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ABSTRACT

Various spaces are singled–out as candidates for the space of all 2-d N=1 and 2 supersymmetric quantum field theories, respectively. This is done by treating the c–function as a Morse–function on these spaces.

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1 Introduction

The study of string theory and that of critical phenomena in two dimensional systems has made the plan of the classification of Conformal Field Theories (CFT's) one of paramount importance. However, the concept of classification is not one that renders itself to a presice definition. The latest program, which in a self-evident way qualifies to be called one of classification, is based on singularity theory [1]. Another, motivated mostly by the hope of gaining some understanding of non-perturbative phenomena in string theory, is the idea of treating CFT's as "special points" in the space, Q, of all 2-d Quantum Field Theories (QFT's) [2]. Although different in approach, there exist certain common grounds in these two ideas. For instance, Renormalization Group (RG)-flows play a central role in both strategies.

The space Q and the RG-flows are addressed in an elegant and unifying fashion in a theorem, due to Zamolodchikov [3], called the "c-theorem". In this theorem, in addition to advocating the existence of a function c, defined over Q, whose critical points correspond to CFT's (i.e. fixed points of the RG-flow), one also demands that the RG-flows decrease the value of the function c. The setting in the c-theorem is one which is most suitable for the application of Morse theory. There, knowledge of the behaviour of a function, defined over a manifold, near its critical points provides one with a great deal of global (topological) information regarding the manifold itself. In this way we can learn about the space Q, solely from the knowledge of the c-function defined on it.

The subspace, $Q_0 \subset Q$, corresponding to N=0 unitary CFT's (i.e. $c < 1$) was treated in this way by C. Vafa [4], and with a sligthly different emphasis, by S.R. Das, et al. [5]. Mostly in the spirit of the former paper, in this letter we apply Morse theory to N=1 and N=2 unitary super-CFT's (i.e. $c < 3/2$ and $c < 3$, respectively). This will allow us to explore the topology of a larger subspace of Q than that of Q_0. We will be cavalier in our handling of the space Q; in addition to the various technicalities which arose in the N=0 case (which were addressed in [4]), the N=1 and 2 cases call for further "restrictions" on the relevant subspaces of Q, which will be discussed below.

2 Morse Theory and N=0 CFT's.

In this section we briefly recall elementary ideas from Morse theory, and review its application to the N=0 case, as treated be Vafa [4].

First, let us develop what is often referred to as "Baby Morse theory" [6]. Given a compact manifold M, and a non-degenerate function f defined on it, one defines a Morse Polynomial (MP) for f by $M(f;t) \equiv \sum_{P_i} t^{n_i}$, where P_i are the critical points of f, at which n_i is the index of f, i.e. the number of negative eigen values of the Hessian (or, in other words, the number of directions in which f decreases). t is an arbitrary parameter between 0 and 1. For the manifold M, one defines a Poincaré Polynomial (PP), $P(M;t) \equiv \sum_{k=0}^{dimM} b_k(M)t^k$, where $b_k(M) \equiv dimH^k(M)$ are the Betti-numbers of M. Now, given the function f the "weak" Morse inequalities place upper-bounds on the b_k's of M: $P(M;t) \leq M(f;t)$. The "strong" form of the inequalities, however, serves our purpose better. It states

$$M(f;t) - P(M;t) = (1+t)Q(t), \tag{1}$$

where the coefficients of $Q(t)$ are non-negative. Notice the following consequence of (1):

$$M(f;-1) = P(M;-1). \tag{2}$$

The Lacunary Principle [6] states that if the product of all the consecutive coefficients in $M(f;t)$ is zero, then $M(f;t) = P(M;t)$. A function for which this equality holds is called a perfect Morse function. Thus, if one can find a function on M whose MP satisfies the Lacunary Principle (i.e. a perfect function), then one can simply read-off the b_k's of M from the coefficients of $M(f;t)$, or simply look-up spaces which have $M(f;t)$ as their PP.

Now, in [4], f and M were identified with the c-function c_0, and the space Q_0, respectively. Also in that case, the critical values, and the index, of c_0 were nothing but $c(m) = 1-6/[m(m+1)]$ with $m \geq 2$, and the number of relevant operators (i.e. $2(m-2)$, excluding the identity) in the N=0 CFT, respectively. Then, $M(c_0;t) = \sum_{m=2}^{\infty} t^{2(m-2)} = 1 + t^2 + t^4 + ... = 1/(1-t^2)$. Since the Lacunary Principle does apply to this MP, c_0 is then perfect, i.e. the PP of the space Q_0 is $P(Q_0;t) = 1/(1-t^2)$. Among the infinity of topologically distinct spaces which share this PP, Vafa discussed 1) The loop-space of $SU(2)$, and 2) CP^∞. Two more examples of such a space are 3) The space of all paths, joining two arbitrary points, on S^3 and 4) the connected-sum of $S^2, S^4, S^6, ...$ We shall generically refer to all these

spaces, which are homologically equivalent, as Q_0, i.e. the space relevant to the N=0 unitary CFT's.

