On the Relation Between Model Parameters and Forecast Parameters

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Abstract

It is well-known that varying model parameters in a NWP model can affect multiple forecast parameters; but exactly how, is difficult to assess. The difficulty arises because of the nonlinearities and interactions inherent in the model. In other words, a change in a single model parameter can affect a number of forecast parameters, and the affect itself depends on the values of all the other model parameters. Here, it is shown that by casting the question in a statistical setting, it is possible to capture the nontrivial relationship between model parameters and forecast parameters. Specifically, it is shown that a "forward" statistical model can display how forecast parameters vary in response to changes in model parameters. Similarly, an "inverse" statistical model, mapping forecast parameters to model parameters, can point to specific values of the latter, given desired values for the former. An operational model (COAMPS®)[1] is utilized to motivate the idea, but the approach is fully illustrated on the Lorenz model. It is shown that some natural forecast parameters in these models are indeed related to model parameters in a manner suggesting a deterministic underlying relation. Multivariate statistical models are then developed to capture that relationship. The utility of the statistical models is shown by deriving results which would not be evident without the statistical models.

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[1]COAMPS is a registered trademark of the Naval Research Laboratory.

1 Introduction

In verification circles it is often stated that diagnosis of forecast errors can be utilized to revise the model so as to improve the forecasts. Examples of forecast errors are 1) Model did not produce a circulation, when one was observed; 2) Model produced a circulation, but it was not as intense as observed; 3) Model produced the wrong amount of precipitation; 4) A front was moving too slowly, or had the wrong shape; etc. But what is a model developer to do with this information?

As important as this type of information is, it is difficult to utilize directly, because most models typically do not have "knobs" that control the cyclogenesis, the amount of precipitation, or the speed of a front. Model developers can often adjust certain parameterizations to improve any of these characteristics; however, frequently with unknown other consequences.

This raises the question of whether there exists a functional relationship between model parameters and forecast parameters, and if so, if this relationship can be adequately quantified to be of practical utility. For example, although it is true that no single model parameter controls the amount of precipitation produced by a model, is it possible that a combination of model parameters do? And if so, what combination, and how?

Furthermore, variations in a single model parameter affect not a single forecast parameter, but several. For example, varying a convective trigger function may affect the formation of convection, but it may also dramatically affect the models ability to sustain a cyclonic storm. Therefore, the knowledge of which model parameters affect a given forecast parameter is not of much practical utility, because variations in those model parameters may adversely affect other forecast parameters. As such, one is interested in the relationship between some number of model parameters, and some number of forecast parameters, simultaneously. Figure 1 illustrates the general situation. The "black box" represents the physical processes underlying the NWP model. Note that the outputs are not NWP outputs but forecast parameters.

The relationship underlying the black box is assumed to be some complex, nonlinear, function with interactions between its inputs. The main goal of this work is to develop a methodology for approximating that relationship with a statistical model. The approach involves varying the model parameters, running the model forward, and recording the resulting forecast parameters. This procedure leads to data involving model and forecast parameters. The resulting data can then be employed to develop two statistical models: 1) a forward statistical model that maps the model parameters to the forecast parameters, and 2) an inverse statistical model which maps forecast parameters to model parameters. The utility of these statistical models is discussed next, but note that their development does not call for observations at all; the statistical models simply represent (or emulate) the relationship between model and forecast parameters in the NWP model.

The practical utility of the forward statistical model to allow a study of the effect, on *multiple* forecast parameters, of changes in *multiple* model parameters. The inverse model has even greater practical utility, because it provides optimal settings for the model parameters. For instance, if an NWP model is producing too much precipitation (e.g., as compared to observations), then the model builder can consult the inverse statistical model to discover the optimal settings of the model parameters in order to reduce the amount of forecast precipitation, without significantly affecting other features of the forecasts. Both the forward and the inverse statistical models also have scientific utility, because they can be used to study the relationship between model and forecast parameters for theoretical purposes.

There are three research activities which may seem to be related to the idea presented here, but there are some important differences. One is sensitivity analysis (Cacuci 2003). In sensitivity analysis, one examines the effect of model parameter variations on the direct output of the model, not forecast parameters. More importantly, sensitivity analysis assumes no interactions between the model parameters; in other words, the sensitivity assigned to one model parameter is assumed to be independent of the values of the other model parameters. By contrast, in the proposed method, no such assumptions are made. Also, sensitivity analysis examines the affect of model parameters on a single forecast parameter, not a number of forecast parameters, simultaneously. Another related analysis is adjoint modeling (Errico 1997), but the main task there is to assess the sensitivity of model output to model input, not model parameters and forecast parameters. Finally, statistical models have been employed to emulate complex numerical models (Krasnopolsky et al. 2008), but there the inputs and outputs of the statistical models are precisely the inputs and outputs of the numerical model. Here, the statistical models emulate the relationship between model parameters and forecast parameters.

To illustrate the methodology, first, an operational model is employed to motivate the methodology. Then, the Lorenz model (Lorenz 1963) is utilized to delve deeper into the approach. In the last section, we return to COAMPS for some preliminary results.

2 COAMPS

The numerical weather prediction system chosen for this experiment is the Naval Research Laboratories Coupled Ocean Atmosphere Prediction System (COAMPS) (Chen et al. 2003) running in an atmosphere only mode. The atmosphere component of COAMPS is a state-of-the-art, fully non-hydrostatic mesoscale weather model capable of resolving atmospheric phenomena to sub-kilometer scales and hence a highly complex system of algorithms. The US Navy currently uses COAMPS for operational weather prediction and for research purposes. For this study, the model is run at fairly coarse resolution over the S. Indian Ocean.

The coordinates of the four corners of the domain under study are $20.9°S, 50.9°E$ (upper left), $20.9°S, 121.8°E$ (upper right), $49.5°S, 27.6°E$ (lower left), and $49.5°S, 145.1°E$ (lower right). The grid length is $125\,Km$, which means that the above domain consists of 61 grid points in the East-West direction, and 31 grid points in the North-South direction.

