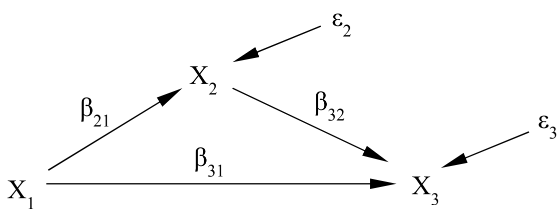
LECTURE 2: DECOMPOSING EFFECTS IN RECURSIVE MODELS

I. A TWO-EQUATION MODEL IN UNSTANDARDIZED FORM.
II. AN OVERIDENTIFIED MODEL IN UNSTANDARDIZED FORM
III. ESTIMATION AND TESTING IN AN UNSTANDARDIZED MODEL
IV. INTERPRETING STRUCTURAL PARAMETERS IN A THREE-EQUATION MODEL.

I. A TWO-EQUATION MODEL IN UNSTANDARDIZED FORM.



X_1 = Father's Income,
X_2 = Education, and
X_3 = Offspring's Income

The structural equations are:

$X_2 = \beta_{21} X_1 + \varepsilon_2$
$X_3 = \beta_{31} X_1 + \beta_{32} X_2 + \varepsilon_3$

with six parameters: σ_{11}, β_{21}, β_{31}, β_{32}, $\sigma_{\varepsilon 2}$, $\sigma_{\varepsilon 3}$. We make the following assumptions:

1. $E(X_1\varepsilon_2) = E(X_1\varepsilon_3) = E(X_2\varepsilon_3) = 0$: Disturbances are uncorrelated with regressors.
2. $\varepsilon_{2i} \sim N(0,\sigma_{\varepsilon 2})$ and $\varepsilon_{3i} \sim N(0,\sigma_{\varepsilon 3})$: Disturbances are normally-distributed with constant variance.
3. $E(\varepsilon_{2i}\, \varepsilon_{2j}) = E(\varepsilon_{3i}\, \varepsilon_{3j}) = 0$: Disturbances are not serially correlated.
4. $E(\varepsilon_2\, \varepsilon_3) = 0$: The two disturbances are uncorrelated.

Again, we characterize the three random variables in terms of observable moments:

$$\underset{3\times 3}{\Sigma} = \begin{bmatrix} \sigma_{11} & & \\ \sigma_{21} & \sigma_{22} & \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Now let's express our moments in terms of parameters:

$X_2 = \beta_{21} X_1 + \varepsilon_2$
$X_3 = \beta_{31} X_1 + \beta_{32} X_2 + \varepsilon_3$

1. $\sigma_{11} = E(X_1^2) = \sigma_{11}$

$\sigma_{21} = E(X_1X_2) = E[X_1(\beta_{21}X_1 + \varepsilon_2)] = \beta_{21}E(X_1^2) + E(X_1\varepsilon_2)$

2. $\sigma_{21} = \beta_{21}\sigma_{11}$

$\sigma_{31} = E(X_1X_3) = E[X_1(\beta_{31} X_1 + \beta_{32}X_2 + \varepsilon_3)] = \beta_{31}E(X_1^2) + \beta_{32}E(X_1X_2) + E(X_1\varepsilon_3) = \beta_{31}\, \sigma_{11} + \beta_{32}\sigma_{21}$

3. $\sigma_{31} = \beta_{31}\, \sigma_{11} + \beta_{32}\sigma_{21}$ (almost - substitute for σ_{21} and get $\sigma_{31} = \beta_{31}\, \sigma_{11} + \beta_{32}\, \beta_{21}\, \sigma_{11}$)

$\sigma_{32} = E(X_2X_3) = E[X_2(\beta_{31} X_1 + \beta_{32}X_2 + \varepsilon_3)] = \beta_{31}E(X_1X_2) + \beta_{32}E(X_2X_2) + E(X_2\varepsilon_3) = \beta_{31}\, \sigma_{21} + \beta_{32}\sigma_{22}$

4. $\boldsymbol{\sigma_{32} = \beta_{31}\, \sigma_{21} + \beta_{32}\sigma_{22}}$ [almost - $\sigma_{32} = \beta_{31}\, \beta_{21}\, \sigma_{11} + \beta_{32}\, (\beta_{21}^2\sigma_{11} + \sigma_{\varepsilon 22})$]

$\sigma_{22} = E(X_2^2) = E[(\beta_{21}X_1 + \varepsilon_2)(\beta_{21}X_1 + \varepsilon_2)] = \beta_{21}^2E(X_1^2) + 2\beta_{21}E(X_1\varepsilon_2) + E(\varepsilon_2\varepsilon_2)$

5. $\boldsymbol{\sigma_{22} = \beta_{21}^2\sigma_{11} + \sigma_{\varepsilon 22}}$

$\sigma_{33} = E(X_3^2) = E[(\beta_{31}X_1 + \beta_{32}X_2 + \varepsilon_3)(\beta_{31}X_1 + \beta_{32}X_2 + \varepsilon_3)] = \beta_{31}^2E(X_1^2) + 2\beta_{31}\beta_{32}E(X_2X_1) +$

$2\beta_{31}E(X_1\varepsilon_3) + \beta_{32}^2E(X_2^2) + 2\beta_{32}E(X_2\varepsilon_3) + E(\varepsilon_3\varepsilon_3)$

$\sigma_{33} = \beta_{31}^2\sigma_{11} + 2\beta_{31}\beta_{32}\sigma_{21} + \beta_{32}^2\sigma_{22} + \sigma_{\varepsilon 33}$

But a more useful result for σ_{33} results from multiplying the X_3 equation by X_3:

$E(X_3\, X_3) = \beta_{31}\, E(X_3X_1) + \beta_{32}\, E(X_3X_2) + E(X_3\varepsilon_3)$

$= \beta_{31}\, \sigma_{31} + \beta_{32}\, \sigma_{32} + E(X_3\varepsilon_3)$ But, multiply the X_3 equation by ε_3: $E(X_3\varepsilon_3) = \beta_{31}\, E(X_1\varepsilon_3) + \beta_{32}\, E(X_2\varepsilon_3) + E(\varepsilon_3^2) = \sigma_{\varepsilon 33}$

6. $\boldsymbol{\sigma_{33} = \beta_{31}\, \sigma_{31} + \beta_{32}\, \sigma_{32} + \sigma_{\varepsilon 33}}$

Now we can rearrange these equations into a population moment (covariance) matrix implied by the population structural equation model. This gives moments almost in terms of parameters. Again, if the model is correct, this matrix should generate the observed covariance matrix.

$$\underset{3\times 3}{\underset{\Sigma}{\begin{bmatrix} \sigma_{11} & & \\ \sigma_{21} & \sigma_{22} & \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}} = \underset{3\times 3}{\underset{\Sigma(\theta)}{\begin{bmatrix} \sigma_{11} & & \\ \beta_{21}\,\sigma_{11} & \beta_{21}^2\,\sigma_{11} + \sigma_{\varepsilon 22} & \\ \beta_{31}\,\sigma_{11} + \beta_{32}\,\sigma_{21} & \beta_{31}\,\sigma_{21} + \beta_{32}\,\sigma_{22} & \beta_{31}\,\sigma_{31} + \beta_{32}\,\sigma_{32} + \sigma_{\varepsilon 33} \end{bmatrix}}}$$

Let's express parameters in terms of moments:

1. $\boldsymbol{\sigma_{11} = \sigma_{11}}$

2. $\boldsymbol{\beta_{21} = \sigma_{21}/\sigma_{11}}$

For β_{31}, begin with (3)

$\sigma_{31} = \beta_{31}\, \sigma_{11} + (\beta_{32})\, \sigma_{21}$ from (4) $\sigma_{32} = \beta_{31}\, \sigma_{21} + \beta_{32}\sigma_{22}$; so $\beta_{32} = (\sigma_{32} - \beta_{31}\, \sigma_{21})/\sigma_{22}$

$\sigma_{31} = \beta_{31}\, \sigma_{11} + \sigma_{21}(\sigma_{32} - \beta_{31}\, \sigma_{21})/\sigma_{22}$ substitute for β_{32}

$\sigma_{31} - \beta_{31}\, \sigma_{11} = (\sigma_{21}\, \sigma_{32} - \beta_{31}\, \sigma_{21}^2)/\sigma_{22}$ subtract $\beta_{31}\, \sigma_{11}$ from both sides

$\sigma_{22}\, \sigma_{31} - \beta_{31}\, \sigma_{22}\, \sigma_{11} = \sigma_{21}\, \sigma_{32} - \beta_{31}\, \sigma_{21}^2$ multiply both sides by σ_{22}

$\sigma_{22}\, \sigma_{31} - \sigma_{21}\, \sigma_{32} = \beta_{31}\, \sigma_{22}\, \sigma_{11} - \beta_{31}\, \sigma_{21}^2$ add $\beta_{31}\, \sigma_{22}\, \sigma_{11}$ to both sides; subtract $\sigma_{21}\, \sigma_{32}$ from both sides

$\sigma_{22}\,\sigma_{31} - \sigma_{21}\,\sigma_{32} = \beta_{31}\,(\sigma_{22}\,\sigma_{11} - \sigma_{21}^2)$ factor out β_{31} from right side

3. $\boldsymbol{\beta_{31} = (\sigma_{22}\,\sigma_{31} - \sigma_{21}\,\sigma_{32})/(\sigma_{11}\,\sigma_{22} - \sigma_{21}^2)}$ divide both sides by $(\sigma_{22}\,\sigma_{11} - \sigma_{21}^2)$

4. $\boldsymbol{\beta_{32} = (\sigma_{11}\,\sigma_{32} - \sigma_{21}\,\sigma_{31})/(\sigma_{11}\,\sigma_{22} - \sigma_{21}^2)}$ Same as above, but using (4) for (3) and (3) for (4)

For the disturbances,

$\sigma_{22} = \beta_{21}^2\sigma_{11} + \sigma_{\varepsilon 22}$ from (5)

$\sigma_{\varepsilon 22} = \sigma_{22} - \beta_{21}^2\sigma_{11} = \sigma_{22} - (\sigma_{21}^2/\sigma_{11}^2)\sigma_{11}$

5. $\boldsymbol{\sigma_{\varepsilon 22} = \sigma_{22} - (\sigma_{21}^2/\sigma_{11})}$

$\sigma_{33} = \beta_{31}\,\sigma_{31} + \beta_{32}\,\sigma_{32} + \sigma_{\varepsilon 33}$ from (6)

6. $\boldsymbol{\sigma_{\varepsilon 33} = \sigma_{33} - \beta_{31}\,\sigma_{31} - \beta_{32}\,\sigma_{32}}$

We can compute the reduced-form by replacing X_2 in our second structural equation with the right-hand side of our first equation.

