma_{x_qy_p} \end{bmatrix} \quad E(YX') = \underset{p \times q}{\Sigma_{YX'}} = \begin{bmatrix} \sigma_{y_1x_1} & \sigma_{y_1x_2} & \cdots & \sigma_{y_1x_q} \\ \sigma_{y_2x_1} & \sigma_{y_2x_2} & \cdots & \sigma_{y_2x_q} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{y_px_1} & \sigma_{y_px_2} & \cdots & \sigma_{y_px_q} \end{bmatrix}$$

Let's put these results together by partitioning our vector of variables into x and y:

$$\underset{(p+q) \times 1}{Z} = \begin{bmatrix} Y \\ (p \times 1) \\ X \\ q \times 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \\ \text{---} \\ x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix}$$

$$\underset{[(p+q) \times 1]}{Z}\underset{[1 \times (p+q)]}{Z'} = \begin{bmatrix} Y \\ (p \times 1) \\ \\ X \\ (q \times 1) \end{bmatrix} \begin{bmatrix} Y' & X' \\ (1 \times p) & (1 \times q) \end{bmatrix} = \begin{bmatrix} YY' & YX' \\ (p \times p) & (p \times q) \\ \\ XY' & XX' \\ (q \times p) & (q \times q) \end{bmatrix}$$

Now that we have Σ, let's compute moments in terms of parameters to obtain Σ(θ). Let's begin with the lower right-hand partition of Σ, Σ_{xx}, which gives covariances among the xs. Begin with the x equation:

$$\underset{(q \times 1)}{x} = \underset{(q \times n)}{\Lambda_x} \underset{(n \times 1)}{\xi} + \underset{(q \times 1)}{\delta}$$

We can take the transpose of x: (A + B)' = A' + B'

$$\underset{(1 \times q)}{x'} = \underset{(1 \times n)}{\xi'} \underset{(n \times q)}{\Lambda_x'} + \underset{(1 \times q)}{\delta'}$$

(A B)' = B' A'

Multiply these together:

$$\underset{(qxq)}{x\,x'} = \underset{(qxn)(nx1)(1xn)(nxq)}{\Lambda_x \xi \xi' \Lambda_x'} + \underset{(qxq)}{\delta\delta'} + \underset{(qxn)(nx1)(1xq)}{\Lambda_x \xi' \delta'} + \underset{(qx1)(1xn)(nxq)}{\delta \xi \Lambda_x'}$$

Take expectations:

$$\underset{(qxq)}{E(x\,x')} = \underset{(qxn)(nxn)(nxq)}{\Lambda_x E(\xi \xi') \Lambda_x'} + \underset{(qxq)}{E(\delta\delta')} + \Lambda_x \overbrace{E(\xi' \delta')}^{0} + \overbrace{E(\delta\, \xi)}^{0} \Lambda_x'$$

$$\underset{(qxq)}{\Sigma_{xx}} = \underset{(qxn)}{\Lambda_x} \underset{(nxn)}{\Phi} \underset{(nxq)}{\Lambda_x'} + \underset{(qxq)}{\Theta_\delta}$$

We can get an expression for Σ_{XY}, the covariances between xs and ys, by multiplying our x equation by our y equation and taking expectations:

$$\underset{(p \times 1)}{y} = \underset{(p \times m)}{\Lambda_y} \; \underset{(m \times 1)}{\eta} + \underset{(p \times 1)}{\varepsilon}$$

Take the transpose of y:

$$\underset{(1 \times p)}{y'} = \underset{(1 \times m)}{\eta'} \; \underset{(m \times p)}{\Lambda_y'} + \underset{(1 \times p)}{\varepsilon'}$$

$$\underset{(qx1)(1xp)}{x\, y'} = \underset{(qxn)(nx1)(1xm)(mxp)}{\Lambda_x \; \xi \; \eta' \; \Lambda_y'} + \underset{(qx1)(1xp)}{\delta \; \varepsilon'} + \underset{(qxn)(nx1)(1xp)}{\Lambda_x \; \xi \; \varepsilon'} + \underset{(qx1)(1xm)(mxp)}{\delta \; \eta' \; \Lambda_y'}$$