In the version of COAMPS currently available to the authors, a small number of model parameters are available to vary. Of these, only two are are selected: a sponge layer timing parameter (rdtime), and the frequency of executing the radiation package (dtrad). Their default values are rdtime=240 and dtrad=3600. They are varied over the sets 240,300,360,420,480 and 900,1800,2700,3600,4500,5400,6300, respectively. As forecast parameters, the mean Sea-Level Pressure (SLP) and mean Boundary-Layer Height (BLH) are selected. It is important to emphasize that the number of model parameters and forecast parameters is selected to be two only for presentation purposes. In practice the number of model and forecast parameters may be much larger. Moreover, the number of model parameters may be less than, equal to, or greater than the number of forecast parameters.

Figure 2 shows the dependence of SLP and BLH on dtrad and rdtime, for the forecasts initiated on Oct. 2, 2007. The multiple curves in each panel correspond to values of the model parameter not shown on the x-axis. Note that SLP (top-left) generally increases with increasing dtrad, for all 5 values of rdtime. BLH (top-right) behaves similarly, but with more fluctuations as a function of both parameters. The bottom two panels in Figure 2 display the same information, but they allow for an additional observation: Although SLP generally increase with rdtime, for all values of dtrad, BLH appears to be mostly insensitive the rdtime. Evidently, even with only two model and forecast parameters, the relationship between them is complex. Further complexity and fluctuation in the relationship is revealed when this analysis is repeated for other days (not shown). In short, there exists some type of a relationship between the model and forecast parameters, but in order to capture/summarize that relationship some sort of a statistical model must be used.

One may argue that the relationships displayed in Figure 2 are specific to the date under study. However, similar patterns exist for other dates as well (not shown). In order to better capture the true relationship between model parameters and forecast parameters, one may want to consider multiple days; however, as will be shown in the next section, the fluctuations of SLP and BLH for a given day are comparable to those across days. As such, it is not necessary to run the model multiple times for different days.

3 The Lorenz Model

In order to better illustrate the idea, the Lorenz model (Lorenz 1963) is employed. It is defined as

$$dX/dt = -s(X-Y) \quad (1)$$

$$dY/dt = rX - Y - XZ \quad (2)$$

$$dZ/dt = XY - bZ \quad (3)$$

where the model parameters are s, r, and b, respectively, the Prandtl number, the Raleigh number, and b is a function of the wavenumber. The state space variables X, Y, and Z measure intensity of convective motion, and horizontal and vertical temperature variation, respectively (Bellomo and Preziosi 1995). Figure 3 shows the standard 2-lobe state space with the default Lorenz settings for $s = 10$, $r = 28$, and $b = 8/3$. In this case, the initial values of the state space variables are set to the conventional values -14, -13, and 47.

Although there are three model parameters to vary in the Lorenz model, only two - r and b - are varied in order to allow a graphic expression of the "black box" in Figure 1. The choice of forecast parameters in the Lorenz model is not unique; but given that X is proportional to the intensity of convective motion, its maximum could be loosely interpreted as maximum amount of precipitation produced by the model. In fact, the maxima of all three state space variables serve as natural choices of forecast parameters. Again, for the purposes of visual display, only two - X_{max} and Z_{max} - are considered as forecast parameters in this study.

4 The "Data"

The proposed methodology calls for varying the model parameters over some range, and recording the resulting forecast parameters, thereby generating "data." The quotes indicate that the data are not "real data" involving obervations made on some natural phenomenon. They are data only in the sense that they are used for developing both forward and inverse statistical models. Figure 4 shows X_{max} as a function of r, for five different initial values of the state space variables. For each setting of the initial values and model parameters, the model is run forward for 40 time steps of size 0.02. Evidently, within the range of r values, there are three distinct phases represented by distinct functional dependence between X_{max} and r. It is worth mentioning, however, that within each phase, the functional dependence of X_{max} on r can be viewed as deterministic, modulo error. In other words, one can model the relationship between X_{max} and r using regression techniques. The vertical jumps correspond to well-known critical values of r at which bifurcations occur. For the remainder of the analysis only the region with the largest r values is considered, i.e., $27 \leq r \leq 50$. A similar analysis suggests that a reasonable range of b values is $8/3 \pm 0.8$.

Limiting the range of model parameters may seem arbitrary, but the choice is based partly on the patterns seen in Figure 2 for COAMPS. Additionally, the choice to limit the range of possible r values does not restrict the methodology, because some of the statistical models developed below are capable of fitting the entire range of data shown in Figure 4, including the discontinuities.

There is another important lesson that can be learned from Figure 4. In the language of Analysis of Variance (Devore and Farnum 2005), the fluctuations in X_{max} within initial conditions are comparable to those between. In other words, for a given initial condition of the state variables, the fluctuations in X_{max} are comparable to the fluctuations that occur for different initial conditions. This is important, because in making the transition from the Lorenz model (a toy model) to COAMPS (i.e., an operational model) different *days* in the latter can be thought of as different initial state variables in the former. As such, it is sufficient to run the Lorenz model forward only one time, i.e., for one set of initial state variables. Although additional runs would produce more data for the statistical model, the relationship between the model and forecast parameters can be captured from a single run.

Although for the purpose of generating Figure 4, r was varied systematically from

1 to 50, in steps of 1, for the development of statistical models this is unnecessary. It is sufficient to only sample the parameter space. In this article, 500 samples are taken from a bivariate uniform distribution for r and b. Figure 5 shows the resulting values of X_{max} and Z_{max}. Each panel shows two scatterplots corresponding to the lowest and highest values of the parameter not varied. For example, the top-left panel shows X_{max} as a function of r for $b = 8/3 - 0.8$ (lower) and $b = 8/3 + 0.8$ (upper). Note that the relations are mostly linear, with the exception of a slight nonlinearity in the relationship between Z_{max} and r. This linearity justifies the use of linear statistical models; although nonlinear models are also examined. Also, note the parallel nature of the scatterplots in each panel; from the point of view of statistical model building, this observation implies that the statistical model does not require an interaction term between the predictors. Still, models with interaction terms are also developed here, motivated by the non-parallel nature of the two "curves" in the lower-right panel in Figure 5.

