SOME DEFINITIONS FOR FLUID FLOW;
REYNOLDS TRANSPORT THEOREM; CONTINUITY

Definitions of a few terms that are used when quantifying flows:

- ***Mass flow rate*** ($\dot{m}$, mass/time), ***volumetric flow rate*** (Q, volume/time), and ***weight flow rate*** ($\dot{w}$, weight/time). These quantities are related by: $\dot{m} = Q\rho$, $\dot{w} = g\,\dot{m} = g\rho Q$.

  Although we frequently treat these quantities as scalars describing the magnitude of the flow rate in the direction of bulk flow, they really are vectors. In some cases, we might be interested in the rate at which a fluid crosses a specified surface (e.g., a control surface) that is not perpendicular to the bulk flow. In that case, the velocity vector can be decomposed into components parallel to and perpendicular to the surface. The volumetric and mass flow rates across the surface can then be computed based solely on the perpendicular component of the velocity:

$$Q = V_{tot} A_{tot} \cos\theta = \vec{\mathbf{V}} \bullet \vec{\mathbf{A}} = A_{tot} \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} \qquad (1)$$

$$\dot{m} = \rho V_{tot} A_{tot} \cos\theta = \rho \vec{\mathbf{V}} \bullet \vec{\mathbf{A}} = \rho A_{tot} \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} \qquad (2)$$

  where the area vector, $\vec{\mathbf{A}}$, is defined to have the same magnitude as the total area of interest (i.e., the area through which the flow of interest occurs) and a direction that is perpendicular to that area, and θ is the angle formed by the velocity and area vectors. Note that, if $\theta = 0^{o}$, cos $\theta = 1$, and $Q = V_{tot}A_{tot}$, and if $\theta = 90^{o}$, cos $\theta = 0$, and $Q = 0$. $\vec{\mathbf{n}}$ is a unit vector (magnitude equal to 1.0) with the same direction as $\vec{\mathbf{A}}$, so that $A_{tot}\vec{\mathbf{n}} = \vec{\mathbf{A}}$.

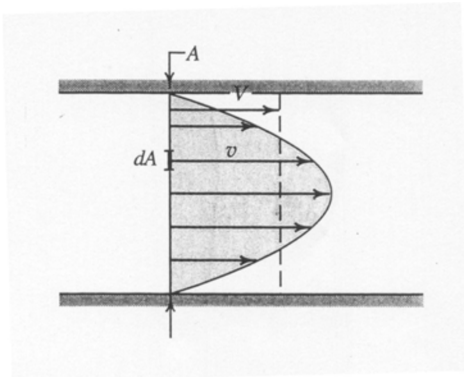
- ***Mean velocity, V***: the velocity that, if applied over the entire cross-section of flow, would yield the same resultant flow rate as the actual flow. Thus:

$$V \equiv \frac{Q}{A_{tot}} = \frac{\int v dA}{A_{tot}} \qquad (3)$$

  The idea is that we move around the area of interest, compute $v\,dA$ ($= dQ$) in each differential unit of area, add up all those terms to find Q, and divide by A_{tot} to find the mean velocity, V. Note that the averaging is based on flow (the product of velocity and area), not on velocity or area alone. For example, if the water velocity in one half of a channel is 10 m/s and that in the other half is 20 m/s, the mean velocity is not 15 m/s, but faster, because twice as much water is flowing at 20 m/s as at 10 m/s. (The mean velocity would therefore be 16.67 m/s, meaning that the total flow through the channel is the same as if the velocity were that value everywhere.)

  Example. The velocity profile in a circular tube with laminar flow is parabolic and is characterized by the following equation characterizing and schematic. Compute the mean velocity in such a tube.

