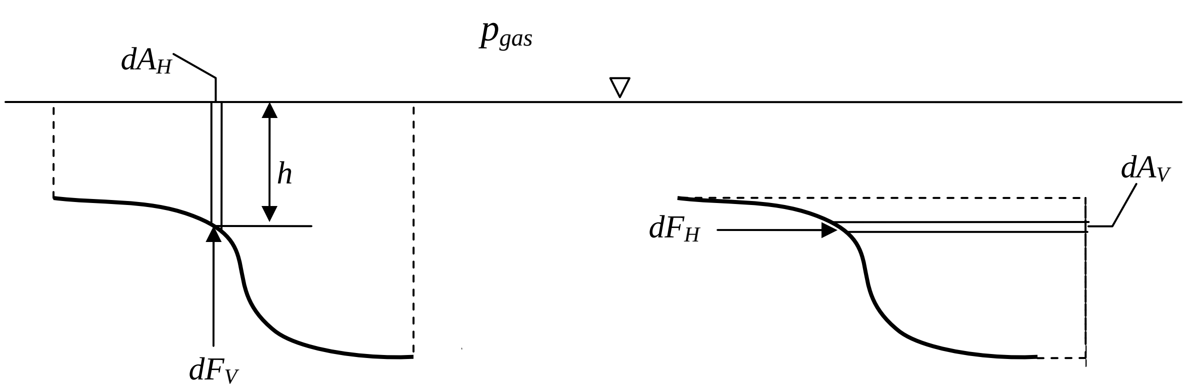
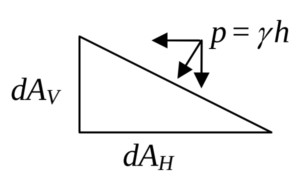
FLUID FORCES ON CURVED SURFACES; BUOYANCY

The principles applicable to analysis of pressure-induced forces on planar surfaces are directly applicable to curved surfaces. As before, the total force on the surface is computed as $\int dF$, or $\int p\, dA$, and, in each differential area, the force is perpendicular to the surface. The complication is therefore simply geometric, because we need to account for the different directions of the dF terms. However, it turns out that this complexity can be eliminated, based on the following analysis. (Note: the analysis addresses ways of finding the magnitude and direction of the force; finding the location of action is not covered here.)

Consider the forces acting on one side of an infinitely thin, but curved, surface that is completely exposed to a fluid at rest. For simplicity, assume that the surface is not curved under itself, so that the vertical component of the fluid pressure is downward everywhere on one side of the sheet, and upward everywhere on the other side.

We can analyze the force downward on the upward-facing side of the sheet, based on a free-body analysis around a mass of fluid defined by the boundaries shown in the figure on the left below. The bottom boundary is the curved surface, the upper boundary is a horizontal plane at the water surface, and the side boundaries are vertical planes at the edges of the surface. Because this mass of fluid is at rest, the net vertical force on it must be zero. Consider a differential area of the curved surface, with length in the curved direction of dl and width equal to the full width of the sheet, at a depth h below the water. This area can be decomposed into vertical and horizontal components, dA_V and dA_H, respectively, as shown by the triangle below the figure.





Because pressure at a given depth is the same in all directions, and because the triangle is of differential size, the pressure exerted on all three faces of the triangle is the same, equal to $p_{air} + \gamma h$. Therefore, a free-body analysis of the water column above the area dA shows that the force downward on dA (and, equivalently, the force upward that the area dA exerts on the water) is:

$$dF_V = \left(p_{gas} + \gamma h \right) dA_H \qquad (1)$$

Integrating over the whole surface, we find the vertical force downward on the sheet, and that the sheet exerts upward on the water, is:

$$F_V = \int p_{gas} dA_H + \gamma \int h dA_H \qquad (2)$$

$$= p_{gas} A_H + \gamma V_{above} = p_{gas} A_H + W_{above} \qquad (3)$$

where W_{above} is the weight of the water above the surface of the sheet. That is, the total downward force on the curved surface equals the sum of (1) the product of the pressure of the overlying gas phase and the projected area of the curved surface onto a horizontal plane, and (2) the weight of water that would fill the volume directly above the curved surface. Note that the analysis is based on the pressure exerted on different points of the surface, which in turn depends on γ and h. Thus, the result shown in Equation 3 is valid even if some object (*e.g.*, a fish) is in the fluid above the surface of interest. That is, as stated, the term W_{above} equals the weight of water that would fill the volume directly above the curved surface, regardless of whether that volume is actually occupied by water or something else.