5 Results: The Statistical Models

As mentioned previously, the main purpose of the data described in the previous section is to allow the development of two sets of statistical models: one for the forward data, and another for the inverse. For each set, two statistical models are developed: A polynomial regression model of order two, and a neural network with four hidden nodes. In the current application, the order of the polynomial and the number of hidden nodes of the neural network are mostly arbitrary, and not intended to be optimal.[2] As will be shown below, this level of model complexity is sufficient to capture the gross features of the data noted in Figure 5. Model selection methods such as Cross-validation or the Bootstrap (Hastie, Tibshirani, and Friedman 2001) can be employed for selecting the optimal complexity of the statistical models, but this is not done in the current paper.

[2]Details of polynomial regression or neural networks are unimportant here. Suffice it to say that they are both regression models capable of fitting nonlinear patterns in data.

Without specifying the parameters explicitly, the equations of the polynomial regression are

$$y_1 = 1 + x_1 + x_2 + x_1^2 + x_2^2 + x_1 x_2 \ , \quad (4)$$

$$y_2 = 1 + x_1 + x_2 + x_1^2 + x_2^2 + x_1 x_2 \ , \quad (5)$$

where the two mean squared errors for the two responses (y_1, y_2) are minimized separately. The neural net has a more complicated functional form (see Bishop 1995), and its parameters are estimated by minimizing the sum of the two mean squared errors. Again, the details of the neural net or the parameter estimation method (i.e., the training algorithm) are not important here. For the forward statistical model, $y_1 = X_{max}, y_2 = Z_{max}, x_1 = r, x_2 = b$, while for the inverse statistical model, $y_1 = r, y_2 = b, x_1 = X_{max}, x_2 = Z_{max}$.

The reason a polynomial regression is selected as one of the statistical models here is that it allows for inference. The regression coefficients for the forward and inverse statistical models are shown in Table 1. The p-values are for a two-sided test of the hypothesis that the corresponding population parameter values are zero. Consider the forward model, first: Given the extremely small p-values appearing in the Table, all but one of the regression coefficients are nonzero. The exception is the coefficient of the b^2 term in the equation for Z_{max}, which may be zero. Therefore, the complexity of the polynomial regression is appropriate, given the available data. ¿From a substantive point of view, the non-zero regression coefficients imply that the underlying functions are nonlinear and with interaction.

The p-values appearing in the inverse model are all below 0.05. As such, the inverse model is highly nonlinear. The regression coefficients themselves have also been presented in the Table, although they have no natural interpretation, given the presence of the statistically significant interaction term in the model.

The choice of neural network as the second statistical model is based on the fact that in realistic situations involving a large number of model and forecast parameters, a neural network is less prone to overfitting than polynomial regression (Bishop 1995). (Of course, the converse is true for data sets involving a small number of model and forecast parameters, as in the current application.) The draw-back, of course is that no statistical inference can be performed. In short, polynomial regression is more diagnostic, but it can overfit the data; on the other hand, a neural network is less prone to overfitting, but it is nondiagnostic. In practice, the ultimate choice between the two depends on the availability of data. Here, given that the Lorenz model is "cheap" in terms of generating data, overfitting is of no concern, and so both statistical models are examined. But for COAMPS, only the polynomial regression model is developed.

The fits to the forward data according to the two statistical models are shown in Figure 6. To allow visual comparison, all the variables have been standardized (i.e., the mean is subtracted out, and the result is divided by the standard deviation). It can be seen that the dependence of X_{max} and Z_{max} on b and r is complex. From these figures a user can visually assess how changes in r and b affect X_{max} and Z_{max}. Of course, in a realistic situation involving multiple variables, a different means of visual assessment must be employed. One possibility is put forth in the discussion section.

Note that the fit resulting from polynomial regression is smoother than that of the neural net. This is a simple consequence of the higher flexibility of the latter. By the same token, the increased complexity of the neural net fit may simply be a reflection of overfitting. As mentioned previously, this article does not concern itself with model selection.

The inverse fits according to polynomial regression and neural network are shown in Figure 7. Again, the displayed relationship is complex, and far from being intuitive. The polynomial regression fit (top row) is smoother than that of the neural network (bottom row), but the gross features are similar. For example, for a desired high value of X_{max} and Z_{max}, the model parameter r must be set to a high value, while b must be set to a low value. The specific values of r and b are, of course, produced by the statistical models. As another example, one can see that increasing X_{max} and Z_{max} simultaneously, affects r significantly, but does not appreciably affect b.

As in the forward model, the neural net may be overfitting the inverse data as well; but the gross features of the polynomial regression fit and the neural network fit are comparable. In general, the relationship is complex. Without these results, it would be very difficult for a user to know how to set the model parameters in order to have some desired values for *both* forecast parameters.

It is worth mentioning that the surfaces in Figure 7 are plotted over a grid defined by the range of X_{max} and Z_{max}; recall that these ranges are not determined directly by the user, but rather by the response of the Lorenz model to the range of r and b values. In fact, not all X_{max} and Z_{max} values occur. That fact also accounts for some of the differences between the polynomial regression and the neural net fits; the differences occur where there is little or no data on X_{max} and Z_{max} in the training data.

Figure 8 displays the polynomial regression fits to the forward and inverse data from COAMPS. Evidently, the fits to the forward data (top row) are mostly linear, and this is confirmed by the p-values (not shown) associated with the regression coefficients. Specifically, the b^2 term in the equation for SLP and BLH both have large p-values, indicating that such a term is probably unnecessary. However, in the equations for the inverse model, all terms appear with highly significant p-values. In other words, the underlying fit is nonlinear, and this is evident in the bottom row of Figure 8.

As in the Lorenz model, the inverse statistical model representing COAMPS is complex. Although one can certainly explain the behavior of the surfaces in Figure 8, it would have been difficult to anticipate them prior to the actual development of the inverse model. At this stage, a user would simply provide the statistical model desired values of SLP and BLH, and the statistical model would return optimal values of rdtime and dtrad.