$$v(r) = v_{\max}\left(1 - \frac{r^2}{R^2}\right) \tag{4}$$



Solution. Define R as the radius of the tube, r as the distance from the center, and $v(r)$, as the velocity at r, and v_{max} as the velocity at the center of the tube (equal to $v(0)$). The mean velocity can therefore be computed based on Equation 3 as:

$$V = \frac{\int_0^R v(r)\,dA}{A_{tot}} = \frac{\int_0^R \left[v_{\max}\left(1 - \frac{r^2}{R^2}\right)\right]\left(2\cancel{\pi} r dr\right)}{\cancel{\pi} R^2} = \frac{2v_{\max}}{R^2}\int_0^R \left(r - \frac{r^3}{R^2}\right)dr$$

$$= \frac{2v_{\max}}{R^2}\left(\frac{1}{2}r^2 - \frac{1}{4}\frac{r^4}{R^2}\right)_0^R = \frac{2v_{\max}}{R^2}\left(\frac{1}{2}R^2 - \frac{1}{4}\frac{R^4}{R^2}\right) = \frac{2v_{\max}}{R^2}\left(\frac{1}{4}R^2\right)$$

$$= \frac{v_{\max}}{2} \tag{5}$$

Thus, the mean velocity in the tube is one-half of the maximum velocity.

Terms describing two important categories of fluid properties:

- ***Intensive*** and ***extensive*** properties. *Intensive* properties are those that retain the same value when more mass, with identical properties to the mass already in the system, is added to the system; examples of such properties include temperature, density, and any property normalized to mass (e.g., energy per unit mass). Correspondingly, *extensive* properties have values that are proportional to the mass in the system (*e.g.*, total energy, total weight, or total momentum of the fluid in a system).

A general approach for analyzing changes of conservative properties in fluid systems: The Reynolds Transport Theorem

The objective of this section is to develop an important, general relationship known as the Reynolds Transport Theorem (RTT). The RTT provides a way to analyze/ interpret the changes in conservative parameters (in particular, mass, energy, momentum) in fluid systems. The key idea is that, if we define a CV, these parameters change within that CV if and only if fluid that contains some of the parameter crosses the CS (by ***advection***), or if the parameter is "injected" or "withdrawn" from the CV in some identifiable way that is not associated with advection.

The RTT is expressed in terms of the vectors and property types defined above. Consider an extensive, conservative parameter *P* that is applicable to a fluid of interest; as noted above, in this course, we will consider three items as possibilities for *P*: mass, energy, and momentum. The amount of any of these parameters possessed by the fluid under consideration (which might range from a differential to a very large amount) is an extensive parameter, which we will designate E_P. For example, the total energy of an aliquot of fluid is an extensive parameter that we will designate E_{energy} (or, for conciseness, E_{en}). The corresponding extensive parameters for the total mass and total momentum of the fluid of interest will be designated E_{mass} and E_{mom}, respectively.

Each parameter E_P is related to an intensive parameter, i_P, defined as the amount of that parameter per unit mass of fluid (E_{mass}, or simply m); thus $i_{en} = E_{en}/m$, and $i_{mom} = E_{mom}/m$. When this idea is applied to fluid mass, the intensive parameter 'i_{mass}' is defined as the mass per unit mass, so it is simply an identity ($i_{mass} = 1.0$).

Based on Equation 2, the mass flow rate across any differential unit of surface is $d\dot{m} = \rho\vec{\mathbf{V}} \bullet d\vec{\mathbf{A}}$. Therefore, the product $i_P\, d\dot{m} = \rho i_P \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}}$ is the rate at which parameter *P* is carried across the differential surface by advection. Above, we defined $\vec{\mathbf{A}}$ as having a direction that is normal to the surface, but we did not specify its direction. By convention, when analyzing flow relative to a control volume, the vector $\vec{\mathbf{A}}$ on any part of the control surface (CS) is defined as pointing out of the CV. As a result, if $\theta = 0$, the implication is that the velocity crossing the CS is perpendicular to and outward across the boundary of the CV; correspondingly, if $\theta = 180^{\circ}$, the flow is perpendicular to and into the CV. Thus, the product $\rho\vec{\mathbf{V}} \bullet d\vec{\mathbf{A}}$ yields the mass flow rate *leaving* the CV. Correspondingly, $i_P \rho \vec{v} \bullet d\vec{A}$ indicates the rate at which parameter *P* is being removed from the CV by advective flow across area dA, with units of amount of mass, energy, or momentum per unit time. By integrating this quantity around the whole CS, we obtain the rate at which advection is causing the amount of the parameter of interest to *decrease* inside the CV:

$$\begin{pmatrix} \text{Rate at which } P \text{ is leaving} \\ \text{the CV by advection} \\ \text{(amount of } P \text{ per time)} \end{pmatrix} = \oint_{CS} i_P \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \qquad (6)$$

where the symbol $\oint_{CS}$ indicates that we are integrating completely around the control surface.