$$\underset{(qxp)}{E(x\, y')} = \underset{(qxn)}{\Lambda_x} \; \underset{(nx1)(1xm)}{E(\xi\, \eta')} \; \underset{(mxp)}{\Lambda_y'} + \underset{(qx1)(1xp)}{E(\delta\, \varepsilon')} + \underset{(qxn)}{\Lambda_x} \; \underset{(nx1)(1xp)}{E(\xi\, \varepsilon')} + \underset{(qx1)(1xm)}{E(\delta\, \eta')} \; \underset{(mxp)}{\Lambda_y'}$$

$$\underset{(qxp)}{\Sigma_{xy}} = \underset{(qxn)}{\Lambda_x} \; \underset{(nxm)}{E(\xi\, \eta')} \; \underset{(mxp)}{\Lambda_y'}$$

We need an expression for $E(\xi\, \eta')$:

$$\underset{(1xm)}{\eta'} = (\underset{(1xn)(nxm)}{\xi' \; \Gamma'} + \underset{(1xm)}{\zeta'}) \; \underset{(mxm)}{(I - B)^{-1\,'}}$$

$$\underset{(nx1)(1xm)}{\xi\, \eta'} = \xi \; (\underset{(nx1)(1xn)(nxm)}{\xi' \; \Gamma'} + \underset{(1xm)}{\zeta'}) \; \underset{(mxm)}{(I - B)^{-1'}}$$

$$\underset{(nxm)}{E(\xi\, \eta')} = [\underset{(nxn)}{E(\xi\, \xi')} \; \underset{(nxm)}{\Gamma'} + \underset{(nx1)(nxm)}{E(\xi\, \zeta')}] \; \underset{(mxm)}{(I - B)^{-1'}}$$

$$\underset{(nxm)}{E(\xi\, \eta')} = \underset{(nxn)}{\Phi} \; \underset{(nxm)}{\Gamma'} \underset{(mxm)}{(I - B)^{-1'}}$$

Now replace $E(\xi\, \eta')$ above:

$$\underset{(qxp)}{\Sigma_{xy}} = \underset{(qxn)}{\Lambda_x} \; \underset{(nxn)}{\Phi} \; \underset{(nxm)}{\Gamma'} \; \underset{(mxm)}{(I - B)^{-1'}} \; \underset{(mxp)}{\Lambda_y'}$$

We can take the transpose of Σ_{XY}: $(A\, B\, C)' = C'\, B'\, A'$

$$\underset{(pxq)}{\Sigma_{yx}} = \underset{(pxm)}{\Lambda_y} \; \underset{(mxm)}{(I - B)^{-1}} \; \underset{(mxn)}{\Gamma} \; \underset{(nxn)}{\Phi} \; \underset{(nxq)}{\Lambda_x'}$$

Finally, we can express the covariance matrix of ys, Σ_{YY}, in terms of parameters. Begin with our equation for y and its transpose y':

$$\underset{(p \times 1)}{y} = \underset{(p \times m)}{\Lambda_y} \; \underset{(m \times 1)}{\eta} + \underset{(p \times 1)}{\varepsilon}$$

$$\underset{(1 \times p)}{y'} = \underset{(1 \times m)}{\eta'} \; \underset{(m \times p)}{\Lambda_y'} + \underset{(1 \times p)}{\varepsilon'}$$

Multiply the two equations together:

$$\underset{\text{(pxp)}}{y\,y'} = \underset{\text{(pxm)}}{\Lambda_y}\;\underset{\text{(mx1)}}{\eta}\;\underset{\text{(1xm)}}{\eta'}\;\underset{\text{(mxp)}}{\Lambda_y'} + \underset{\text{(pxp)}}{\varepsilon\,\varepsilon'} + \underset{\text{(pxm)}}{\Lambda_y}\;\underset{\text{(mx1)}}{\eta}\;\underset{\text{(1xp)}}{\varepsilon'} + \underset{\text{(px1)}}{\varepsilon}\;\underset{\text{(1xm)}}{\eta'}\;\underset{\text{(mxp)}}{\Lambda_y'}$$