Also, as in the Lorenz model, in the inverse data many SLP and BLH values do not appear at all. Indeed, SLP and BLH turn out to be correlated, and so the actual training data for the inverse model resides only along the diagonal on the SLP-BLH plane.

6 Summary and Discussion

In summary, it is shown that a statistical model can capture the underlying relationship between several model parameters and several forecast parameters, simultaneously. A forward statistical model, mapping model parameters to forecast parameters, serves as a diagnostic tool for examining how the latter change in response to changes in the former. The inverse statical model, by virtue of mapping forecast parameters to model parameters, can produce specific values of the latter that lead to some desirable values for the forecast parameters. The idea is illustrated on the Lorenz model as well as on COAMPS. The Lorenz model, although only a toy model, allows a thorough examination. The utility of the conclusions derived from an operational model like COAMPS provides further evidence that the proposed approach has great theoretical as well as practical utility.

Given the stochastic nature of the Lorenz model it is not surprising that a small change in any of the model parameters brings about an unpredictable change in the forecast parameters. However, as seen in Figure 4, for example, systematic changes in the model parameters do indeed reveal a clear deterministic relationship between the model and forecast parameters. In fact, that relationship is only slightly nonlinear, and so can be easily modeled statistically. The relationships found between model parameters and forecast parameters in COAMPS are in many ways even simpler. However, the approach is sufficiently flexible to allow any complicated relationship.

It was mentioned that certain visual tools must be developed to allow a clear display of the complex relationship between model and forecast parameters, especially when large numbers of the two are considered. One possibility is a 2-dimensional array of "pixels", with the number of rows equal to the number of model parameters, and the number of columns equal to the number of forecast parameters, with each pixel's color displaying the value of the corresponding forecast parameter. Such an interactive tool would allow the user to "turn the knob" on a model parameter(s), and literally watch the affect on all the forecast parameters. Indeed, this same tool can be used for displaying the inverse model, i.e., allowing the user to change the forecast parameter(s) of interest and view the effect on model parameters.

Finally, in dealing with inverse problems, some problems display nonuniqueness. For example, suppose the plot of a forecast parameter versus a model parameter is a parabola opening to top. In other words, as the model parameter increases, the forecast parameter decreases, but only up to a certain value, beyond which it increases. Although this situation did not arise in the data considered here, it is possible. The inverse data, then, would appear like a parabola opening to the right. As such, a given value of the forecast parameter would be associated with *two* values of the model parameter. Although such multi-valued behavior of the inverse problem cannot be captured by either type of statistical model considered here, it is possible to represent it using mixture models (Bishop 1996).

Acknowledgments

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References

Bellomo, N., and L. Preziosi, 1995: *Modelling Mathematical Methods and Scientific Computation*. CRC press. 497 pp.

Bishop, C. M., 1996: *Neural networks for pattern recognition.* Clarendon Press, Oxford, pp. 482.

Cacuci, D.G., 2003: Sensitivity and Uncertainty Analysis, Theory, Chapman and Hall/CRC, New York.

Chen, S., J. Cummings, J. Doyle, R. Hodur, T. Holtt, C.S. Liou, M. Liu, J. Ridout. J. Schmidt, W. Thompson, A. Mirin, and G. Sugiyama, 2003: COAMPS version 3 model description - general theory and equations. Naval Research Laboratory Technical Report, NRL/ PU7500-04-448. 141 pp.

Devore J., and N. Farnum, 2005: *Applied Statistics for Engineers and Scientists*. Thomson Learning, Inc.

Errico, R. M., 1997: What is an Adjoint Model? *Bull. Amer. Met. Soc.*, **78**, 2577-2587.

Hastie, T., R. Tibshirani, and J. Friedman, 2001: *The elements of statistical learning*, Springer Series in Statistics.

Hodur, R. M., 1997: The Naval Research Laboratorys Coupled Ocean/Atmosphere Mesoscale Prediction System (COAMPS). Mon. Wea. Rev. 125, 1414-1430.

Krasnopolsky, V.M., M.S. Fox-Rabinovitz, H.L. Tolman, and A. A. Belochitski, 2008: Neural Network Approach for Robust and Fast Calculation of Physical Processes in Numerical Environmental Models: Compound Parameterization with a Quality Control of Larger Errors. *Neural Networks*, **21**, 535543.

Lorenz, E. N., 1963: Deterministic non-periodic flow. J. Atmos. Sci., 20, 130-141.

Lorenz, E. N., K.A. Emanuel, 1998: Optimal Sites for Supplementary Weather Observations: Simulation with a Small Model. J. Atmos. Sci., 55, 399-414.

Figure Captions

Figure 1. A complex numerical model can be viewed as a "black box" representing the complex relationship between its parameters and its forecasts.

Figure 2. SLP (left) and BLH (right) as a function of the parameters dtrad (top) and rdtime (bottom). The multiple curves in each panel correspond to values of the model parameter not shown on the x-axis.

Figure 3. The 2-lobe state space of the Lorenz model with parameters $s = 10$, $r = 28$, and $b = 8/3$. The initial values for the the state space variables are -14, -13, and 47.

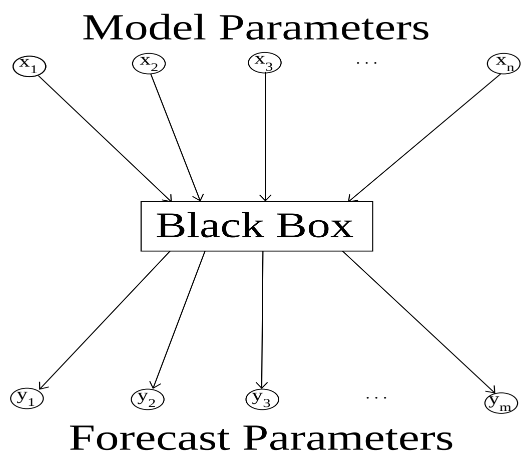
Figure 4. The forecast parameter X_{max} as a function of the model parameter r, for 5 different initial conditions (in different colors).