Energy and momentum, but not mass, can be added to or lost from the fluid inside the CV by means other than advection. For example, energy can be injected into a fluid by pressurizing the fluid in a pump, it can be extracted by passing the fluid through a turbine, and it can be added or lost by injecting or removing heat, respectively. Similarly, momentum can be injected or extracted by applying various external forces on the fluid (recall that an object gains momentum at a rate equal to any external force that is applied). Such processes are most easily quantified in terms of the overall rate at which the parameter is injected into or removed from the CV; i.e., in terms of extensive parameters. We will write the rate at which *P* is added by non-advective inputs and withdrawals as $\dot{E}_{P,non-adv}$, with dimensions of the amount of *P* per unit time:

$$\begin{pmatrix} \text{Rate at which } P \text{ appears} \\ \text{in the CV by non-advective} \\ \text{processes (amount of } P \text{ per time)} \end{pmatrix} = \dot{E}_{P,non-adv} \tag{7}$$

The total amount of *P* in the CV at any instant can be determined by integrating the amount of *P* per unit volume throughout the entire CV:

$$\begin{pmatrix} \text{Total amount of } P \text{ in} \\ \text{the CVat any instant} \end{pmatrix} = \int_{\text{CV}} \rho i_P d\cancel{V} \tag{8}$$

where $\cancel{V}$ is volume. The sense of this integration is that we move all throughout the CV, considering differential units of volume, determine the amount of *P* in each, and then sum up all those quantities. The rate of change of the summation in Equation 8 yields the net rate at which *P* is increasing in the CV, i.e.:

$$\begin{pmatrix} \text{Net rate at which amount} \\ \text{of } P \text{ in the CV increases} \\ \text{(amount of } P \text{ per time)} \end{pmatrix} = \frac{\partial}{\partial t} \int_{\text{CV}} \rho i_P d\cancel{V} \tag{9}$$

Finally, we can combine the terms developed above to find:

$$\begin{pmatrix} \text{Net rate at which amount} \\ \text{of } P \text{ in the CV increases} \\ \text{(amount of } P \text{ per time)} \end{pmatrix} =$$

$$-\begin{pmatrix} \text{Rate at which } P \text{ is leaving} \\ \text{the CV by advection} \\ \text{(amount of } P \text{ per time)} \end{pmatrix} + \begin{pmatrix} \text{Rate at which } P \text{ enters the CV} \\ \text{by non-advective processes} \\ \text{(amount of } P \text{ per time)} \end{pmatrix}$$

$$\frac{\partial}{\partial t} \int_{\text{CV}} \rho i_P d\cancel{V} = -\oint_{\text{CS}} i_P \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} + \dot{E}_{P,non-adv} \tag{10}$$

Equation 10 is the general form of the Reynolds Transport Theorem and can be used to solve many important problems in fluid mechanics.[1] Although the theorem looks mathematically complex, it is relatively easy to understand conceptually (i.e., in terms of the word expressions that precede it), and in many cases, simplifying assumptions can be made that allow the equation to be fairly easy to use quantitatively as well. For instance, in the commonly encountered situation where the system of interest has reached a steady state, the amount of P in the CV remains constant over time. As a result, the time derivative on the left-hand side of Equation 10 becomes zero, and the equation simplifies to:

RTT for steady-state systems:
$$0 = -\oint_{\mathrm{CS}} i_P \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} + \dot{E}_{P,non-adv} \qquad (11)$$

If, in addition to having a steady-state system, the property of interest is mass, $\dot{E}_{P,non-adv} = 0$ and $i_P = 1$, so the equation becomes even simpler:

RTT for mass in steady-state systems:
$$0 = -\oint_{\mathrm{CS}} \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} = \oint_{\mathrm{CS}} \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \qquad (12)$$

Whenever the RTT is applied to mass in a system with steady flow, it is referred to as the ***continuity equation***; this equation is discussed in more detail subsequently.