$X_2 = \beta_{21}\,X_1 + \varepsilon_2 = \pi_{21}\,X_1 + \pi_{\varepsilon 2}$ Only exogenous variables in structural form, so reduced form = structural form

$X_3 = \beta_{31}\,X_1 + \beta_{32}\,(X_2) + \varepsilon_3$

$= \beta_{31}\,X_1 + \beta_{32}\,(\beta_{21}\,X_1 + \varepsilon_2) + \varepsilon_3$

$= \beta_{31}\,X_1 + \beta_{32}\,\beta_{21}\,X_1 + \beta_{32}\,\varepsilon_2 + \varepsilon_3$

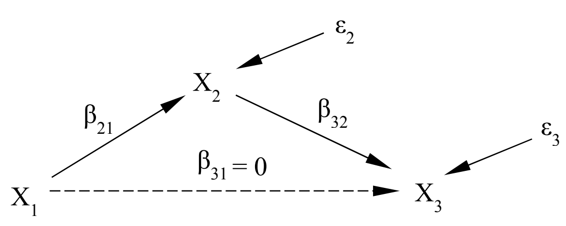
$= (\beta_{31} + \beta_{32}\,\beta_{21})\,X_1 + \beta_{32}\,\varepsilon_2 + \varepsilon_3$

$X_3 = \pi_{31}\quad X_1 \quad + \quad \pi_{\varepsilon 3}$ where $\pi_{31} = \beta_{31} + \beta_{32}\,\beta_{21}$

$\pi_{\varepsilon 3} = \beta_{32}\,\varepsilon_2 + \varepsilon_3$

II. AN OVERIDENTIFIED MODEL IN UNSTANDARDIZED FORM

Now consider Model II in unstandardized form:



X_1 = Father's Income,
X_2 = Education, and
X_3 = Income

$X_2 = \beta_{21}\,X_1 + \varepsilon_2$
$X_3 = \beta_{32}\,X_2 + \varepsilon_3$ Note: $\beta_{31} = 0$

We can obtain expressions for moments in terms of parameters by simply constraining $\beta_{31} = 0$ in the above normal equations:

1. $\sigma_{11} = \sigma_{11}$ 4. $\sigma_{21} = \beta_{21}\,\sigma_{11}$

2. $\sigma_{22} = \beta_{21}^2\,\sigma_{11} + \sigma_{\varepsilon 22}$ 5. $\sigma_{31} = \beta_{32}\,\sigma_{21}$

3. $\sigma_{32} = \beta_{32}\,\sigma_{22}$ 6. $\sigma_{33} = \beta_{32}^2\,\sigma_{22} + \sigma_{\varepsilon 33}$ or $\sigma_{33} = \beta_{32}\,\sigma_{32} + \sigma_{\varepsilon 33}$ (from constraining Model I)

Or in our covariance structure matrix form:

$$\underset{3\times 3}{\Sigma} = \begin{bmatrix} \sigma_{11} & & \\ \sigma_{21} & \sigma_{22} & \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & & \\ \beta_{21}\,\sigma_{11} & \beta_{21}^2\,\sigma_{11} + \sigma_{\varepsilon 22} & \\ \beta_{32}\,\sigma_{21} & \beta_{32}\,\sigma_{22} & \beta_{32}^2\,\sigma_{22} + \sigma_{\varepsilon 33} \end{bmatrix} = \underset{3\times 3}{\Sigma(\theta)}$$

From these normal equations, we can express parameters in terms of moments (again, notice there is one more equation than parameter, since we eliminated β_{31}.

1. $\sigma_{11} = \sigma_{11}$

2. $\beta_{21} = \sigma_{21}/\sigma_{11}$

3. $\beta_{32} = \sigma_{31}/\sigma_{21} = \sigma_{32}/\sigma_{22}$ from (3) and (5)

$\sigma_{\varepsilon 22} = \sigma_{22} - (\beta_{21}^2\,\sigma_{11}) = \sigma_{22} - (\sigma_{21}^2/\sigma_{11}^2\,\sigma_{11}) = \sigma_{22} - \sigma_{21}^2/\sigma_{11}$

4. $\sigma_{\varepsilon 22} = \sigma_{22} - \sigma_{21}^2/\sigma_{11}$

$\sigma_{\varepsilon 33} = \sigma_{33} - \beta_{32}^2\,\sigma_{22} = \sigma_{33} - (\sigma_{31}^2/\sigma_{21}^2)\,\sigma_{22} = \sigma_{33} - (\sigma_{32}^2/\sigma_{22}^2)\,\sigma_{22}$ (because of two ways of computing β_{32})

5. $\sigma_{\varepsilon 33} = \sigma_{33} - (\sigma_{31}^2/\sigma_{21}^2)\,\sigma_{22} = \sigma_{33} - (\sigma_{32}^2/\sigma_{22})$

Note that this model implies one overidentifying restriction, which resulted from our assumption that $\beta_{31} = 0$ in the population. We can compute it from either (3) or (5) by setting the two equivalent ways of computing β_{32} or $\sigma_{\varepsilon 33}$ to be equal, and cross-multiplying: $\sigma_{31}/\sigma_{21} = \sigma_{32}/\sigma_{22}$; therefore,

$\sigma_{31}\,\sigma_{22} = \sigma_{32}\,\sigma_{21}$

If the model is true in the population this constraint on the observable moments (covariances) will hold exactly. If it doesn't hold exactly, the model is wrong. Stated differently, if $\beta_{31} \neq 0$, then the model is wrong, and our overidentifying constraint on moments will not hold. Note the parallel with our standardized model.

III. ESTIMATION AND TESTING IN AN UNSTANDARDIZED MODEL

Since we rarely have access to population moments, we must typically rely on sample moments to estimate structural parameters.

$$\underset{3\times 3}{S} = \begin{bmatrix} s_{11} & & \\ s_{21} & s_{22} & \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

$$\underset{3\times 3}{\Sigma(\hat{\theta})} = \begin{bmatrix} s_{11} & & \\ \hat{\beta}_{21}\,s_{11} & \hat{\beta}_{21}^2\,s_{11} + s_{\varepsilon 22} & \\ \hat{\beta}_{32}\,s_{21} & \hat{\beta}_{32}\,s_{22} & \hat{\beta}_{32}^2\,s_{22} + s_{\varepsilon 33} \end{bmatrix}$$

1. $s_{11} = s_{11}$

2. $\hat{\beta}_{21} = s_{21}/s_{11}$

3. $\hat{\beta}_{32} = s_{31}/s_{21} = s_{32}/s_{22}$

4. $s_{\varepsilon 22} = s_{22} - s_{21}^2/s_{11} = s_{33} - (s_{32}^2/s_{22})$

5. $s_{\varepsilon 33} = s_{33} - (s_{31}^2/s_{21}^2)\, s_{22} = s_{33} - (s_{32}^2/s_{22})$

And the overidentifying restriction on sample moments is: $s_{31}\, s_{22} = s_{32}\, s_{21}$. As in the standardized case, there are two ways of estimating β_{32}. Which method do we use? Again, we could use s_{31}/s_{21} or s_{32}/s_{22}, since both are unbiased $E(s_{31}/s_{21}) = E(s_{32}/s_{22}) = \beta_{32}$. Or we could use some weighted average of the two $w_1(s_{31}/s_{21}) + w_2(s_{32}/s_{22})$, which will also be unbiased $E[w_1(s_{31}/s_{21}) + w_2(s_{32}/s_{22})] = \beta_{32}$. All three will be unbiased, but one will be the most efficient. Theorems of ordinary least squares estimation tells us that $\hat{\beta}_{32} = s_{32}/s_{22}$ is efficient (minimum variance):

$VAR(\hat{\beta}_{32}) \leq VAR(\hat{\beta}_{32}{*})$, where $\hat{\beta}_{32}{*}$ is any other linear unbiased estimator of β_{32} and $VAR(\hat{\beta}_{32}) = E[\hat{\beta}_{32} - E(\hat{\beta}_{32})]^2$ and $VAR(\hat{\beta}_{32}{*}) = E[\hat{\beta}_{32}{*} - E(\hat{\beta}_{32}{*})]^2$

Aside: the reason is that $VAR(\hat{\beta}_{32})$ is equivalent to the Cramer-Rao lower bound, which is the lower bound of the variance of an unbiased estimator under mild regularity conditions, see Theil (1971) or Greene (2003). Thus, to get an efficient estimator in this case use weights $w_1 = 0$ and $w_2 = 1.0$. Again, this weighting scheme only yields efficient estimators in recursive models in observables (and the assumptions of linearity, model specification, and disturbance specification hold). In general, we will have to use a different weighting scheme to get efficient estimators.

