Modeling Microbial Growth Dynamics: Two Key Equations

Michaelis-Menten equation characterizes food ('substrate') consumption rate:

\[-\frac{r_S}{X} = \frac{k_{max}S}{K_S + S}\]

\(-r_S\): rate of substrate utilization (mg BOD$_5$/L-d)

\(X\): biomass concentration (mg VSS/L)

\(-r_S/X\): specific rate of substrate utilization (F/M)

\(= (mg \text{ BOD}_5/ mg \text{ VSS-d})\)

\(k_{max}\): maximum specific substrate utilization rate

\(= (mg \text{ BOD}_5/ mg \text{ VSS-d})\)

\(K_S\): ‘half-saturation’ constant (mg BOD$_5$/L)

VSS: Volatile suspended solids, a measure of biomass
Michaelis-Menten equation indicates that the substrate consumption rate per gram of biomass is proportional to substrate concentration when $S$ is low ($<<K_S$) and independent of $S$ when $S$ is high ($>>K_S$)

\[- \frac{r_S}{X} = \frac{k_{max}S}{K_S + S}\]
Modeling Microbial Growth Dynamics: Two Key Equations

Monod equation relates substrate consumption to microbial growth:

\[ r_X = -Yr_S - k_d X \]

- \( r_X \): rate of biomass formation (mg VSS/L-d)
- \( Y \): yield coefficient (mg VSS formed/ mg BOD\(_5\) consumed)
- \( k_d \): biomass decay rate (d\(^{-1}\))

Monod equation indicates that the biomass grows in proportion to the substrate (food) consumption, but also decays at a rate proportional to the biomass concentration.
\[ r_X = -Yr_S - k_d X \]

\[ r_X = +Y \frac{k_{\text{max}} SX}{K_S + S} - k_d X \]

\[ \frac{r_X}{X} \equiv \mu = Y \frac{k_{\text{max}} S}{K_S + S} - k_d = \frac{\mu_{\text{max}} S}{K_S + S} - k_d \]

\[ r_{X/X} \text{ or } \mu : \text{specific growth rate (d}^{-1}) \]

\[ \mu_{\text{max}} : \text{maximum specific growth rate, } Yk_{\text{max}} (d^{-1}) ; \]

\[ \mu = \mu_{\text{max}} \text{ when } S >> K_S \]
Batch Microbial Growth Dynamics

Mass balances on $S$ and $X$:

\[ V \frac{dS}{dt} = \varphi (S_{in} - S_{out}) + r_S V \]

\[ V \frac{dX}{dt} = \varphi (X_{in} - X_{out}) + r_X V \]

\[ \frac{dS}{dt} = -r_S = \frac{k_{max} SX}{K_S + S} \]

\[ \frac{dX}{dt} = r_X = -Y r_S - k_d X \]
Batch Microbial Growth Dynamics

Example:  \( S(0) = 200 \text{ mg BOD/L} \)  
\( k_{\text{max}} = 15 \text{ mg BOD/mg VSS-d} \)  
\( Y = 0.4 \text{ mg VSS/mg BOD} \)  
\( X(0) = 1 \text{ mg VSS/L} \)  
\( K_S = 20 \text{ mg BOD/L} \)  
\( k_d = 0.12 \text{ d}^{-1} \)
**Microbial Growth in a Steady-State CSTR**

Mass balance on S:

\[
V \frac{dS}{dt} = 0 = Q\left(S_{in} - S_{out}\right) + r_S V
\]

\[
0 = Q\left(S_{in} - S\right) - \frac{k_{max} SX}{K_S + S} V
\]

Rearrange, write Q/V as \(1/\tau\):

\[
\frac{1}{\tau} \frac{S_{in} - S}{X} = \frac{k_{max} S}{K_S + S}
\]
**Mass balance on X:**

\[
V \frac{dX}{dt} = 0 = Q \left( X_{in} - X_{out} \right) + r_X V
\]

\[
0 = Q \left( X_{in} - X \right) + QY \left( S_{in} - S \right) - k_d XV
\]

\[
\frac{Q}{V} \frac{S_{in} - S}{X} = \frac{1}{Y} \left[ \frac{Q}{V} \left( 1 - \frac{X_{in}}{X} \right) + k_d \right]
\]