Finally, if Equation 12 applies and the fluid is incompressible, ρ is constant for the fluid crossing any portion of the CS. In that case, the RTT simplifies to:

RTT for mass of incompressible fluids in steady-state systems:
$$0 = \oint_{\mathrm{CS}} \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \qquad (13)$$

In words, Equation 12 is a formal, mathematical statement of the fact that, in steady-state systems, the amount of mass in the CV does not change over time, and that therefore the rate at which mass enters the CV must equal the rate at which it leaves. Equation 13 then applies this result to systems with fixed fluid densities and indicates that, for such systems, the volumetric flow rate into the CV must equal the volumetric flow rate out.

Additional simplifications can be made if fluid enters or leaves the CV at only a few distinct locations. In that case, we can convert the integrals in the above expressions into summations as follows:

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[1] The final term in Equation 10 can be thought of as the total rate at which parameter P is added to the fluid as it passes through the CV and can therefore be written as the *substantial* or *material* derivative of P, i.e., as DE_P/Dt. This derivative is the expression that is used for the non-advective input rate of Pin the RTT as presented in the Munson *et al*. text.

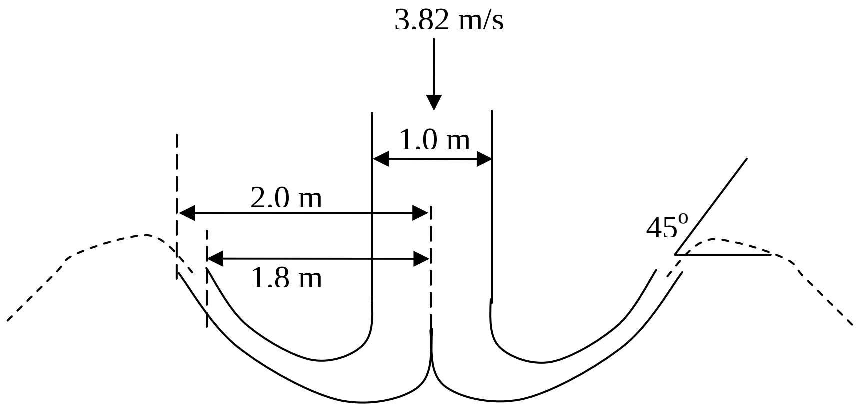
RTT for steady-state systems with discrete inlets / outlets:	$0 = -\sum_{\substack{inlets/\\outlets}} \left(i_P \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \right) + \dot{E}_{P,non-adv}$	(14)
RTT for mass in steady-state systems with discrete inlets / outlets:	$0 = -\sum_{\substack{inlets/\\outlets}} \left(\rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \right)$	(15)
RTT for mass of incompressible fluids in steady-state systems with discrete inlets / outlets:	$0 = -\sum_{\substack{inlets/\\outlets}} \left(\vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \right)$	(16)

If we choose the boundaries of the CV in such a way that they are perpendicular to the CV at each inlet and outlet, the equations can be simplified even more, as follows:

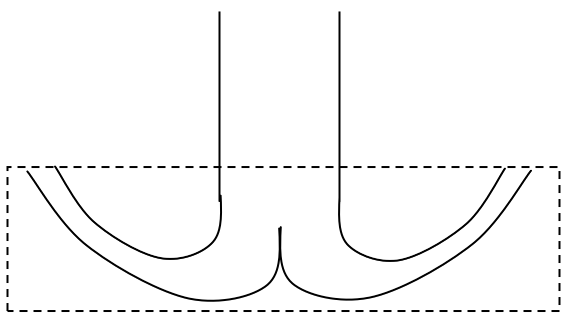
RTT for incompressible fluids in steady-state systems with discrete inlets / outlets and flow perpendicular to CS:	$0 = \sum_{inlets} \left(i_P \rho VA \right) - \sum_{outlets} \left(i_P \rho VA \right) + \dot{E}_{P,non-adv}$	(17)
RTT for mass in steady-state systems with discrete inlets / outlets and flow perpendicular to CS:	$0 = \sum_{inlets} \left(\rho VA \right) - \sum_{outlets} \left(\rho VA \right)$	(18)
RTT for mass of incompressible fluids in steady-state systems with discrete inlets / outlets and flow perpendicular to CS:	$0 = \sum_{inlets} \left(VA \right) - \sum_{outlets} \left(VA \right)$	(19)

Note that Equations 15, 16, 18, and 19 are all forms of the continuity equation.