Figure 5. The dependence of X_{max} (left) and Z_{max} (right) on r (top) and b (bottom). The two "curves" in each panel correspond to the lowest and largest value of the model parameter not varied along the x-axis.

Figure 6. Polynomial regression (left) and neural network (right) fit to the forward data.

Figure 7. Polynomial regression (left) and neural network (right) fit to the inverse data.

Figure 8. Polynomial regression fit to the forward COAMPS data (top) and the inverse COAMPS data (bottom).



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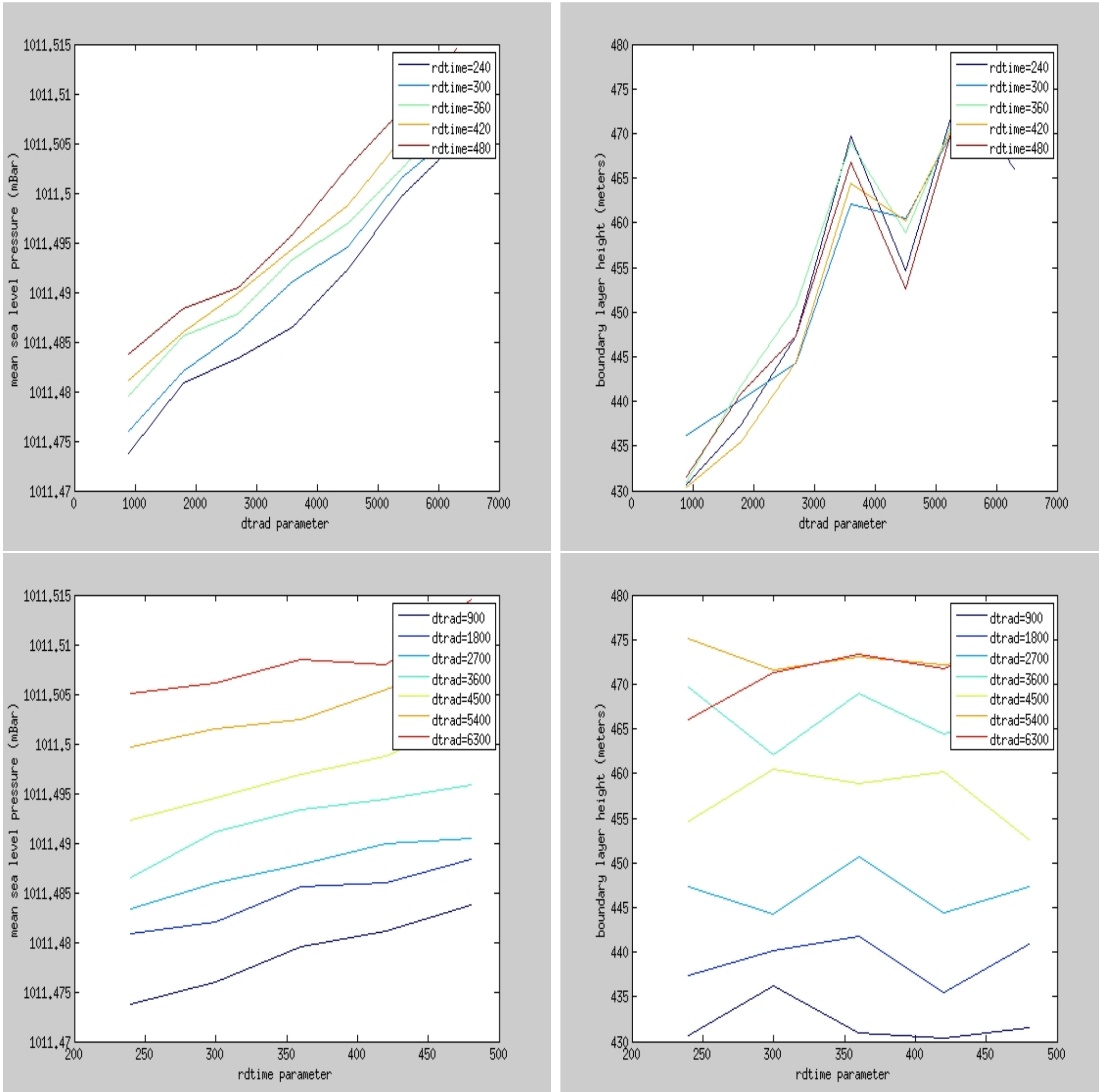

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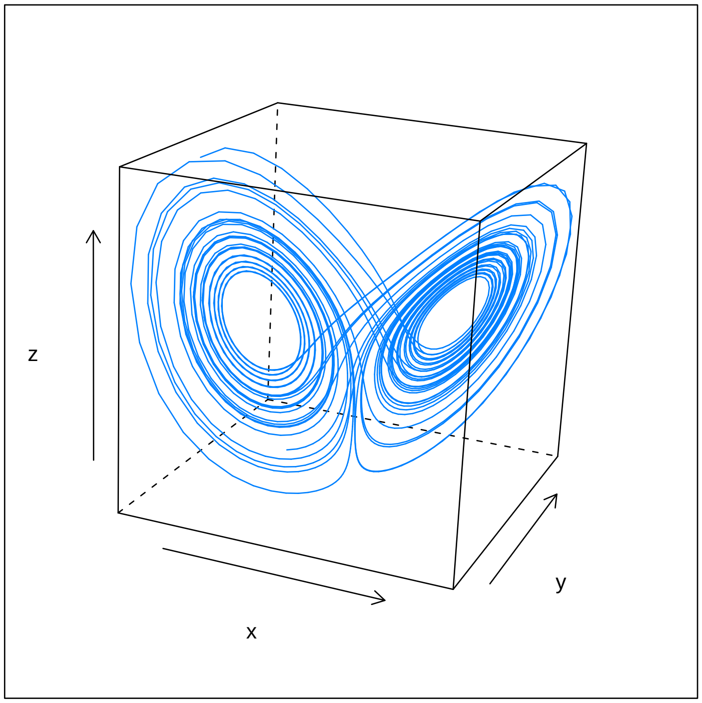