3 Morse Theory and the N=1 super-CFT's.

Let us now apply the paraphernalia of Morse theory to the N=1 super-discrete- series. The space under consideration is then the space of all 2-d N=1 supersymmetric QFT's, which we shall call Q_1. Of course, as in the N=0 case, we are assuming that the various tachnical difficulties regarding the application of Morse theory (e.g. orbifold singularities, and the infinite-dimensional nature of Q_1, etc.) have been properly taken into account.

In computing the MP it is important to realize what the definition of "the index of the c-function" truly is. For the N=0 case, the index is simply the number of relevant fields in the CFT, since this is the number of RG-trajectories in the direction of which c_0 decreases. In the N=1 case, since we are considering RG-flows which preserve N=1 SUSY, the index of the c-function c_1 must be identified with the number of relevant *super*-fields (excluding the identity). Also note that since there are no SUSY-preserving marginal operators here, c_1 is non-degenerate. From [7] and [8] it can be seen that this number is $(m-2)$. Thus the MP for c_1 is

$$M(c_1;t) = \sum_{m=2}^{\infty} t^{m-2} = 1 + t + t^2 + t^3 + ... = \frac{1}{1-t}. \tag{3}$$

In contrast to c_0, the perfectness of c_1 can not be decided on the basis of the Lacunary Principle, since every power of t is present. Thus it is not possible to identify Q_1 (or the homology of Q_1) uniquely. However, Morse theory does provide us with restrictions on the possible Betti-numbers of Q_1, as shown below.

The weak form of the inequalities already place limits on the b_k's – that $b_0(Q_1) = 1$, and $b_k(Q_1) = 0$ or 1 for $\forall k \neq 0$. The former relation implies that Q_1 is connected. To further explore the possible allowed 0/1 configurations of the $b_k(Q_1)$, we must go on to consider the strong form of the inequalities.

The strong form of the Morse inequalities place strong restrictions on the possible values of the $b_k(Q_1)$. Substituting (3) in (1) gives

$$b_k = 1 - q_k - q_{k-1}, \tag{4}$$

where for clarity of notation I have supressed the Q_1 appearing in the argument of $b_k(Q_1)$. Since (4) can be inverted:

$$q_n = \begin{cases} \sum_{k=0}^{n}(-1)^k b_k & \text{for } n\text{=odd} \\ \sum_{k=1}^{n}(-1)^{k+1} b_k & \text{for } n\text{=even,} \end{cases} \tag{5}$$

the non-negativeness of the q_k's allows us to develop an algorithm for enumerating all the possible 0/1 configurations of the b_k's. The result is as follows: If we indicate the 0 *or* 1 value of a b_k by "$\left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right.$", then in the sequence of b_k's (starting with b_0=1), any value is followed by a $\left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right.$, except the 0 in $\left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right.$ which is followed by a single 0 (not 1). This can best be seen, schematically, in figure 1.

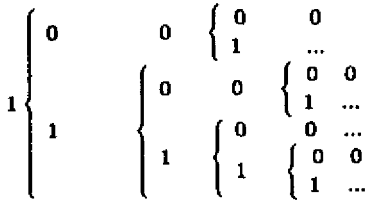

Figure 1. From left to right, the columns correspond to the allowed values of $b_0(Q_1)$, $b_1(Q_1)$, $b_2(Q_1)$, etc.

Although, at this stage we can not innumerate all the possibilities, for there are infinite of them, we see that there are stringent restrictions on the allowed Betti-numbers. For example, as seen from figure 1, spaces whose PP begin with $1 + t^2$, or with $1 + t + t^3$ are not allowed. The former implies that the Q_0-candidates, in particular, are not legitimate as candidates for Q_1. The latter corresponds to examples of non-simply-connected spaces which are again not allowed (however, see below).