Example. Water is flowing steadily through a "mushroom cap," as shown in the following diagram. The water enters at a velocity of 3.82 m/s through a center pipe that is 1.0 m in diameter, and is then redirected so that it exits at a 45^{o} angle through an annular space between two concentric circles with radii of 1.8 and 2.0 m. Find the average velocity of the water at the exit point.



Solution. We can define a CV that encircles cuts across the cap in such a way that the only flows that cross the CS are in the inlet pipe and the annular outlet, as shown below.



To apply the RTT to this CV, we first note that, since the system has steady flow of an incompressible fluid, Equation 13 applies. Furthermore, since the CV has only one inlet and one outlet, Equation 16 applies as well. Therefore, we can write the RTT for the system as follows:

$$0 = \oint_{CS} \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} = -\sum_{\substack{inlets/\\outlets}} \left(\vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} \right) = -V_{in} A_{in} \cos\theta_{in} - V_{out} A_{out} \cos\theta_{out}$$

The incoming velocity is given, and the area of the CS that it crosses is $\pi d_{pipe}^2 / 4$, or 0.785 m^2. The velocity vector at this location makes a 180^o angle with the normal to the surface, so θ_{in} is 180^o, and cos θ_{in} is –1.0.

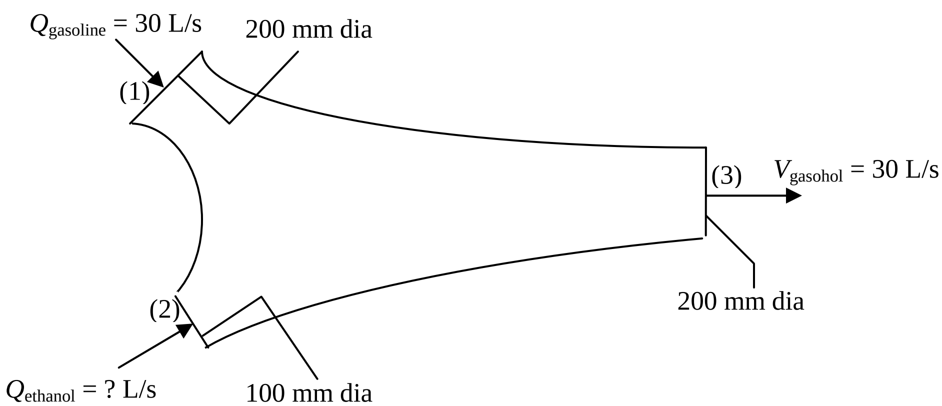
The exiting velocity is unknown, but we know that the area of the CS that it crosses is the annular space between r values of $r_1 = 1.8$ and $r_2 = 2.0$ m. A_{out} is therefore given by $\pi\left(r_2^2 - r_1^2\right)$, which equals 2.39 m^2. We also know that, at the outlet, the velocity vector makes a 45^o angle with the normal to the control surface. Substituting this information into the RTT, we find:

$$0 = -\left(3.82\text{ m/s}\right)\left(0.785\text{ m}^2\right)\cos 180^\circ - V_{out}\left(2.39\text{ m}^2\right)\cos 45^\circ$$

$$= -\left(3.82\text{ m/s}\right)\left(0.785\text{ m}^2\right)\left(-1\right) - V_{out}\left(2.39\text{ m}^2\right)\left(0.707\right)$$

$$V_{out} = \frac{\left(3.82\text{ m/s}\right)\left(0.785\text{ m}^2\right)}{\left(2.39\text{ m}^2\right)\left(0.707\right)} = 1.78\text{ m/s}$$

Example. A mixture of 10% alcohol and 90% gasoline (gasohol) is being prepared by mixing the two individual liquids in the wye pipe shown schematically below. The volumetric flowrate of the gasoline, the velocity of the gasohol stream, and the dimensions of the pipe are shown in the diagram. The densities of the ethanol, gasoline, and gasohol are 788.6, 680.3, and 691.1 kg/m^3, respectively. Find the volumetric flowrate and average velocity of the incoming alcohol.