**Assume X\gg X_{in}, write Q/V as 1/\tau :**

\[
\frac{1}{\tau} \frac{S_{in} - S}{X} = \frac{1}{Y} \left[ \frac{1}{\tau} + k_d \right]
\]
Equate final expressions from the two mass balances and solve for S. Then substitute back into mass balance on S to solve for X:

\[
\frac{k_{\text{max}} S}{K_S + S} = \frac{1}{Y} \left[ \frac{1}{\tau} + k_d \right]
\]

\[
S = \frac{K_S \left( 1 + k_d \tau \right)}{(Yk_{\text{max}} - k_d) \tau - 1}
\]

\[
X = \frac{Y \left( S_{\text{in}} - S \right)}{1 + k_d \tau}
\]
Microbial Growth in a CSTR

Example:  $S(0) = 200 \text{ mg BOD/L}$  \quad  $X(0) = 1 \text{ mg VSS/L}$
$$k_{\text{max}} = 15 \text{ mg BOD/mg VSS-d}$$  \quad  $$K_S = 20 \text{ mg BOD/L}$$
$$Y = 0.4 \text{ mg VSS/mg BOD}$$  \quad  $$k_d = 0.12 \text{ d}^{-1}$$
Microbial Growth in a Steady-State CSTR with Sludge Recycle

Mass balance on $S$:

$$V \frac{dS}{dt} = 0 = Q \left( S_{in} - S_{out} \right) + r_S V$$

$$0 = Q \left( S_{in} - S \right) - \frac{k_{max} SX}{K_S + S} V$$

Rearrange, write $Q/V$ as $1/\tau$:

$$\frac{1}{\tau} \frac{S_{in} - S}{X} = \frac{k_{max} S}{K_S + S}$$
Mass balance on X:

\[ V \frac{dX}{dt} = 0 = QX_{in} - Q_{out}X_{out} - Q_uX_w + r_XV \]

Assume \( Q_wX_w >> Q_{in}X_{in}, \ Q_{out}X_{out} \), write \( Q/V \) as \( 1/\tau \):

\[ Q_wX_w = QY(S_{in} - S) - k_dXV \]

\[ \frac{Q_uX_w}{VX} = QY(S_{in} - S) - k_d \]

\[ \frac{Q_uX_w}{VX} = \frac{1}{\tau}Y(S_{in} - S) - k_d \]
\[ \frac{Q_u X_w}{VX} = \frac{Q}{V} \frac{Y(S_{in} - S)}{X} - k_d \]

\[ \frac{VX}{Q_u X_w} = \frac{\text{biomass in reactor}}{\text{rate of biomass removal from reactor}} \]

\[ = \left( \frac{\text{average residence time}}{\text{of biomass in reactor}} \right) = \text{SRT} \]

**SRT is Solids Retention Time**
\[
\frac{Q_u X_w}{VX} = \frac{Q}{V} \frac{Y (S_{in} - S)}{X} - k_d
\]

\[
\frac{1}{\text{SRT}} = \frac{1}{\tau} \frac{Y (S_{in} - S)}{X} - k_d
\]

\[
\frac{1}{\tau} \frac{(S_{in} - S)}{X} = \frac{1}{Y} \left( \frac{1}{\text{SRT}} + k_d \right)
\]
Equate final expressions from the two mass balances and solve for $S$. Then substitute back into mass balance on $S$ to solve for $X$:

\[
\frac{k_{\text{max}} S}{K_S + S} = \frac{1}{Y} \left( \frac{1}{\text{SRT}} + k_d \right)
\]

\[
S = \frac{K_S (1 + k_d [\text{SRT}])}{(Yk_{\text{max}} - k_d)[\text{SRT}] - 1}
\]

\[
X = \frac{Y (S_{\text{in}} - S)}{1 + k_d [\text{SRT}]} \frac{\text{SRT}}{\tau}
\]
Example: Same conditions as prior examples, and $\tau = 6$ h
• **Key Advantages and Disadvantages of Sludge Recycle**
  - Increases biomass concentration in aeration tank, increasing $r_S$ and decreasing required hydraulic residence time to achieve given level of contaminant removal
  - In theory, allows adjustable control over treatment efficiency
  - Requires extra tank and maintenance of environmental conditions favoring sludge flocculation and good settling

• **Potential of MBRs**
  - Provide benefits of recycle without risk of failure due to poor solid/liquid separation
  - Potential problem with fouling