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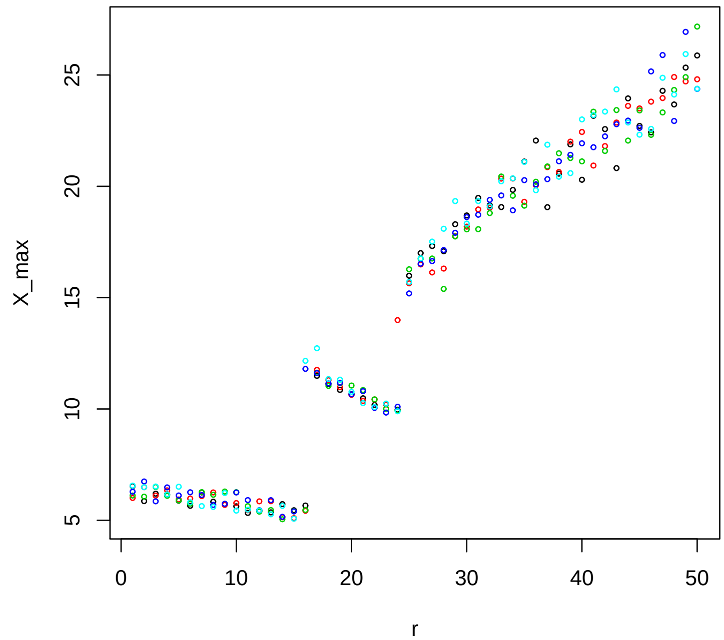

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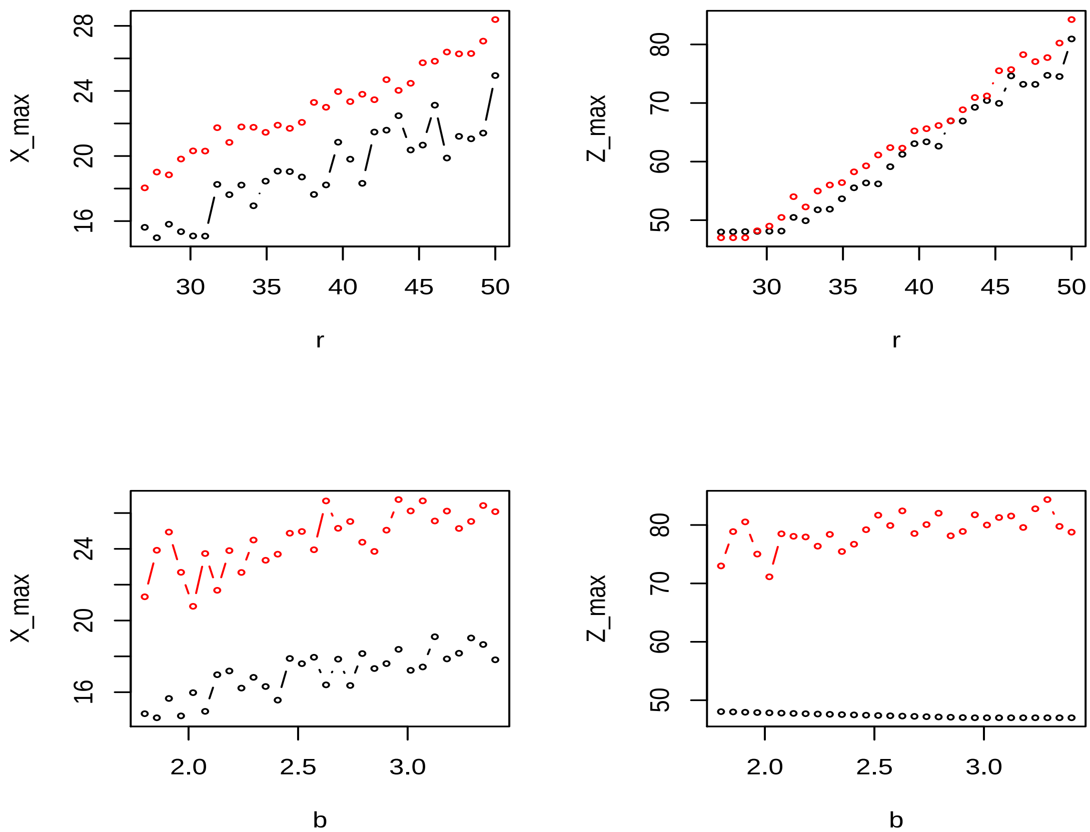