Solution. In this system, the density of the fluid changes as it passes through the CV. At every location where the flow enters the CV, it is perpendicular to the CS, so we can apply Equation 18 to write the RTT as follows:

$$0 = \sum_{inlets}\left(\rho VA\right) - \sum_{outlets}\left(\rho VA\right) = \left(\rho VA_{tot}\right)_1 + \left(\rho VA_{tot}\right)_2 - \left(\rho VA_{tot}\right)_3$$

$$0 = \rho_1 Q_1 + \rho_2 Q_2 - \rho_3 V_3 A_3$$

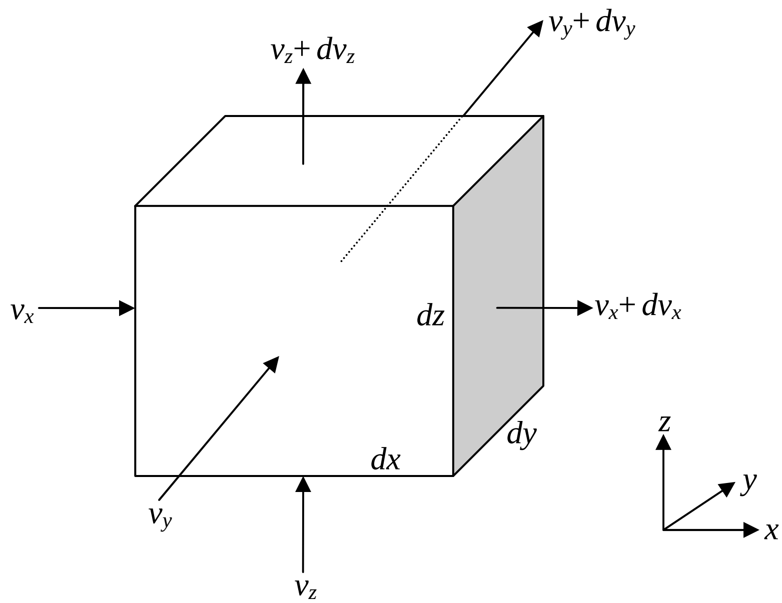
$$Q_2 = \frac{\rho_3 V_3 A_3 - \rho_1 Q_1}{\rho_2}$$

$$= \frac{\left(691.1\frac{\text{kg}}{\text{m}^3}\right)\left(1.08\frac{\text{m}}{\text{s}}\right)\left(\pi\frac{(0.20\text{ m})^2}{4}\right) - \left(680.3\frac{\text{kg}}{\text{m}^3}\right)\left(0.030\frac{\text{m}^3}{\text{s}}\right)}{788.6\frac{\text{kg}}{\text{m}^3}} = 3.4\text{x}10^{-3}\,\frac{\text{m}^3}{\text{s}}$$

$$V_2 = \frac{Q_2}{A_2} = \frac{4.4\text{x}10^{-3}\text{ m}^3/\text{s}}{\pi\frac{(0.100\text{ m})^2}{4}} = 0.43\text{ m/s}$$

Note that the approach for solving this problem utilized the fact that the flow was perpendicular to the CS at each entry and exit point, but did not require any assumption about the flow pattern between those points.

The Continuity Equation from the RTT. The forms of the RTT shown in Equations 10 through 19 are written for a CV of arbitrary shape, and they use a coordinate system that is defined by the shape of the CS. As a result, the direction of the area vector changes from one part of the surface to another, when viewed from a fixed frame of reference outside the CV. When the RTT is applied to mass in steady-state systems, it is instructive to recast the equation in the framework of the conventional Cartesian coordinate system. To develop this alternative form of the equation, we write the RTT for a CV that consists of a differential-sized box, with sides of length dx, dy, and dz. The flow at location (x, y, z) can be moving in any arbitrary direction, and has velocity components (magnitudes only) of v_x, v_y, and v_z; the flow at $(x+dx, y+dy, z+dz)$ has corresponding velocity components $v_x + dv_x$, $v_y + dv_y$, and $v_z + dv_z$, as shown in the following schematic.