Notice that as a result of (2) not all the solutions in figure 1 are allowed. This equation places the following constraint on the b_k's:

$$\sum_{k=0}^{\infty}(-1)^k b_k = \frac{1}{2}\,. \tag{6}$$

However, it is important to realize that the evaluation of the infinite-sum in $M(c_1;t)$ (which, at $t=-1$ gives the 1/2 in (6)) has already taken us into the realm of analysis. In other words, to impose (6) requires some sort of a "regularization" of the infinite-sum on the left-hand side; recall that

the b_k's are 0 or 1. Intuitively, (6) means that the 0's and the 1's must be evenly distributed among the b_k's with k=even and odd.

One way of "regularizing" the sum in (6) is by placing a cut-off on it. This amounts to treating Q_1 as a finite dimensional space – call it Q_1^D. For our purposes it does make sense to treat Q_1 as finite dimensional, since we can consider only those super-CFT's (i.e. all the points of Q_1 at which c_1 is equal to $c_1(m) = \frac{3}{2}[1 - \frac{8}{m(m+2)}]$ with $m \geq 2$) to which a given super-CFT ($m \neq \infty$) flows to; with $m = \infty$ removed, Q_1 can be finite dimensional.

In fact, for Q_1^D, we can discard the constraint (6) altogether, since it is nothing but a consequence of the Morse inequalities (at $t=-1$) which we have already taken into account. Instead, though, we have the Poincaré duality at out disposal (assuming Q_1^D is orientable), i.e. $b_k = b_{D-k}$. Then, of the configurations appearing in figure 1, those which satisfy the Poincaré duality have the following PP's: For D=odd,

$$P(Q_1^D;t) = \begin{cases} 1 + t^D \\ 1 + t + t^{D-1} + t^D \\ 1 + t + t^2 + t^{D-2} + t^{D-1} + t^D \\ \vdots \\ 1 + t + t^2 + t^3 + \ldots + t^D, \end{cases}$$

and for D=even,

$$P(Q_1^D;t) = 1 + t + t^2 + t^3 + \ldots + t^D.$$

Before going any further, we can compute the Euler number of Q_1^D:

$$\chi(Q_1^D) \equiv P(Q_1^D; t=-1) = \begin{cases} 0 & \text{for } D\text{=odd} \\ 1 & \text{for } D\text{=even.} \end{cases} \tag{7}$$

For the D=odd case, except for the first solution which is the PP of S^D, and for the last solution which we shall discuss below, the rest are difficult to identify with "common" spaces. Of course, to visualize their homology one can think in terms of connected-sums of S^N's (see below). Even without knowledge of the specific spaces, we can extract some useful information for D=odd or even. For instance, all of our PP's have $b_0 = 1$, implying that Q_1^D is connected. Except for the case $Q_1^D = S^D$, we have $b_1 = 1$ which means that Q_1^D is in fact not simply connected – there exists exactly one non-contractible curve. In other words, Q_1^D has a hole in it. The Euler-numbers (7) are also helpful in visualizing these spaces. A convenient representation which captures at least the homological structure of these spaces is given in terms of connected-sums of S^N's. Corresponding to the various D=odd solutions we have

$$Q_1^D \approx \begin{cases} S^D \\ S^1 \vee S^{D-1} \vee S^D \\ S^1 \vee S^2 \vee S^{D-2} \vee S^{D-1} \vee S^D \\ \vdots \\ S^1 \vee S^2 \vee S^3 \vee ... \vee S^D, \end{cases}$$

and for the D=even case we have only

$$Q_1^D \approx S^1 \vee S^2 \vee S^3 \vee ... \vee S^D,$$

where "$\approx$" indicates homological equivalence. The actual *topology* of Q_1^D may be quite different from these.

Finally, it is amuzing that the only PP that appears in both D=even and odd case is $P(Q_1^D;t) = 1+t+t^2+t^3+...+t^D$, which is equal to the MP itself, i.e. that c_1, as a function defined on this space, is perfect. A non-rigorous argument for the perfectness of c_1 would follow upon consideration of the $D \to \infty$ limit. In that limit the only consistent solution (i.e. common to both D=even and odd) is

$$P(Q_1;t) = \lim_{D\to\infty} P(Q_1^D;t) = 1+t+t^2+\ ...\ = \frac{1}{1-t} = M(c_1;t).$$

Thus, we see that, although The Lacunary principle could not decide the perfectness of c_1, the MP for c_1 is special enough (i.e. with all coefficients equal to 1) so that one can atleast entertain the possibility of c_1's perfectness. Now, as an infinite dimensional space, other than $S^1 \vee S^2 \vee S^3 \vee ...$, another candidate-space for Q_1 is the space of paths, joining two arbitrary points, on S^2.