Take expectations:

$$\underset{\text{(pxp)}}{E(y\,y')} = \underset{\text{(pxm)}}{\Lambda_y}\;\underset{\text{(mxm)}}{E(\eta\,\eta')}\;\underset{\text{(mxp)}}{\Lambda_y'} + \underset{\text{(pxp)}}{E(\varepsilon\,\varepsilon')} + \underset{\text{(pxm)}}{\Lambda_y}\;\underset{\text{(mxp)}}{E(\eta\,\varepsilon')} + \underset{\text{(pxm)}}{E(\varepsilon\,\eta')}\;\underset{\text{(mxp)}}{\Lambda_y'}$$

$$\underset{\text{(pxp)}}{\Sigma_{yy}} = \underset{\text{(pxm)}}{\Lambda_y}\;\underset{\text{(mxm)}}{E(\eta\,\eta')}\;\underset{\text{(mxp)}}{\Lambda_y'} + \underset{\text{(pxp)}}{\Theta_\varepsilon}$$

We need an expression for $E(\eta\,\eta')$. Start with our equations for η and η':

$$\underset{\text{(mx1)}}{\eta} = \underset{\text{(mxm)}}{(I - B)^{-1}}\;(\underset{\text{(mxn)}}{\Gamma}\;\underset{\text{(nx1)}}{\xi} + \underset{\text{(mx1)}}{\zeta})$$

$$\underset{\text{(1xm)}}{\eta'} = (\underset{\text{(1xn)}}{\xi'}\;\underset{\text{(nxm)}}{\Gamma'} + \underset{\text{(1xm)}}{\zeta'})\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}$$

$$\underset{\text{(mx1)(1xm)}}{E(\eta\,\eta')} = E\{\, [\, \underset{\text{(mxm)}}{(I - B)^{-1}}\;(\underset{\text{(mxn)}}{\Gamma}\;\underset{\text{(nx1)}}{\xi} + \underset{\text{(mx1)}}{\zeta})\,]\;[\,(\underset{\text{(1xn)}}{\xi'}\;\underset{\text{(nxm)}}{\Gamma'} + \underset{\text{(1xm)}}{\zeta'})\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}\,]\,\}$$

$$\underset{\text{(mx1)(1xm)}}{E(\eta\,\eta')} = \underset{\text{(mxm)}}{(I - B)^{-1}}\;E\,[\,(\underset{\text{(mxn)}}{\Gamma}\;\underset{\text{(nx1)}}{\xi} + \underset{\text{(mx1)}}{\zeta})\,]\;[\,(\underset{\text{(1xn)}}{\xi'}\;\underset{\text{(nxm)}}{\Gamma'} + \underset{\text{(1xm)}}{\zeta'})\,]\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}$$

$$\underset{\text{(mx1)(1xm)}}{E(\eta\,\eta')} = \underset{\text{(mxm)}}{(I - B)^{-1}}\;E\,[\,(\underset{\text{(mxn)}}{\Gamma}\;\underset{\text{(nx1)}}{\xi}\;\underset{\text{(1xn)}}{\xi'}\;\underset{\text{(nxm)}}{\Gamma'} + \underset{\text{(mx1)}}{\zeta}\;\underset{\text{(1xm)}}{\zeta'} + \underset{\text{(mxn)}}{\Gamma}\;\underset{\text{(nx1)}}{\xi}\;\underset{\text{(1xm)}}{\zeta'} + \underset{\text{(mx1)}}{\zeta}\;\underset{\text{(1xn)}}{\xi'}\;\underset{\text{(nxm)}}{\Gamma'})\,]\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}$$

$$\underset{\text{(mx1)(1xm)}}{E(\eta\,\eta')} = \underset{\text{(mxm)}}{(I - B)^{-1}}\;\underset{\text{(mxn)(nx1)(1xn)(nxm)}}{\Gamma\;E(\xi\,\xi')\;\Gamma'} + \underset{\text{(mx1)(1xm)}}{E(\zeta\,\zeta')} + \underset{\text{(mxn)}}{\Gamma}\;\underset{\text{(nx1)(1xm)}}{E(\xi\;\zeta')} + \underset{\text{(mx1)(1xn)(nxm)}}{E(\zeta\;\xi')\;\Gamma'})\,]\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}$$

$$\underset{\text{mxm}}{E(\eta\,\eta')} = \underset{\text{mxm}}{(I - B)^{-1}}\;(\underset{\text{(mxn)(nxn)(nxm)}}{\Gamma\;\Phi\;\Gamma'} + \underset{\text{(mxm)}}{\Psi})\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}$$