Since v_x, v_y, and v_z are, by definition, the magnitudes of the velocity in the +*x*, +*y*, and +*z* directions, they each form an angle of 180^o with a line that is normal to and pointing outward from the corresponding surface of the CV.[2] The velocities $v_x + dv_x$, $v_y + dv_y$, and $v_z + dv_z$ are also defined to be positive in the +*x*, +*y*, and +*z* directions, but they form an angle of 0^o with lines pointing directly outward from the CS at the locations where they apply. Thus, applying the RTT to mass in this control volume for a system at steady-state (Equation 15), we can write:

$$0 = \oint \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}}$$

$$= \left[\left(-\rho v_x\right)_x + \left(\rho v_x\right)_{x+dx} \right] dA_{yz} + \left[\left(-\rho v_y\right)_y + \left(\rho v_y\right)_{y+dy} \right] dA_{xz} + \left[\left(-\rho v_z\right)_z + \left(\rho v_z\right)_{z+dz} \right] dA_{xy}$$

If we substitute $\left(\rho v_x\right)_x + \frac{\partial\left(\rho v_x\right)}{\partial x} dx$ for $\left(\rho v_x\right)_{x+dx}$, the quantity $\left[\left(-\rho v_x\right)_x + \left(\rho v_x\right)_{x+dx} \right]$ simplifies to $\frac{\partial\left(\rho v_x\right)}{\partial x} dx$. When analogous substitutions are made for $\left(\rho v_y\right)_{y+dy}$ and $\left(\rho v_z\right)_{z+dz}$, the above equation becomes:

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[2] Note that this does *not* mean that the net flow is in the +*x*, +*y*, and +*z* direction, but only that the signs of v_x, v_y, and v_z are determined based on those directions being defined as positive.

$$0 = \frac{\partial(\rho v_x)}{\partial x} dx\, dA_{yz} + \frac{\partial(\rho v_y)}{\partial y} dy\, dA_{xz} + \frac{\partial(\rho v_z)}{\partial z} dz\, dA_{xy} \qquad (20)$$

$$= \frac{\partial(\rho v_x)}{\partial x} d\cancel{V} + \frac{\partial(\rho v_y)}{\partial y} d\cancel{V} + \frac{\partial(\rho v_z)}{\partial z} d\cancel{V} \qquad (21)$$

Dividing through by $d\cancel{V}$, we obtain:

$$0 = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \qquad (22)$$

Equation 22 is yet another formulation of the ***continuity equation***, and a very commonly encountered one. As a reminder, this equation is a mathematical statement of the principle of conservation of mass for a system with steady flow. The sense of Equation 22 is that, in a system at steady-state, mass that enters the CV by flow in the x direction might leave by flow in the x, y, or z direction. Hence a decrease in the mass flowrate in the x direction as the fluid passes through the CV (corresponding to $\partial(\rho v_x)/\partial x < 0$) must be compensated by an increase in the mass flowrate in at least one of the other directions. Note that ρ and v might vary with location in the system (e.g., along the length or cross-section of a pipe), but by the assumption of steady flow, they do not change at a given location over time.

If the fluid under consideration is incompressible, ρ is constant throughout the system, and the equation simplifies to:

$$0 = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \qquad (23)$$

In the commonly encountered situation in which the flow can be reasonably characterized as 1-D, the continuity equation for compressible and incompressible fluids is simply:

$$\frac{\partial(\rho v_x)}{\partial x} = 0 \qquad (24)$$

$$\frac{\partial v_x}{\partial x} = 0 \qquad (25)$$

The result in Equation 25 indicates that the velocity of an incompressible fluid in 1-D, steady flow is constant in the direction of flow. In such a system, the fluid velocity might change in the direction perpendicular to the flow (e.g., it might vary with location as we move across the cross-section of a pipe), but it cannot change as we move downstream. Furthermore, because the velocity is constant along any line in the downstream direction, the *average* velocity across any cross-section must also be constant between an upstream and a downstream point.