Now, how do we test Model I versus Model II? We could try to develop a confidence interval or hypothesis test of the above restriction on sample moments. But a better way would be to test the parameters that differentiate Model I and Model II directly, since theoretically given our structural model, they are what we're interested in. Let's go back to Model I (the less-restrictive model) and replace population moments with sample moments to arrive at an estimator.

$$\underset{3 \times 3}{\underset{S}{\begin{bmatrix} s_{11} & & \\ s_{21} & s_{22} & \\ s_{31} & s_{32} & s_{33} \end{bmatrix}}} = \underset{3 \times 3}{\underset{\Sigma(\hat{\theta})}{\begin{bmatrix} s_{11} & & \\ \hat{\beta}_{21}\, s_{11} & \hat{\beta}_{21}^2\, s_{11} + s_{\varepsilon 22} & \\ \hat{\beta}_{31}\, s_{11} + \beta_{32}\, s_{21} & \hat{\beta}_{31}\, s_{21} + \hat{\beta}_{32}\, s_{22} & \hat{\beta}_{31}\, s_{31} + \hat{\beta}_{32}\, s_{32} + s_{\varepsilon 33} \end{bmatrix}}}$$

1. $s_{11} = s_{11}$

2. $\hat{\beta}_{21} = s_{11}/s_{21}$

3. $\hat{\beta}_{31} = (s_{22}\, s_{31} - s_{21}\, s_{32})/(s_{11}\, s_{22} - s_{21}^2)$

4. $\hat{\beta}_{32} = (s_{11}\, s_{32} - s_{21}\, s_{31})/(s_{11}\, s_{22} - s_{21}^2)$

5. $s_{\varepsilon 33}{*} = s_{33} - \hat{\beta}_{31}\, s_{31} - \hat{\beta}_{32}\, s_{32}$

Again, since this is a recursive model in observables, these are OLS estimators which are unbiased and efficient. To form confidence intervals and perform hypothesis tests, we need to calculate the standard errors of the estimates. First, since we assumed that $\varepsilon_{33i} \sim N(0,\sigma_{\varepsilon 33})$, we can assume that $\hat{\_}_{31} \sim N(\beta_{31},\sigma_{\beta 31}^2)$.
Second, we need an unbiased estimator of the variance disturbance $\sigma_{\varepsilon 33}$. The estimator, $s_{\varepsilon 33}*$, of equation (5) gives a biased estimate of $\sigma_{\varepsilon 33}$, because it doesn't take into consideration the number of parameters we have estimated in the equation for X_3. Theorems of least squares residuals tell us that an unbiased estimator of $\sigma_{\varepsilon 33}$ is:

$$s_{\varepsilon 33} = (N - 1)/(N - 2)\ (s_{33} - \hat{\beta}_{31}\, s_{31} - \hat{\beta}_{32}\, s_{32})$$

This is the two-variable case of the more general result for K parameters (variables) in the X_3 equation:

$$s_{\varepsilon 33} = (N - 1)/(N - K)\ (s_{33} - \hat{\beta}_{31}\, s_{31} - \hat{\beta}_{32}\, s_{32} - \ldots - \hat{\beta}_{3K}\, s_{3K})$$

And the adjusted $R^2 = 1 - s_{\varepsilon 33}/s_{33}$

> Aside: Note that this is the usual least squares residual estimator (Kmenta *Elements of Econometrics* 1971, p. 361):
>
> $s_{\varepsilon 33} = 1/(N - K)\ (m_{33} - \hat{\beta}_{31}\, m_{31} - \hat{\beta}_{32}\, m_{32} - \ldots - \hat{\beta}_{3K}\, m_{3K})$, where $m_{ij} = (N - 1)\, s_{ij}$ are sums of squares and cross-products.

We can obtain the variances of our estimators (Kmenta 1971):

$$\sigma_{\beta 31}^2 = (\sigma_{\varepsilon 33}\, s_{22})/(s_{11}\, s_{22} - s_{21}^2)$$

$$\sigma_{\beta 32}^2 = (\sigma_{\varepsilon 33}\, s_{11})/(s_{11}\, s_{22} - s_{21}^2)$$

But we usually don't know the value of $\sigma_{\varepsilon 33}$, so we use our unbiased estimator above. The standard error of the estimate is the square root of this:

$$s_{\beta 31} = [(s_{\varepsilon 33}\, s_{22})/(s_{11}\, s_{22} - s_{21}^2)]^{½}$$

$$s_{\beta 32} = [(s_{\varepsilon 33}\, s_{11})/(s_{11}\, s_{22} - s_{21}^2)]^{½}$$

We can then form t-statistics:

$$t_{\beta 31,n-2} = (\hat{\beta}_{31} - \beta_0)/s_{\beta 31}$$

$$t_{\beta 32,n-2} = (\hat{\beta}_{32} - \beta_0)/s_{\beta 32}$$

and perform tests of hypotheses. For example, we can test Model II against Model I by testing H_0: $\beta_{31} = 0$ versus H_1: $\beta_{31} \neq 0$:

Model II H_0: $\beta_{31} = 0$ (Null Hypothesis)
Model I H_1: $\beta_{31} \neq 0$ (Alternative Hypothesis)

$$t_{\beta 31,n-2} = \hat{\beta}_{31}/s_{\beta 31}$$

(Aside: Note that we are *not* testing Model I here—it is the maintained alternative hypothesis.)

Or, as we learned from our regression class, we can use an F-test to test the joint hypothesis: $\beta_{31} = \beta_{32} = 0$. In general, the following is true:

$[SSR/(K-1)]/[SSE/(N-K)] \sim F_{K-1, N-K}$ where $SSE = S_{\varepsilon 33}(N-K)$, $SST = s_{33}(N-1)$, and because SST = SSR + SSE, it follows that SSR = SST - SSE

In this case, $[SSR/(2-1)]/[SSE/(N-2)] \sim F_{1, N-2}$

The important point here is that for estimation and testing in recursive models in observables, we can rely completely on the general linear regression model estimated by ordinary least squares. For multiple equation models, simply use equation-by-equation ordinary least squares regressions, and use conventional t-tests and F-tests to test hypotheses. In the above example, we would run two regressions: regressing X_2 on X_1, and regressing X_3 on X_1 and X_2. To get the reduced form for the second equation, we could regress X_3 on the exogenous variable(s), X_1. We can accomplish this using the regression algorithm in any statistical software package, such as Regression in SPSS or Stata. Given the output, we can decompose total effects into direct and indirect components, compute standardized coefficients, and test hypotheses about specific parameters.

Aside: For those interested in the generalization of these results to the multiparameter single-equation case in matrix algebra, we can related our results to the general linear model. For the general case, the standard errors of the K-explanatory variables can be obtained by taking the square roots of the diagonal elements of the estimated covariance matrix of the estimator (see Kmenta 1971, pp. 357-64):

$$\underset{k \times k}{COV(\hat{\theta})} = \sigma^2 \underset{k \times k}{(X'X)^{-1}}$$

where $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots \hat{\theta}_k]'$ is a k x 1 vector of coefficients, and σ^2 is the variance of the disturbance, and $X'X^{-1}$ is the inverse of the sums of squares and cross-products of X.

We usually don't know σ^2, so we estimate it using the least squares residual estimator s^2, which gives us the estimated covariance matrix of the estimator: $s^2 (X'X)^{-1}$

We can translate this into our notation as follows. Let $\sigma^2 = \sigma_{\varepsilon 33}$, and $s^2 = s_{\varepsilon 33}$. Then let S_x be the k x k submatrix of S representing the covariance matrix of Xs (predictors) only. Therefore, $S_x = (1/N - 1)\, X'X$, and $S_x^{-1} = (N-1)\,(X'X)^{-1}$

$$\underset{k \times k}{COV(\hat{\theta})} = \sigma_{\varepsilon 33}/(N-1)\, \underset{k \times k}{S_x^{-1}}$$

Then, since $s^2 = s_{\varepsilon 33}$, our estimated covariance matrix of $\hat{\theta}$ is: $s_{\varepsilon 33}(N - 1)\, S_x^{-1}$, and the standard errors are the square roots of the diagonal elements of the k x k matrix.

To test overidentifying restrictions on two nested recursive models, use the following F-test:

$$F = \frac{[ESS(R) - ESS(U)]/r}{ESS(U)/(N - K - 1)}$$

with r and N- K - 1 degrees of freedom, where ESS(R) is the error sums of squares from the restricted model ($\beta = \beta_0$, β is an r x 1 vector of restricted coefficients) and ESS(U) is the error sums of squares for the unrestricted model $\beta \neq \beta_0$, and r is the number of restrictions.