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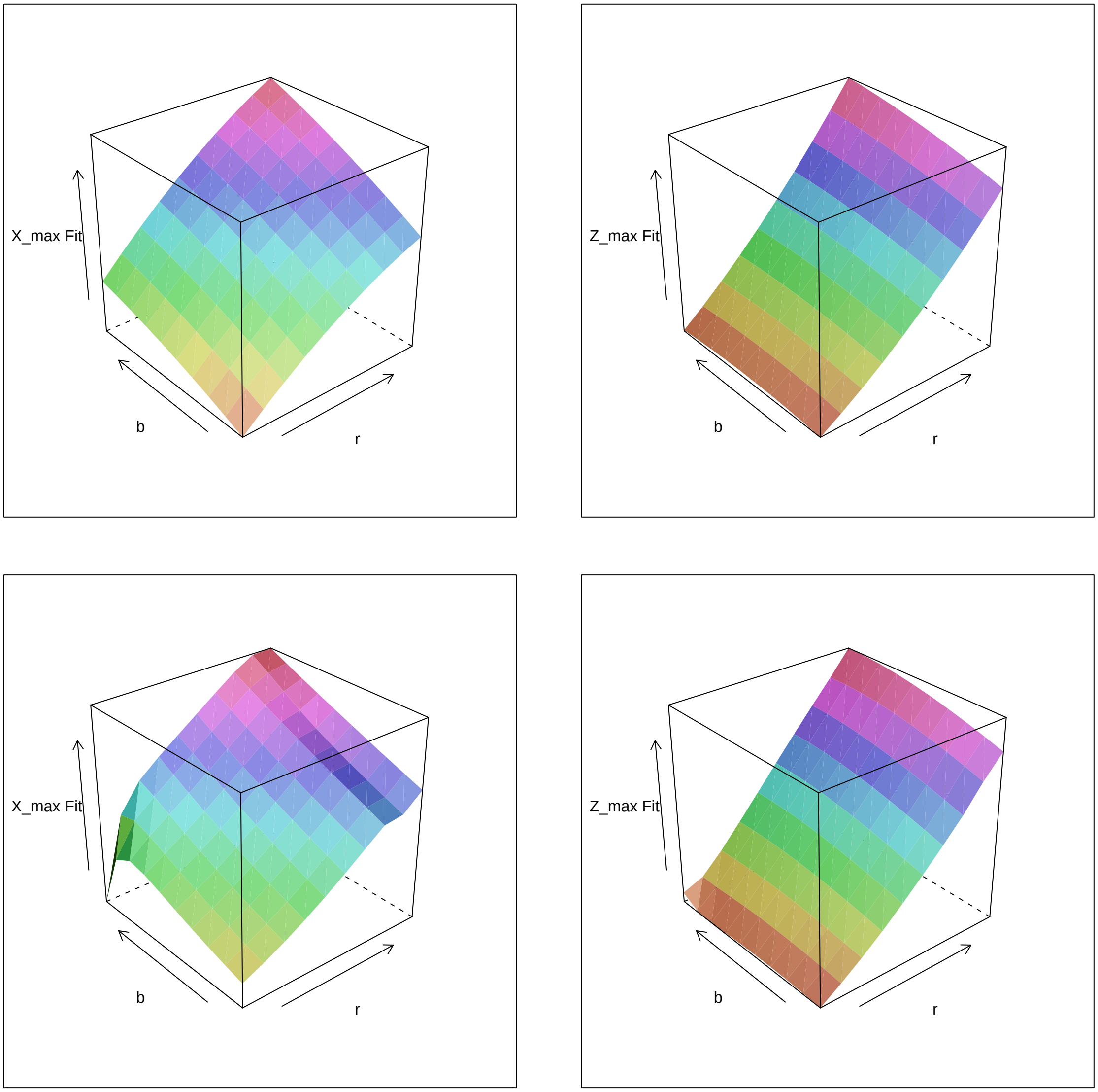

Figure 6. Polynomial regression (top) and neural network (bottom) fit to the forward data on X_{max} (left) and Z_{max} (right)

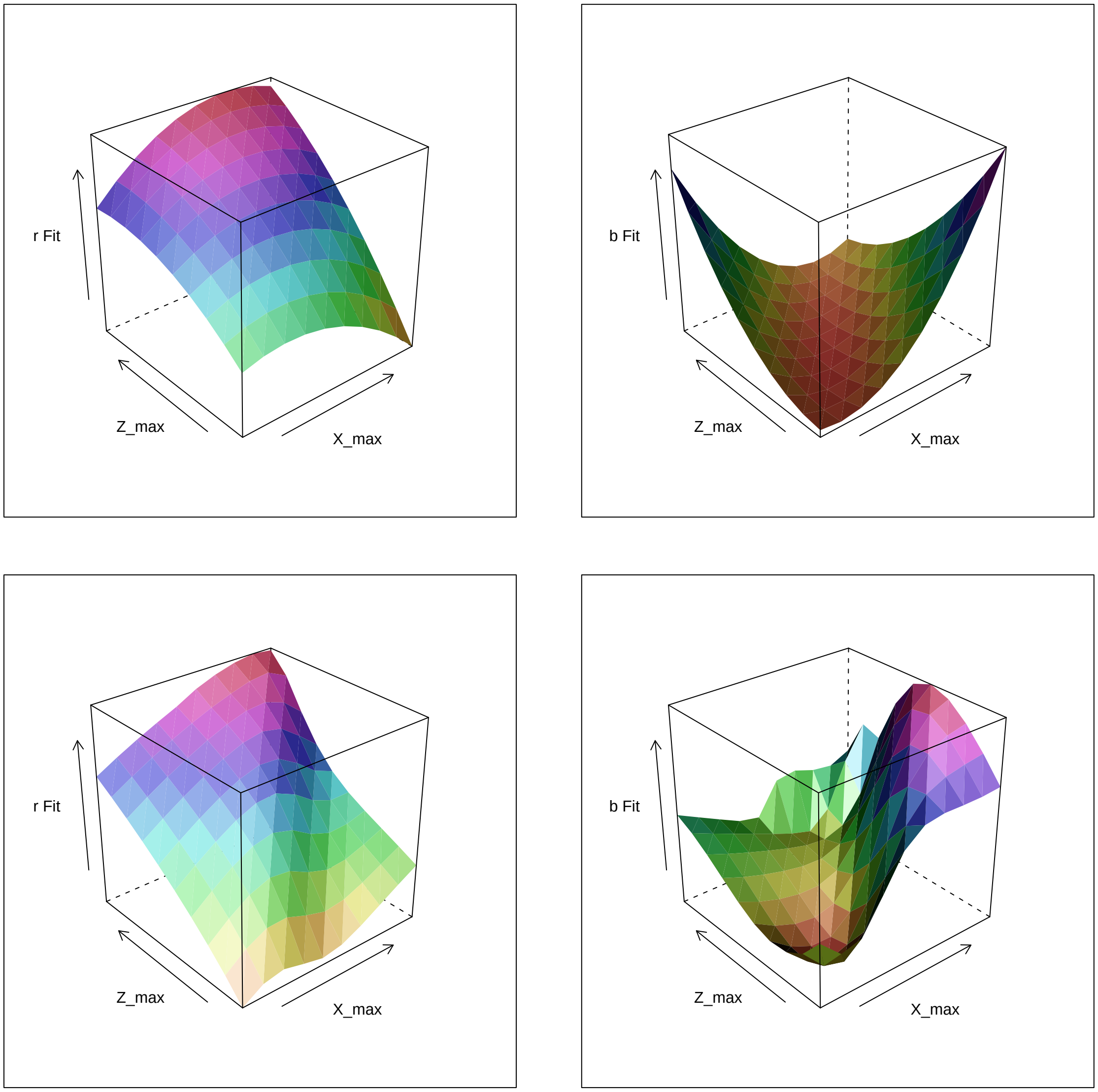

Figure 7. Polynomial regression (top) and neural network (bottom) fit to the inverse data on r (left) and b (right)