The RTT Normalized to Fluid Mass or Weight

Return now to the RTT for steady-state systems for parameters that can have non-advective inputs or withdrawals of a parameter of interest (Equation 11):

$$0 = -\oint i_P \rho \vec{\mathbf{V}} \bullet d\vec{\mathbf{A}} + \dot{E}_{P,non-adv} \qquad (11)$$

Each of the two terms on the right-hand side have dimensions corresponding to an amount of *P* per unit time. In a system at steady state, the mass and weight of fluid passing through the CV are constant over time. For such systems, it is often useful to rewrite Equation 11 in terms of the amount of the parameter of interest per unit mass or weight of fluid passing through the system. To do this, we simply divide by the rate at which mass or weight enters or leaves the CV:

$$\begin{pmatrix} \text{Amount of } P \text{ added} \\ \text{per unit mass of fluid} \\ \text{passing through CV} \end{pmatrix} = \frac{\text{Amount of } P \text{ added to fluid per unit time}}{\text{Mass of fluid entering or leaving CV per unit time}}$$

Although the above equality is valid as a general statement, it is most useful when applied to systems with a single inlet and outlet. In that case, Equation 11 simplifies to a version of Equation 17, as follows:

$$0 = \left(i_P \rho V A\right)_{inlet} - \left(i_P \rho V A\right)_{outlet} + \dot{E}_{P,non-adv} \qquad (26)$$

The mass flow rate through the CV is the product ρVA, which, for a system at steady state, is identical at the inlet and the outlet (based on the continuity equation). Dividing through by this term yields:

$$0 = i_{P,inlet} - i_{P,outlet} + \frac{\dot{E}_{P,non-adv}}{\rho VA} = i_{P,inlet} - i_{P,outlet} + \frac{\dot{E}_{P,non-adv}}{\dot{m}} \qquad (27)$$

The last term on the right of Equation 27 is the non-advective input of *P* per unit mass of fluid as the fluid passes through the CV. We defined the amount of *P* per unit mass of fluid as the intensive parameter i_P. Therefore, $\frac{\dot{E}_{P,non-adv}}{\dot{m}}$ can be written as $i_{P,non-adv}$, and Equation 27 can be written as:

$$0 = i_{P,inlet} - i_{P,outlet} + i_{P,non-adv}$$

$$i_{P,outlet} = i_{P,inlet} + i_{P,non-adv} \qquad (28)$$

Equation 28 simply states that the amount of *P* per unit mass of fluid exiting the CV equals the amount per unit mass entering plus the amount that is added per unit mass by non-advective processes while the fluid is in the CV. If we divide through by *g*, we obtain the analogous expression with all the terms having units of *P* per unit weight of fluid; we will designate these terms with $\hat{i}$:

$$\hat{i}_{P,outlet} = \hat{i}_{P,inlet} + \hat{i}_{P,non-adv} \qquad (29)$$

Summary

The RTT describes a conservation concept: things can appear in a CV if they are either carried in across the boundaries or added by some process that takes place inside the CV; they can disappear from the CV if they are either carried out across the boundaries or are removed by some process occurring inside the boundaries; and they accumulate inside the CV at a rate that equals the difference between the rates at which appear and disappear. In general, the accumulation term is quantified via an integration throughout the volume of the CV, whereas the transport term is quantified via an integration across the boundaries of the CV. The advective term can be written in a concise vector format by defining area to be a vector.

In steady flow, the accumulation term disappears, greatly simplifying the analysis. Further, if transport occurs uniformly over just a few areas (and especially if it is perpendicular to those areas), the integration can be converted to a summation, which can be carried out manually.

The RTT applies for any extensive property (*E*) of a fluid (which can be expressed as an intensive parameter [*i*] by normalizing to mass). When applied to mass, *i* becomes unity, and the RTT becomes a statement of the conservation of mass. Also, for a fluid with constant density, $\dot{E}_{mass} = 0$, so the RTT indicates that advection out minus advection in equals the rate of accumulation. Finally, if we restrict our considerations to systems at steady state, no fluid accumulates within the CV over time, and the RTT simplifies to a statement that advection in equals advection out.

When applied to mass, the RTT often seems like an overly complex way to express an intuitive idea. However, even when we are focusing on mass, the RTT is a useful approach for generalizing the principle of conservation of mass to any shape of CV and CS in any complex velocity field. In reality, though, the power of the RTT becomes apparent when we consider the changes in the amount of energy and momentum that a fluid contains; we consider those parameters next.