Now replace $E(\eta\,\eta')$ above to get Σ_{yy}:

$$\underset{\text{(pxp)}}{\Sigma_{yy}} = \underset{\text{(pxm)}}{\Lambda_y}\;[\,\underset{\text{(mxm)}}{(I - B)^{-1}}\;(\underset{\text{(mxn)(nxn)(nxm)}}{\Gamma\;\Phi\;\Gamma'} + \underset{\text{(mxm)}}{\Psi})\;\underset{\text{(mxm)}}{(I - B)^{-1\prime}}\,]\;\underset{\text{(mxp)}}{\Lambda_y'} + \underset{\text{(pxp)}}{\Theta_\varepsilon}$$

Now, let's put these results together in our partitioned matrix of observable moments, Σ:

$$\begin{bmatrix} \underset{(p\times p)}{\Sigma_{YY}} & \underset{(p\times q)}{\Sigma_{YX}} \\ \underset{(q\times p)}{\Sigma_{YX}} & \underset{(q\times q)}{\Sigma_{XX}} \end{bmatrix} = \begin{bmatrix} \Lambda_y\,(\mathrm{I}-\mathrm{B})^{-1}\,(\Gamma\,\Phi\,\Gamma' + \Psi)(\mathrm{I}-\mathrm{B})^{-1\prime}\Lambda_y' + \Theta_\varepsilon & \Lambda_y\,(\mathrm{I}-\mathrm{B})^{-1}\,\Gamma\,\Phi\,\Lambda_x \\ \Lambda_x\,\Phi\,\Gamma'(\mathrm{I}-\mathrm{B})^{-1}\,\Lambda_y{}' & \Lambda_x\,\Phi\,\Lambda_x' + \Theta_\delta \end{bmatrix}$$

$$\underset{(p+q)\times p+q}{\Sigma} = \underset{(p+q)\times p+q}{\Sigma(\theta)}$$

This is the specification of the LISREL model, which is what we set out to show back on page 1.

The above equations specify the observed population moments, Σ, in terms of the parameters of the general LISREL (covariance structure) model, Σ(θ). It is precisely the same kind of equation we've been expressing this semester, beginning with a bivariate regression model.

Note that in empirical applications—as before—we don't have access to the population moments Σ, but have to rely on sample moments, S. In that case, we can use the above population equation to express the sample moments in terms of sample *estimates* of the model's parameters:

$$\begin{bmatrix} \underset{(p\times p)}{S_{YY}} & \underset{(p\times q)}{S_{YX}} \\ \underset{(q\times p)}{S_{YX}} & \underset{(q\times q)}{S_{XX}} \end{bmatrix} = \begin{bmatrix} \hat{\Lambda}_y\,(\mathrm{I}-\hat{\mathrm{B}})^{-1}\left(\hat{\Gamma}\,\hat{\Phi}\,\hat{\Gamma}' + \hat{\Psi}\right)\left(\mathrm{I}-\hat{\mathrm{B}}\right)^{-1'}\hat{\Lambda}_y' + \hat{\Theta}_\varepsilon & \hat{\Lambda}_y\,(\mathrm{I}-\hat{\mathrm{B}})^{-1}\,\hat{\Gamma}\,\hat{\Phi}\,\hat{\Lambda}_x \\ \hat{\Lambda}_x\,\hat{\Phi}\,\hat{\Gamma}'(\mathrm{I}-\hat{\mathrm{B}})^{-1}\,\hat{\Lambda}_y{}' & \hat{\Lambda}_x\,\hat{\Phi}\,\hat{\Lambda}_x' + \hat{\Theta}_\delta \end{bmatrix}$$

$$\underset{(p+q)\times p+q}{\mathrm{S}} = \underset{(p+q)\times p+q}{\Sigma(\hat{\theta})}$$

As before, for just-identified or over-identified recursive models in observables, we can rely on the method of moments to obtain estimates of parameters in the matrix on the right-hand side. But for more complicated over-identified models, we need a way of reconciling more than one way of computing parameters from sample moments. And there is the additional problem of obtaining correct standard errors and test statistics for the estimates. We'll rely mainly on the principle of maximum likelihood for this.