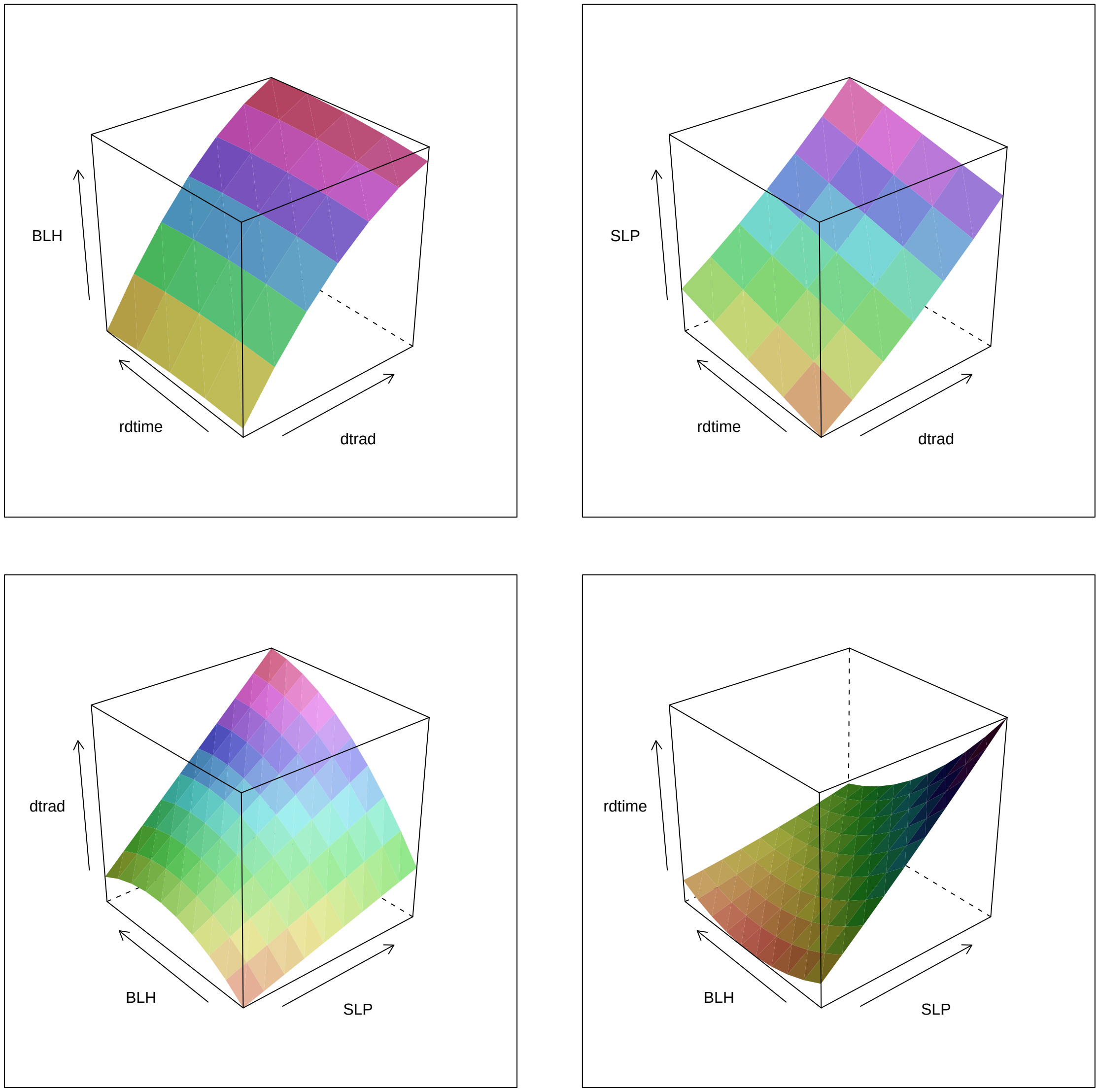

Figure 8. Polynomial regression fit to the forward COAMPS data (top) and the inverse COAMPS data (bottom).

	Forward statistical model			
	X_{max}	p-value	Z_{max}	p-value
Intercept	0.13326	0.0601	-0.05516	0.0549
r	0.81644	$< 2 \times 10^{-16}$	0.97841	$< 2 \times 10^{-16}$
b	0.43703	$< 2 \times 10^{-16}$	0.10568	3.59×10^{-10}
r^2	-0.09812	0.0230	0.08390	4.37×10^{-06}
b^2	-0.03648	0.3923	-0.02819	0.1046
$r\,b$	0.06262	0.0974	0.06486	4.49×10^{-5}
	Inverse statistical model			
	r	p-value	b	p-value
Intercept	0.17029	1.35×10^{-7}	-0.5780	2.46×10^{-6}
X_{max}	-0.1512	2.41×10^{-5}	1.5964	$< 2 \times 10^{-16}$
Z_{max}	1.13018	$< 2 \times 10^{-16}$	-1.3285	$< 2 \times 10^{-16}$
X_{max}^2	-0.17995	0.00160	1.1431	6.11×10^{-7}
Z_{max}^2	-0.18606	0.00134	1.1637	5.97×10^{-7}
$X_{max}\,Z_{max}$	0.22635	0.03006	-2.0103	1.99×10^{-6}

Table 1. Polynomial regression coefficients for the forward statistical model (top), mapping (r, b) to (X_{max}, Z_{max}), and the inverse statistical model (bottom), mapping (X_{max}, Z_{max}) to (r, b). The p-values are from a two-sided test of the hypothesis that the corresponding population values are zero.