4 Morse Theory and the Chiral N=2 Super-CFT's.

Q_2, the space of all 2-d, chiral, N=2 super-CFT's requires a slightly more careful analysis of the pattern of RG-flows. We have restricted our attention to the subspace of chiral theories, because it is these whose algebra's chiral rings have a correspondence with the local rings of the respective super potentials, thereby equipping one with a rigorous geometric foundation [1].

Also, it is known that the unitary, chiral, theories have a Landau-Ginzburg (LG) description [1], which makes the study of RG-flows, and the matter of counting the number of relevant operators an easy task.

It turns out that the LG-superpotentials for chiral N=2 super-CFT's fall into the ADE pattern that appears in the classification of modality-zero, complex singularities [1]. This fact is more than sufficient to provide us with all the information we need for constructing the MP. Firstly, with the LG-superpotentials at hand, it is easy to see that, as in N=0 and 1, RG-flows keep one within the A-series [9]. Therefore we are justified, again, to compute the MP for only the A-series. Secondly, the number of relevant chiral superfields (including the identity) is simply equal to the index of the singularity, as a result of the above-mentioned ring-correspondence; for the case of the A_{k+1}-series, with $k \geq 0$, this number is $(k+1)$ (or simply k, excluding the identity). Thirdly, the modality-zero condition, i.e. the absence of (SUSY-preserving) marginal operators, assures us that the c-function is non-degenerate. The final point is in regards to the complex nature of the superfield; as a result of this, the coupling-constant in front of the relevant perturbation can also be complex. Recall that Q, in general, can be considered a space of coupling-constants, *a la* Zamalodchikov [7]. Thus, Q_2 is a complex space.

It is tempting to interpret the number of relevant chiral superfields, k, as the index of the c-function c_2. However, recalling that the index of a function is the number of negative eigen values of the Hessian (i.e. a quadratic -form), the attempt of defining an "index" for a complex function, defined over a complex space, becomes hopelessly obstructed. This obstruction is essentially due to the fact that the field of complex numbers can not be totally ordered [1]. Although one can define critical points for a complex function, one can not decide the maximum/minimum behaviour of the function at the critical points. In other words, whether c_2 increases or decreases, in a given (complex) direction, can not be decided upon. This nonexistence of an analog of Morse theory, for complex spaces [2], destroys all hope of learning about the complex cohomology (in the sense of Dolbeault cohomology - $H^{p,q}(Q_2)$) of Q_2.

The situation is however not entirely bleak, for since the value of c_2 at the critical points is real (this being $c_2(k) = \frac{3k}{k+2}$ with $k \geq 0$), we can

[1] On this point, we acknowledge many fruitful discussions with M. Mahdavi-H.

[2] There does exist an analog of Morse theory for complex spaces, and it is called the Picard-Lefschetz theory (see [10]). However, it lacks the analog of Morse inequalities.

shift our attention from Q_2 to its real "covering", Q_{2R}. Indeed, since every compact complex (D-dimensional) manifold is equivalent to a compact real ($2D$-dimensional) manifold, we can still apply Morse theory and extract information about the real (de Rham) cohomology of the space Q_{2R}. In short then, we now have a real space, Q_{2R}, and a real-valued c-function, c_{2R}, defined on it, available for the application of Morse theory. The only subtlety is that the index of c_{2R} at a critical point is not simply k (see above), but rather $2k$ [3], since at a given critical point there are now 2 independent real directions, associated to each relevant chiral superfield, in which c_{2R} decreases.

Finally, the MP for c_{2R} is $M(c_{2R};t) = \sum_{k=0}^{\infty} t^{2k} = 1 + t^2 + t^4 + ... = 1/(1-t^2)$. We see that, as in the N=0 case, the Lacunary Principle implies that c_{2R} is perfect which in turn means that $P(Q_{2R};t) = 1/(1-t^2)$. In fact, this is the same as the PP of Q_0 (i.e. the N=0 space)! Thus, although Q_0 and Q_{2R} can be topologically different spaces, they are homologically equivalent. So, one may recall the same Q_0-candidate-spaces for Q_{2R} – loop space of $SU(2)$, space of paths on S^3 joining two arbitrary points, $S^2 \vee S^4 \vee S^6 \vee ...$, and CP^{∞} [4].

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[3] We would like to thank C. Vafa for pointing-out this crucial fact to us.

[4] This latter space is, in fact, suggested by the relation between the minimal N=2 models and coset constructions of them based on $SU(N)/SU(N-1) \times U(1) \approx CP^{(N-1)}$. This was Pointed-out to us by C. Vafa.

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