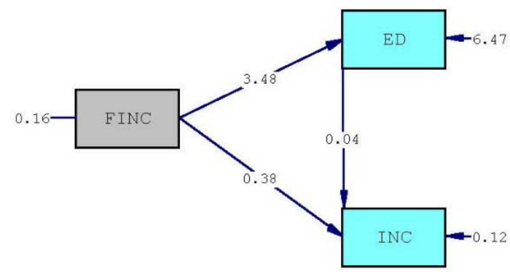
LISREL EXAMPLE:

Let's estimate our model using the following hypothetical data:

$$\underset{3\times 3}{\Sigma_\rho} = \begin{array}{ccc} \text{FINC} & \text{ED} & \text{INC} \end{array} \begin{bmatrix} 1.0 & & \\ .48 & 1.0 & \\ .50 & .45 & 1.0 \end{bmatrix}$$

SD' = .40 2.9 .41

LISREL gives us a path diagram:



Chi-Square=0.00, df=0, P-value=1.00000, RMSEA=0.000

```
 The following lines were read from file
H:\529 examples\2-eq model.LS8:

 TWO EQUATION PATH  MODEL
 DA NI=3 NO=578

 SD
 *
 .40    2.9     .41
 KM
 *
 1.0
 .48   1.0
 .50   .45   1.0
 LA
 *
 FINC ED INC
 SE
 2 3 1/
 MO NX=1 NY=2 GA=FU,FR BE=FU,FI PS=DI, FR
 FR BE 2 1
 VA 0.3 GA 1 1
 PD
 OU ME=ML  SC

 TWO EQUATION PATH MODEL

                          Number of Input Variables  3
                          Number of Y - Variables    2
                          Number of X - Variables    1
                          Number of ETA - Variables  2
                          Number of KSI - Variables  1
                          Number of Observations   578

 TWO EQUATION PATH MODEL

         Covariance Matrix

                  ED        INC       FINC
            --------   --------   --------
       ED       8.41
      INC       0.54       0.17
     FINC       0.56       0.08       0.16


 TWO EQUATION PATH MODEL

 Parameter Specifications

         BETA

                  ED        INC
            --------   --------
       ED          0          0
      INC          1          0

         GAMMA
```

```
             FINC
         --------
      ED          2
     INC          3

        PHI

             FINC
         --------
                4

        PSI

               ED        INC
         --------   --------
                5          6


 TWO EQUATION PATH MODEL

 Number of Iterations =  4

 LISREL Estimates (Maximum Likelihood)

        BETA

               ED        INC
         --------   --------
      ED      - -        - -
     INC     0.04        - -
           (0.01)
             6.90

        GAMMA

             FINC
         --------
      ED     3.48
           (0.27)
            13.13
     INC     0.38
           (0.04)
             9.34

        Covariance Matrix of Y and X

               ED        INC       FINC
         --------   --------   --------
      ED     8.41
     INC     0.54       0.17
    FINC     0.56       0.08       0.16

        PHI

             FINC
         --------
             0.16
           (0.01)
            16.97

        PSI
        Note: This matrix is diagonal.

               ED        INC
         --------   --------
```

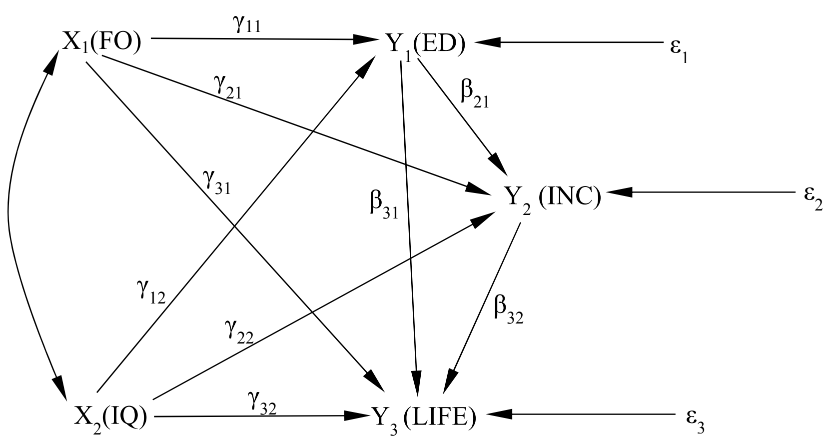
```
             6.47       0.12
            (0.38)     (0.01)
            16.97      16.97

        Squared Multiple Correlations for Structural Equations

               ED        INC
           --------   --------
             0.23       0.31

        Squared Multiple Correlations for Reduced Form

               ED        INC
           --------   --------
             0.23       0.25

        Reduced Form

             FINC
           --------
      ED       3.48
              (0.27)
              13.13

     INC       0.51
              (0.04)
              13.86


                          Goodness of Fit Statistics

                             Degrees of Freedom = 0
              Minimum Fit Function Chi-Square = 0.00 (P = 1.00)
      Normal Theory Weighted Least Squares Chi-Square = 0.00 (P = 1.00)

                 The Model is Saturated, the Fit is Perfect !


 TWO EQUATION PATH MODEL

 Standardized Solution

        BETA

               ED        INC
           --------   --------
      ED       - -        - -
     INC      0.27        - -

        GAMMA

             FINC
           --------
      ED      0.48
     INC      0.37

        Correlation Matrix of Y and X

               ED        INC       FINC
           --------   --------   --------
      ED      1.00
     INC      0.45       1.00
    FINC      0.48       0.50       1.00

        PSI
        Note: This matrix is diagonal.

               ED        INC
```

```
           --------   --------
               0.77       0.69

         Regression Matrix Y on X (Standardized)

               FINC
           --------
      ED       0.48
     INC       0.50

                           Time used:    0.031 Seconds
```

IV. INTERPRETING STRUCTURAL PARAMETERS IN A THREE-EQUATION MODEL.

Let's decompose total (reduced-form) effects into direct and indirect effects using path analysis, which was developed by the population geneticist, Sewall Wright. In this exercise, we are in the population (we are God), and don't have to worry about sample data, estimation, statistical inference or testing). We just compute the population parameters.

Consider the following model:



I've changed notation slightly, using gamma, γ, as a parameter relating exogenous variables (Xs) to endogenous variables (Ys), while reserving betas, βs, as parameters relating endogenous variables to other endogenous variables.

$$Y_1 = \gamma_{11} X_1 + \gamma_{12} X_2 + \varepsilon_1$$

$$Y_2 = \gamma_{21} X_1 + \gamma_{22} X_2 + \beta_{21} Y_1 + \varepsilon_2$$

$$Y_3 = \gamma_{31} X_1 + \gamma_{32} X_2 + \beta_{31} Y_1 + \beta_{32} Y_2 + \varepsilon_3$$

Again, assume we're in deviation scores, and that we've made the usual assumptions about the disturbances, plus the following additional assumptions:

$$E(\varepsilon_1 \varepsilon_2) = E(\varepsilon_1 \varepsilon_3) = E(\varepsilon_2 \varepsilon_3) = 0$$

$$E(X_1 \varepsilon_1) = E(X_1 \varepsilon_2) = E(X_1 \varepsilon_3) = E(X_2 \varepsilon_1) = E(X_2 \varepsilon_2) = E(X_2 \varepsilon_3) = 0$$

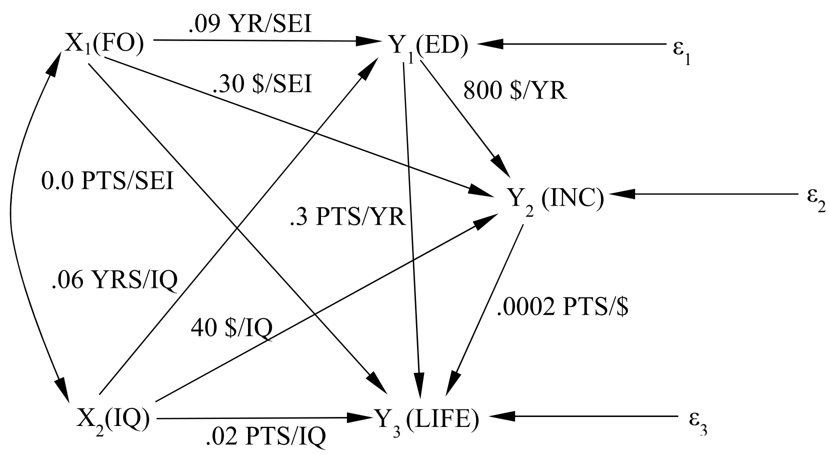
$$E(Y_1 \varepsilon_2) = E(Y_1 \varepsilon_3) = E(Y_2 \varepsilon_3) = 0$$

We begin with the population (covariances) moments among observables:

$$\underset{5\times 5}{\Sigma} = \begin{array}{c} \begin{array}{ccccc} X_1 & X_2 & Y_1 & Y_2 & Y_3 \end{array} \\ \begin{bmatrix} 400 & & & & \\ 150 & 225 & & & \\ 45 & 27 & 9 & & \\ 54 & 29 & 9.6 & 36 & \\ 27.6 & 19.8 & 5.2 & 10.9 & 9 \end{bmatrix} \end{array} \qquad \sigma = \begin{bmatrix} 20\ SEI \\ 15\ IQ \\ 3\ YRS \\ \$6K \\ 3PTS \end{bmatrix}$$

Note: square roots of diagonal elements of Σ gives a vector of standard deviations:

If we computed parameters in terms of moments, we'd come up with the following values for structural parameters:



Our three equations (in their structural form) look as follows (standardized parameters appear in parentheses):

$$\begin{aligned} Y_1 &= \underset{(.60)}{.09\, X_1} + \underset{(.30)}{.06\, X_2} + \varepsilon_1 \\ Y_2 &= \underset{(.10)}{30\, X_1} + \underset{(.10)}{40\, X_2} + \underset{(.40)}{800\, Y_1} + \varepsilon_2 \\ Y_3 &= \underset{(.00)}{0.0\, X_1} + \underset{(.10)}{.02\, X_2} + \underset{(.30)}{.30\, Y_1} + \underset{(.40)}{.0002\, Y_2} + \varepsilon_3 \end{aligned}$$

What do these parameter values of direct effects tell us about stratification and quality of life? Note that if we were working with sample data, we could use multiple regression to obtain OLS estimates of the parameters of our three equations. But we can also decompose total effects into direct and indirect effects, which would tell us more about the specific mechanisms operating here. Recall that to get total effects we compute reduced forms from the above structural forms. The structural form of the Y_1 equation is also the reduced form (since it doesn't contain any endogenous predictors). Let's begin with Y_2 equation:

$$Y_2 = \gamma_{21} X_1 + \gamma_{22} X_2 + \beta_{21} Y_1 + \varepsilon_2$$

Replace Y_1 with the structural equation for $Y_1 = \gamma_{11} X_1 + \gamma_{12} X_2 + \varepsilon_1$:

$$Y_2 = \gamma_{21} X_1 + \gamma_{22} X_2 + \beta_{21} (\gamma_{11} X_1 + \gamma_{12} X_2 + \varepsilon_1) + \varepsilon_2 \qquad \text{Substitute for } Y_1$$

$$= \gamma_{21} X_1 + \gamma_{22} X_2 + \gamma_{11} \beta_{21} X_1 + \gamma_{12} \beta_{21} X_2 + \beta_{21} \varepsilon_1 + \varepsilon_2 \quad \text{Multiply terms}$$

$$= \gamma_{21} X_1 + \gamma_{11} \beta_{21} X_1 + \gamma_{22} X_2 + \gamma_{12} \beta_{21} X_2 + \beta_{21} \varepsilon_1 + \varepsilon_2 \quad \text{Rearrange terms}$$

$$= \underbrace{(\gamma_{21} + \gamma_{11} \beta_{21})}_{} X_1 + \underbrace{(\gamma_{22} + \gamma_{12} \beta_{21})}_{} X_2 + \underbrace{(\beta_{21} \varepsilon_1 + \varepsilon_2)}_{} \qquad \text{Factor terms by } X_1 \text{ and } X_2$$

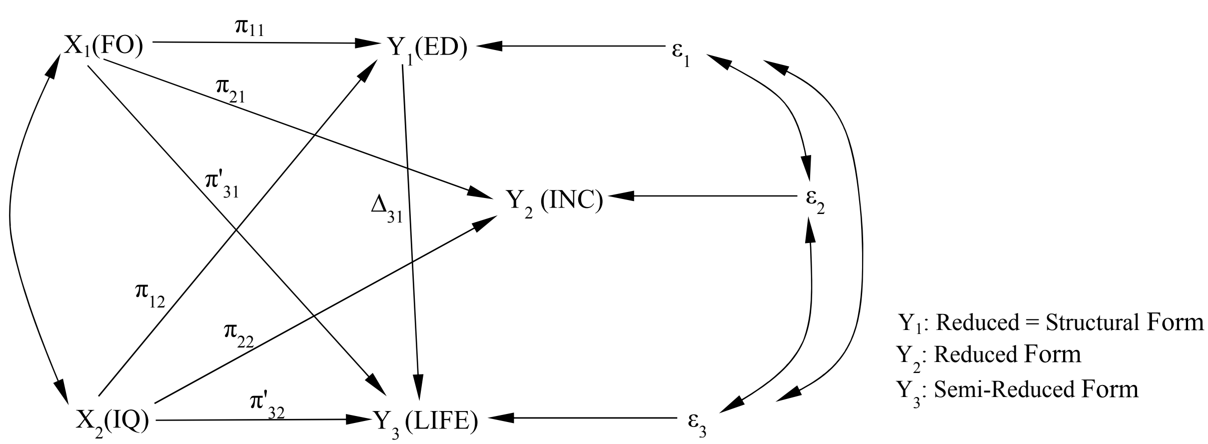
$$Y_2 \quad = \quad \pi_{21} \; X_1 \quad + \quad \pi_{22} \quad X_2 \quad + \quad \pi_{\varepsilon 2}$$

> Aside: If we were working with sample data and had to estimate the reduced-form parameters, πs, we could simply regress Y_2 on X_1 and X_2 using OLS to get unbiased and efficient estimates, since $E(\pi_{\varepsilon 2} X_1) = E(\pi_{\varepsilon 2} X_2) = 0$.

If you look closely at the πs, notice that they contain a direct effect of X plus indirect effects of X (products of parameters) through intervening variables. In general, reduced-form coefficients give total effects, which can be decomposed into direct and indirect effects. In our example,

For the effect of X_1 (FO) on Y_2 (INC)					For the effect of X_2 (IQ) on Y_2 (INC)				
π_{21}	=	γ_{21}	+	$\gamma_{11} \quad \beta_{21}$	π_{22}	=	γ_{22}	+	$\gamma_{12} \quad \beta_{21}$
Total Effect (of FO on INC)	=	Direct Effect	+	Indirect Effect (FO through ED)	Total Effect (of IQ on INC)	=	Direct Effect	+	Indirect Effect (IQ through ED)
102 \$/SEI (.34)	=	30\$ / SEI (.30)	+	72 \$/SEI (.04)	88 \$/IQ (.22)	=	40 \$/IQ (.10)	+	48 \$/IQ (.12)

What does this tell us substantively?



Here is a path diagram of the reduced form of Y_1 and Y_2, and the semi-reduced form for Y_3:

We can do the same for our equation for quality of life, Y_3. But here we have two endogenous variables, Y_1 and Y_2. We can do two substitutions. The first, substituting Y_2 with its equation gives us the semi-reduced form:

$$Y_3 = \gamma_{31} X_1 + \gamma_{32} X_2 + \beta_{31} Y_1 + \beta_{32} Y_2 + \varepsilon_3$$

$$= \gamma_{31} X_1 + \gamma_{32} X_2 + \beta_{31} Y_1 + \beta_{32} (\gamma_{21} X_1 + \gamma_{22} X_2 + \beta_{21} Y_1 + \varepsilon_2) + \varepsilon_3$$

$$= \underbrace{(\gamma_{31} + \gamma_{21} \beta_{32})}_{} X_1 + \underbrace{(\gamma_{32} + \gamma_{22} \beta_{32})}_{} X_2 + \underbrace{(\beta_{31} + \beta_{21} \beta_{32})}_{} Y_1 + \underbrace{(\beta_{32} \varepsilon_2 + \varepsilon_3)}_{}$$

$$Y_3 = \pi'_{31} X_1 + \pi'_{32} X_2 + \Delta_{31} Y_1 + \pi'_{\varepsilon_3}$$

where π' means semi-reduced form and Δ refers to the total effect of an endogenous variable. For our example, we again can compute direct effects of our predictors, and the indirect effects through schooling:

For the effect of X_1 (FO) on Y_3 (LIFE)

$$\pi'_{31} = \gamma_{31} + \gamma_{21}\beta_{32}$$

Semi-Total Effect (of FO on LIFE) = Direct Effect + Indirect Effect (FO through INC)

.006 PTS/SEI (.04) = 0 PTS/SEI (0.0) + .006 PTS/SEI (.04)

For the effect of X_2 (IQ) on Y_3 (LIFE)

$$\pi'_{32} = \gamma_{32} + \gamma_{22}\beta_{32}$$

Semi-Total Effect (of IQ on LIFE) = Direct Effect + Indirect Effect (IQ thru INC)

.028 PTS/IQ (.14) = .02 PTS/IQ (.10) + .008 PTS/IQ (.04)

For the effect of Y_1 (ED) on Y_3 (LIFE)

$$\Delta_{31} = \beta_{31} + \beta_{21}\beta_{32}$$

Total Effect (of ED on LIFE) = Direct Effect + Indirect Effect (ED through INC)

.46 PTS/YR (.46) = .30 PTS/YR (.30) + .16 PTS/YR (.16)

Again, what does this tell us substantively? Finally, we can compute the reduced-form of our Y_3 equation. We can start with our semi-reduced form equation (which already substituted for Y_2) and substitute for Y_1.

$$Y_3 = (\gamma_{31} + \gamma_{21}\beta_{32}) X_1 + (\gamma_{32} + \gamma_{22}\beta_{32}) X_2 + (\beta_{31} + \beta_{21}\beta_{32}) Y_1 + (\beta_{32}\varepsilon_2 + \varepsilon_3)$$

Now substitute for $Y_1 = \gamma_{11} X_1 + \gamma_{12} X_2 + \varepsilon_1$

$$Y_3 = (\gamma_{31} + \gamma_{21}\beta_{32}) X_1 + (\gamma_{32} + \gamma_{22}\beta_{32}) X_2 + (\beta_{31} + \beta_{21}\beta_{32})(\gamma_{11} X_1 + \gamma_{12} X_2 + \varepsilon_1) + (\beta_{32}\varepsilon_2 + \varepsilon_3)$$

If we multiply this out, which is very tedious, we arrive at:

$$Y_3 = (\gamma_{31} + \gamma_{21}\beta_{32}) X_1 + (\gamma_{32} + \gamma_{22}\beta_{32}) X_2 + \gamma_{11}\beta_{31} X_1 + \gamma_{12}\beta_{31} X_2 + \beta_{31}\varepsilon_1 + \gamma_{11}\beta_{21}\beta_{32} X_1 +$$
$$\gamma_{12}\beta_{31}\beta_{32} X_2 + \beta_{21}\beta_{32}\varepsilon_1 + (\beta_{32}\varepsilon_2 + \varepsilon_3)$$

$$Y_3 = \underbrace{(\gamma_{31} + \gamma_{11}\beta_{31} + \gamma_{21}\beta_{32} + \gamma_{11}\beta_{21}\beta_{32})}_{\pi_{31}} X_1 + \underbrace{(\gamma_{32} + \gamma_{12}\beta_{31} + \gamma_{22}\beta_{32} + \gamma_{12}\beta_{31}\beta_{32})}_{\pi_{32}} X_2 + \underbrace{(\beta_{21}\beta_{32}\varepsilon_1 + \beta_{32}\varepsilon_2 + \varepsilon_3)}_{\pi_{\varepsilon 3}}$$

For the effect of X_1 (FO) on Y_3 (LIFE)

$$\pi_{31} = \gamma_{31} + \gamma_{11}\beta_{31} + \gamma_{21}\beta_{32} + \gamma_{11}\beta_{21}\beta_{32}$$

Total Effect (of FO on LIFE) = Direct Effect + Indirect Effects (through ED & INC)

.048 PTS/SEI (.32) = 0.0 PTS/SEI (0.0) + .048 PTS/SEI (.32)

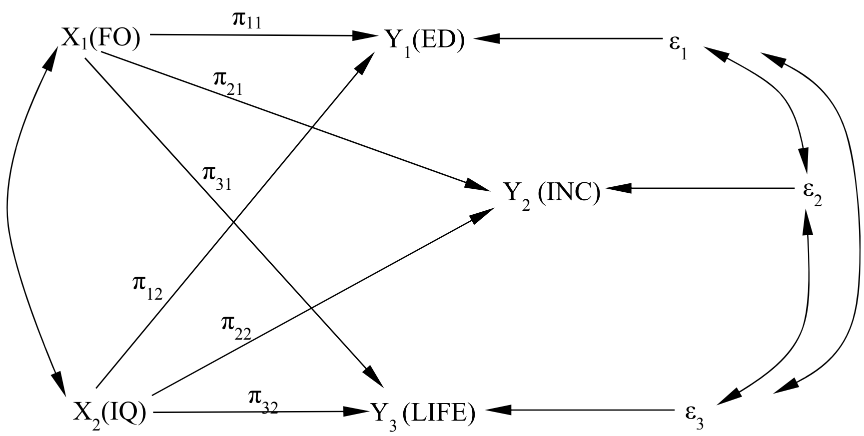
For the effect of X_2 (IQ) on Y_3 (LIFE)

$$\pi_{32} = \gamma_{32} + \gamma_{12}\beta_{31} + \gamma_{22}\beta_{32} + \gamma_{12}\beta_{21}\beta_{32}$$

Total Effect (of IQ on LIFE) = Direct Effect + Indirect Effects (through ED & INC)

.056 PTS/IQ (.28) = .02 PTS/IQ (.10) + .036 PTS/IQ (.18)

Give a substantive interpretation to these direct and indirect effects. Here is a path diagram of the model in its reduced form for all equations:



Y_1: Reduced = Structural Form
Y_2: Reduced Form
Y_3: Reduced Form

We can place these parameter values in a table:

Equation	Dependent Variable	Predetermined Variables: X_1 FO	X_2 IQ	Y_1 ED	Y_2 INC	σ_ε	R^2
1.	Y_1 ED	.09 (.60)	.06 (.30)	--	--	1.82	.63
2.	Y_2 INC	102 (.34)	88 (.22)	--	--	5235	.24
3.	Y_2 INC	30 (.10)	40 (.10)	800 (.40)	--	5057	.30
4.	Y_3 LIFE	.048 (.32)	.056 (.28)	--	--	2.56	.27
5.	Y_3 LIFE	.006 (.04)	.028 (.14)	.46 (.46)	--	2.41	.35
6.	Y_3 LIFE	.000 (.00)	.020 (.10)	.30 (.30)	.0002 (.40)	2.20	.46

Note: Standardized coefficients appear in parentheses.

In this table, the single equation (line 1) for education is both structural and reduced forms. The equation for income appears in reduced-form in line 2 and structural form in line 3. The equation for quality of life appears in reduced-form in line 2, semi-reduced form in line 5, and structural form in line 6. As an exercise, study the relationships between this table, the path diagram, and the above equations.

Here is the model run in LISREL 8.8, with the output annotated:

1. First the full model:

```
The following lines were read from file C:\529 examples\status2.ls8:

STATUS ATTAINMENT AND LIFE SATISFACTION
DA NI=5 NO=1000
SD
*
20 15 3 60.00 3
```

(DA)ta line NI means number of input observable variables. NO is number of observations. I'm inputting standard deviations (SD) and a correlation matrix (KM)

```
KM
*
1.00
.500 1.00
.750 .600 1.00
.450 .390 .535 1.00
.460 .440 .580 .605 1.00

LA
```

LA stands for labels; * means free format

```
*
FOCC IQ ED INCOME LIFE
SE
3 4 5 1 2 \
```

Here I'm (SE)lecting variables to analyze; y's first, x's next. The backslash says stop here.

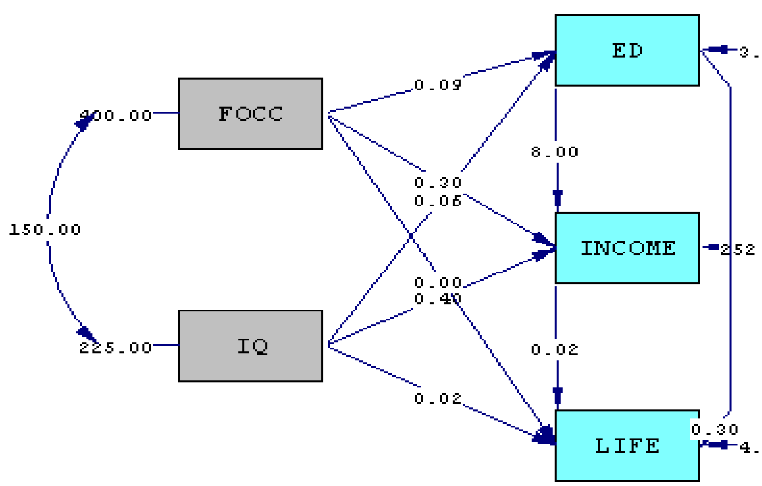
```
MO NX=2 NY=3 GA=FU,FR BE=FU,FI PS=DI,FR
FR BE 3 2 BE 3 1 BE 2 1
PD
OU ME=ML RS EF SC
```

(MO)del parameters. NX is # Xs; NY is # Ys. BE is full & fixed; PS is diagonal and free. I then free select Bs. (OU)put. PD asks for a path diagram.

Method of estimation is (M)aximum (L)ikelihood. RS asks for (R)esiduals and *fitted* moment matrix $\Sigma(\hat{\theta})$. EF asks for total and indirect effects, SC for completely standardized solution.

```
STATUS ATTAINMENT AND LIFE SATISFACTION

                    Number of Input Variables   5
                    Number of Y - Variables     3
                    Number of X - Variables     2
                    Number of ETA - Variables   3
                    Number of KSI - Variables   2
                    Number of Observations   1000
```

Check to be sure data are read correctly.



Chi-Square=0.00, df=0, P-value=1.00000, RMSEA=0.000

STATUS ATTAINMENT AND LIFE SATISFACTION

Covariance Matrix **= S**

	ED	INCOME	LIFE	FOCC	IQ
ED	9.00				
INCOME	96.30	3600.00			
LIFE	5.22	108.90	9.00		
FOCC	45.00	540.00	27.60	400.00	
IQ	27.00	351.00	19.80	150.00	225.00

STATUS ATTAINMENT AND LIFE SATISFACTION

Parameter Specifications **Note: 0 means the matrix elements is a constant (fixed). Parameters to be estimated (free) are numbered.**

BETA = **Matrix of regression coefficients among endogenous variables (etas – η) (here ys)**

	ED	INCOME	LIFE
ED	0	0	0
INCOME	1	0	0
LIFE	2	3	0

$$\begin{matrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \beta_{21} & \mathbf{0} & \mathbf{0} \\ \beta_{31} & \beta_{32} & \mathbf{0} \end{matrix} = \mathbf{B}$$

GAMMA = **Matrix of coefficients from regressing etas (η) on ksis (ξ) – or in this case x on y**

	FOCC	IQ
ED	4	5
INCOME	6	7
LIFE	8	9

$$\begin{matrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{21} & \gamma_{23} \\ \gamma_{31} & \gamma_{23} & \gamma_{33} \end{matrix} = \Gamma$$

PHI = **Covariance matrix among exogenous variables (ksis ξ); E(ξ ξ') = Φ**

	FOCC	IQ
FOCC	10	
IQ	11	12

$$\begin{matrix} \varphi_{11} & \\ \varphi_{21} & \varphi_{11} \end{matrix} = \Phi$$

PSI = **Covariance matrix of disturbances (psi ζ); E(ζ ζ') = Ψ**

ED	INCOME	LIFE
13	14	15

$$\begin{matrix} \psi_{11} & & \\ \mathbf{0} & \psi_{22} & \\ \mathbf{0} & \mathbf{0} & \psi_{33} \end{matrix} = \Psi$$

STATUS ATTAINMENT AND LIFE SATISFACTION

Number of Iterations = 0

LISREL Estimates (Maximum Likelihood)

Note: Matrix entries are unstandardized estimates, standard errors (in parentheses) and t-values. Also recall that ML in fully-recursive linear models in observables is exactly equation-by-equation OLS.

```
         BETA
                  ED     INCOME       LIFE
            --------   --------   --------
       ED        - -        - -        - -

   INCOME       8.00        - -        - -
              (0.87)
                9.17

     LIFE       0.30       0.02        - -
              (0.04)     (0.00)
                7.65      14.57

         GAMMA

                FOCC         IQ
            --------   --------
       ED       0.09       0.06
              (0.00)     (0.00)
               26.97      13.49

   INCOME       0.30       0.40
              (0.12)     (0.13)
                2.48       3.00

     LIFE       0.00       0.02
              (0.01)     (0.01)
                0.02       3.43

         Covariance Matrix of Y and X

                  ED     INCOME       LIFE       FOCC         IQ
            --------   --------   --------   --------   --------
       ED       9.00
   INCOME      96.30    3600.00
     LIFE       5.22     108.90       9.00
     FOCC      45.00     540.00      27.60     400.00
       IQ      27.00     351.00      19.80     150.00     225.00

         PHI

                FOCC         IQ
            --------   --------
     FOCC     400.00
             (17.92)
               22.33

       IQ     150.00     225.00
             (10.62)    (10.08)
               14.12      22.33

         PSI
         Note: This matrix is diagonal.

                  ED     INCOME       LIFE
            --------   --------   --------
                3.33    2527.20       4.82
              (0.15)   (113.19)     (0.22)
               22.33      22.33      22.33


         Squared Multiple Correlations for Structural Equations
```

R-squareds

ED	INCOME	LIFE
0.63	0.30	0.46

NOTE: Rý for Structural Equations are Hayduk's (2006) Blocked-Error Rý

Reduced Form

These are Pi's in our notation π_{yx}

	FOCC	IQ
ED	0.09	0.06
	(0.00)	(0.00)
	26.97	13.49
INCOME	1.02	0.88
	(0.10)	(0.13)
	10.66	6.90
LIFE	0.05	0.06
	(0.00)	(0.01)
	10.24	8.96

Squared Multiple Correlations for Reduced Form **(Note the magnitudes compared to the structural form.)**

ED	INCOME	LIFE
0.63	0.24	0.27

Goodness of Fit Statistics

Degrees of Freedom = 0
Minimum Fit Function Chi-Square = 0.0 (P = 1.00)
Normal Theory Weighted Least Squares Chi-Square = 0.00 (P = 1.00)

The Model is Saturated, the Fit is Perfect !

STATUS ATTAINMENT AND LIFE SATISFACTION

Standardized Solution

BETA

e.g., $\beta\sigma_x/\sigma_y = P_{yx}^{\beta}$

	ED	INCOME	LIFE
ED	- -	- -	- -
INCOME	0.40	- -	- -
LIFE	0.30	0.40	- -

GAMMA

e.g., $\gamma\,\sigma_x/\sigma_y = P_{yx}^{\gamma}$

	FOCC	IQ
ED	0.60	0.30
INCOME	0.10	0.10
LIFE	0.00	0.10

Correlation Matrix of Y and X e.g., ρ_{xy}

	ED	INCOME	LIFE	FOCC	IQ
ED	1.00				
INCOME	0.53	1.00			
LIFE	0.58	0.60	1.00		
FOCC	0.75	0.45	0.46	1.00	
IQ	0.60	0.39	0.44	0.50	1.00

PSI
Note: This matrix is diagonal.

ED	INCOME	LIFE
0.37	0.70	0.54

Regression Matrix Y on X (Standardized)

e.g., $\pi_{yx}\sigma_x / \sigma_y = P_{yx}^{\pi}$

	FOCC	IQ
ED	0.60	0.30
INCOME	0.34	0.22
LIFE	0.32	0.28

STATUS ATTAINMENT AND LIFE SATISFACTION

Total and Indirect Effects

Total Effects of X on Y

	FOCC	IQ
ED	0.09	0.06
	(0.00)	(0.00)
	26.97	13.49
INCOME	1.02	0.88
	(0.10)	(0.13)
	10.66	6.90
LIFE	0.05	0.06
	(0.00)	(0.01)
	10.24	8.96

Reduced Form **π's**

$$\begin{matrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \\ \pi_{31} & \pi_{32} \end{matrix} = \Pi$$

Indirect Effects of X on Y

	FOCC	IQ		
ED	- -	- -		
INCOME	0.72	0.48	$(\gamma_{11}\beta_{21})$	$(\gamma_{12}\beta_{21})$
	(0.08)	(0.06)		
	8.68	7.58		
LIFE	0.05	0.04	$(\gamma_{31}+\gamma_{21}\beta_{32}+\gamma_{11}\beta_{31}+\gamma_{11}\beta_{21}\beta_{32})$	$(\gamma_{32}+\gamma_{22}\beta_{32}+\gamma_{12}\beta_{31}+\gamma_{12}\beta_{21}\beta_{32})$
	(0.00)	(0.00)		
	11.36	9.01		

Total Effects of Y on Y

	ED	INCOME	LIFE		
ED	- -	- -	- -		
INCOME	8.00	- -	- -	β_{21}	
	(0.87)				
	9.17				
LIFE	0.46	0.02	- -	$\beta_{31} + \beta_{32}\,\beta_{21}$	β_{32}
	(0.04)	(0.00)			
	11.08	14.57			

Largest Eigenvalue of B*B' (Stability Index) is 64.092

Indirect Effects of Y on Y

	ED	INCOME	LIFE	
ED	- -	- -	- -	
INCOME	- -	- -	- -	
LIFE	0.16	- -	- -	$\beta_{32}\,\beta_{21}$
	(0.02)			
	7.76			

STATUS ATTAINMENT AND LIFE SATISFACTION

Standardized Total and Indirect Effects

Standardized Total Effects of X on Y

	FOCC	IQ
ED	0.60	0.30
INCOME	0.34	0.22
LIFE	0.32	0.28

Standardized Indirect Effects of X on Y

	FOCC	IQ
ED	- -	- -
INCOME	0.24	0.12
LIFE	0.32	0.18

Standardized Total Effects of Y on Y

	ED	INCOME	LIFE
ED	- -	- -	- -
INCOME	0.40	- -	- -
LIFE	0.46	0.40	- -

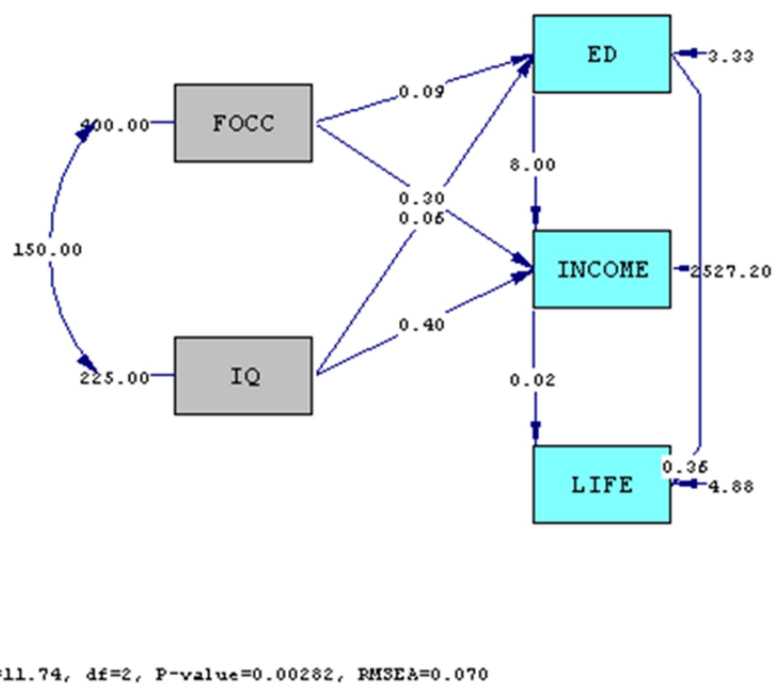
Standardized Indirect Effects of Y on Y

	ED	INCOME	LIFE
ED	- -	- -	- -
INCOME	- -	- -	- -

```
     LIFE       0.16         - -         - -

                             Time used:     0.031 Seconds
```

2. Now let's run a constrained model, in which we constrain $\gamma_{31} = \gamma_{21} = 0$:

```
The following lines were read from file C:\529 examples\status6.ls8:

STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0
DA NI=5 NO=1000
SD
*
20 15 3 60.00 3

KM
*
1.00
.500 1.00
.750 .600 1.00
.450 .390 .535 1.00
.460 .440 .580 .605 1.00

LA
*
FOCC IQ ED INCOME LIFE

SE
3 4 5 1 2 \
MO NX=2 NY=3 GA=FU,FR BE=FU,FI PS=DI,FR
FR BE 3 2 BE 3 1 BE 2 1
FI GA 3 1 GA 3 2             Here I'm fixing GA 3 1 and GA 3 2 so they're not estimated
VA 0 GA 3 1 GA 3 2           I'm giving the fixed parameters values of zero
PD
OU ME=ML RS EF SC
```



```
 STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0

                          Number of Input Variables  5
                          Number of Y - Variables    3
                          Number of X - Variables    2
                          Number of ETA - Variables  3
                          Number of KSI - Variables  2
                          Number of Observations  1000

 STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0

         Covariance Matrix

                  ED     INCOME       LIFE       FOCC         IQ
            --------   --------   --------   --------   --------
      ED        9.00
  INCOME       96.30    3600.00
    LIFE        5.22     108.90       9.00
    FOCC       45.00     540.00      27.60     400.00
      IQ       27.00     351.00      19.80     150.00     225.00

Parameter Specifications

         BETA
                  ED     INCOME       LIFE
            --------   --------   --------
      ED           0          0          0
  INCOME           1          0          0
    LIFE           2          3          0

         GAMMA
                FOCC         IQ
            --------   --------
      ED           4          5
  INCOME           6          7
    LIFE           0          0                  Note effects of FOCC and IQ are fixed.

         PHI
               FOCC          IQ
            --------   --------
    FOCC           8
      IQ           9         10

         PSI
                  ED     INCOME       LIFE
            --------   --------   --------
                  11         12         13

Number of Iterations = 12

 LISREL Estimates (Maximum Likelihood)

         BETA
                  ED     INCOME       LIFE
            --------   --------   --------
      ED         - -        - -        - -
  INCOME        8.00        - -        - -
              (0.87)
                9.17

    LIFE        0.36       0.02        - -
              (0.03)     (0.00)
               13.01      14.96
```

```
        GAMMA

                FOCC          IQ
            --------    --------
      ED        0.09        0.06
              (0.00)      (0.00)
               26.97       13.49

  INCOME        0.30        0.40
              (0.12)      (0.13)
                2.48        3.00

    LIFE         - -         - -
```

`**Note: No estimate for GA 31 and GA 3 2**

```
        Covariance Matrix of Y and X

                  ED      INCOME        LIFE        FOCC          IQ
            --------    --------    --------    --------    --------
      ED        9.00
  INCOME       96.30     3600.00
    LIFE        5.22      108.90        9.00
    FOCC       45.00      540.00       27.31      400.00
      IQ       27.00      351.00       16.94      150.00      225.00

        PHI
                FOCC          IQ
            --------    --------
    FOCC      400.00
             (17.92)
               22.33

      IQ      150.00      225.00
             (10.62)     (10.08)
               14.12       22.33

        PSI
        Note: This matrix is diagonal.

                  ED      INCOME        LIFE
            --------    --------    --------
                3.33     2527.20        4.88
              (0.15)    (113.19)      (0.22)
               22.33       22.33       22.33

        Squared Multiple Correlations for Structural Equations

                  ED      INCOME        LIFE
            --------    --------    --------
                0.63        0.30        0.46

 NOTE: Rý for Structural Equatioms are Hayduk's (2006) Blocked-Error Rý

        Reduced Form

                FOCC          IQ
            --------    --------
      ED        0.09        0.06
              (0.00)      (0.00)
               26.97       13.49

  INCOME        1.02        0.88
              (0.10)      (0.13)
               10.66        6.90

    LIFE        0.05        0.04
              (0.00)      (0.00)
```

```
             16.04      10.65

        Squared Multiple Correlations for Reduced Form

                ED     INCOME       LIFE
           --------   --------   --------
              0.63       0.24       0.24

                          Goodness of Fit Statistics

                             Degrees of Freedom = 2
             Minimum Fit Function Chi-Square = 11.81 (P = 0.0027) 2df test of γ31 γ32 = 0
     Normal Theory Weighted Least Squares Chi-Square = 11.74 (P = 0.0028)
               Estimated Non-centrality Parameter (NCP) = 9.74
           90 Percent Confidence Interval for NCP = (2.45 ; 24.50)

                        Minimum Fit Function Value = 0.012
             Population Discrepancy Function Value (F0) = 0.0098
           90 Percent Confidence Interval for F0 = (0.0025 ; 0.025)
           Root Mean Square Error of Approximation (RMSEA) = 0.070
          90 Percent Confidence Interval for RMSEA = (0.035 ; 0.11)
             P-Value for Test of Close Fit (RMSEA < 0.05) = 0.16

                Expected Cross-Validation Index (ECVI) = 0.038
          90 Percent Confidence Interval for ECVI = (0.031 ; 0.053)
                       ECVI for Saturated Model = 0.030
                      ECVI for Independence Model = 2.94

    Chi-Square for Independence Model with 10 Degrees of Freedom = 2918.03
                            Independence AIC = 2928.03
                                Model AIC = 37.74
                              Saturated AIC = 30.00
                           Independence CAIC = 2957.57
                               Model CAIC = 114.54
                             Saturated CAIC = 118.62

                          Normed Fit Index (NFI) = 1.00
                        Non-Normed Fit Index (NNFI) = 0.98
                     Parsimony Normed Fit Index (PNFI) = 0.20
                        Comparative Fit Index (CFI) = 1.00
                        Incremental Fit Index (IFI) = 1.00
                         Relative Fit Index (RFI) = 0.98

                             Critical N (CN) = 784.72

                      Root Mean Square Residual (RMR) = 0.74
                             Standardized RMR = 0.016
                        Goodness of Fit Index (GFI) = 1.00
                   Adjusted Goodness of Fit Index (AGFI) = 0.96
                  Parsimony Goodness of Fit Index (PGFI) = 0.13

 STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0
```

Fitted Covariance Matrix $= \sum(\hat{\theta})$

```
                ED     INCOME       LIFE       FOCC         IQ
           --------   --------   --------   --------   --------
       ED      9.00
   INCOME     96.30    3600.00
     LIFE      5.22     108.90       9.00
     FOCC     45.00     540.00      27.31     400.00
       IQ     27.00     351.00      16.94     150.00     225.00
```

Fitted Residuals $= \sum(\hat{\theta}) - S$

```
                 ED     INCOME       LIFE       FOCC         IQ
            --------   --------   --------   --------   --------
      ED       - -
  INCOME      0.00        - -
    LIFE      0.00       0.00       0.00
    FOCC       - -        - -       0.29        - -
      IQ       - -       0.00       2.86        - -        - -

 Summary Statistics for Fitted Residuals

 Smallest Fitted Residual =    0.00
   Median Fitted Residual =    0.00
  Largest Fitted Residual =    2.86

 Stemleaf Plot

 - 0|0000000000000
   0|3
   1|
   2|9

         Standardized Residuals

                 ED     INCOME       LIFE       FOCC         IQ
            --------   --------   --------   --------   --------
      ED       - -
  INCOME       - -        - -
    LIFE       - -        - -        - -
    FOCC       - -        - -       0.32        - -
      IQ       - -        - -       3.42        - -        - -

 Summary Statistics for Standardized Residuals

 Smallest Standardized Residual =    0.00
   Median Standardized Residual =    0.00
  Largest Standardized Residual =    3.42

 Stemleaf Plot

   0|00000000000003
   1|
   2|
   3|4
 Largest Positive Standardized Residuals
 Residual for       IQ and     LIFE   3.42

 STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0

 Standardized Solution

         BETA

                 ED     INCOME       LIFE
            --------   --------   --------
      ED       - -        - -        - -
  INCOME      0.40        - -        - -
    LIFE      0.36       0.41        - -

         GAMMA

               FOCC         IQ
            --------   --------
      ED      0.60       0.30
  INCOME      0.10       0.10
    LIFE       - -        - -

         Correlation Matrix of Y and X
```

	ED	INCOME	LIFE	FOCC	IQ
ED	1.00				
INCOME	0.54	1.00			
LIFE	0.58	0.60	1.00		
FOCC	0.75	0.45	0.46	1.00	
IQ	0.60	0.39	0.38	0.50	1.00

PSI
Note: This matrix is diagonal.

ED	INCOME	LIFE
0.37	0.70	0.54

Regression Matrix Y on X (Standardized)

	FOCC	IQ
ED	0.60	0.30
INCOME	0.34	0.22
LIFE	0.36	0.20

STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0

Total and Indirect Effects

Total Effects of X on Y

	FOCC	IQ
ED	0.09	0.06
	(0.00)	(0.00)
	26.97	13.49
INCOME	1.02	0.88
	(0.10)	(0.13)
	10.66	6.90
LIFE	0.05	0.04
	(0.00)	(0.00)
	16.04	10.65

Indirect Effects of X on Y

	FOCC	IQ
ED	- -	- -
INCOME	0.72	0.48
	(0.08)	(0.06)
	8.68	7.58
LIFE	0.05	0.04
	(0.00)	(0.00)
	16.04	10.65

Total Effects of Y on Y

	ED	INCOME	LIFE
ED	- -	- -	- -
INCOME	8.00	- -	- -
	(0.87)		
	9.17		

```
    LIFE       0.52       0.02        - -
             (0.03)     (0.00)
              17.65      14.96

   Largest Eigenvalue of B*B' (Stability Index) is  64.129

        Indirect Effects of Y on Y

                 ED     INCOME       LIFE
           --------   --------   --------
      ED        - -        - -        - -

  INCOME        - -        - -        - -

    LIFE       0.17        - -        - -
             (0.02)
               7.82

 STATUS ATTAINMENT AND LIFE SATISFACTION GA31=GA32=0

 Standardized Total and Indirect Effects

        Standardized Total Effects of X on Y

               FOCC         IQ
           --------   --------
      ED       0.60       0.30
  INCOME       0.34       0.22
    LIFE       0.36       0.20

        Standardized Indirect Effects of X on Y

               FOCC         IQ
           --------   --------
      ED        - -        - -
  INCOME       0.24       0.12
    LIFE       0.36       0.20

        Standardized Total Effects of Y on Y

                 ED     INCOME       LIFE
           --------   --------   --------
      ED        - -        - -        - -
  INCOME       0.40        - -        - -
    LIFE       0.52       0.41        - -

        Standardized Indirect Effects of Y on Y

                 ED     INCOME       LIFE
           --------   --------   --------
      ED        - -        - -        - -
  INCOME        - -        - -        - -
    LIFE       0.17        - -        - -

                          Time used:    0.016 Seconds
```