A very similar analysis can be carried out to determine the horizontal force on the upper curved surface. In this case, the boundaries for the free-body analysis include horizontal planes that are tangent to the highest and lowest points of the curved surface, vertical planes at the edges defining the width of the surface, a vertical plane an arbitrary distance to the right of the surface, and the surface itself. The horizontal force on a differential area and on the whole surface are then given by:

$$dF_H = (p_{gas} + \gamma h) dA_V$$

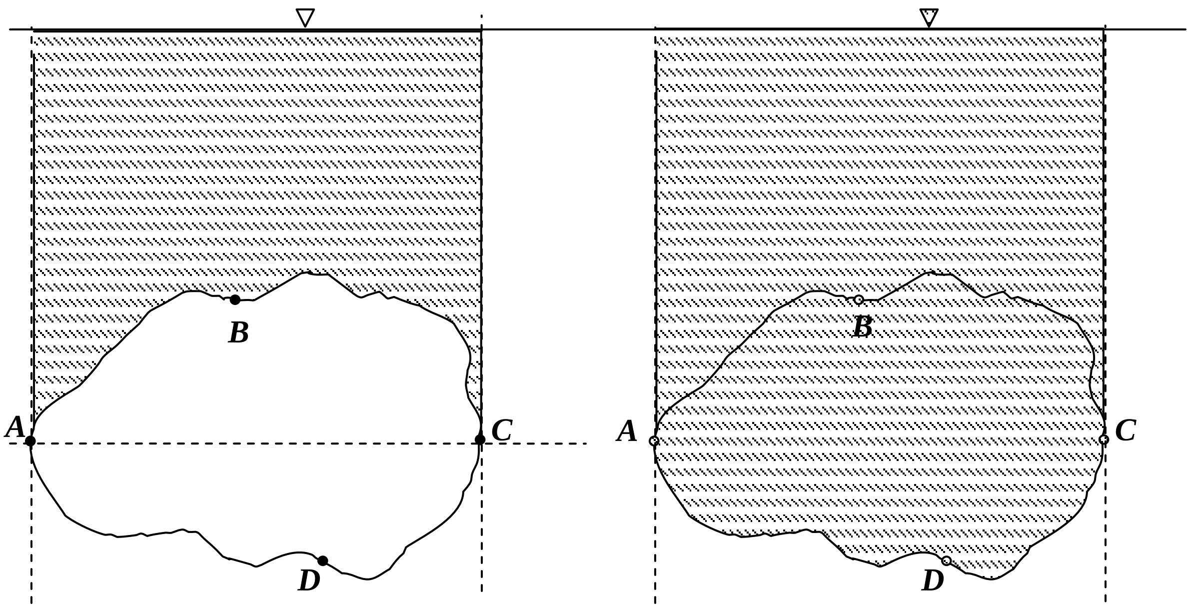
$$F_H = \int p_{gas} dA_V + \gamma \int h dA_V \qquad (4)$$

The integral $\int h dA_V$ is the moment of the area projected onto the vertical plane and is therefore the centroid of that area, h_c. Integrating the first term in Equation 4 and making the above substitution for the second term yields:

$$F_H = \left(p_{gas} + \gamma h_c \right) A_V \qquad (5)$$

This result can be stated in words as: the horizontal force on a curved surface is the product of the projected area of the surface on a vertical plane with the sum of the pressure of the overlying air and the water pressure at the centroid of the projected area.

Now consider how the analysis applies to a solid object that is submerged in the fluid. We can consider the vertical forces on such an object in two parts: one part accounts for all the downward forces on the upper surface of the object, and the other part accounts for all the upward forces on its lower surface. Two copies of a schematic of such an object are shown below; the differences in shading are explained shortly.



The analysis of the vertical force on this object is virtually identical to that leading to Equation 3, but now the pressure-based force upward on the bottom surface is greater than the downward pressure-based on the top surface. In the schematic, this difference is indicated by the shading, which indicates W_{above}: the weight of water above the top surface (shown by the shading in the diagram on the left) is less than the corresponding weight above the bottom surface (shown by the shading in the diagram on the right). The difference between these two forces is the net upward (buoyant) force on the object. The effect of the air pressure on the two surfaces cancels out, leaving us with:

$$F_{buoyant} = F_{V,bottom} - F_{V,top}$$

$$= \left(p_{gas} A_H + W_{\substack{above \\ bottom}} \right) - \left(p_{gas} A_H + W_{\substack{above \\ top}} \right)$$

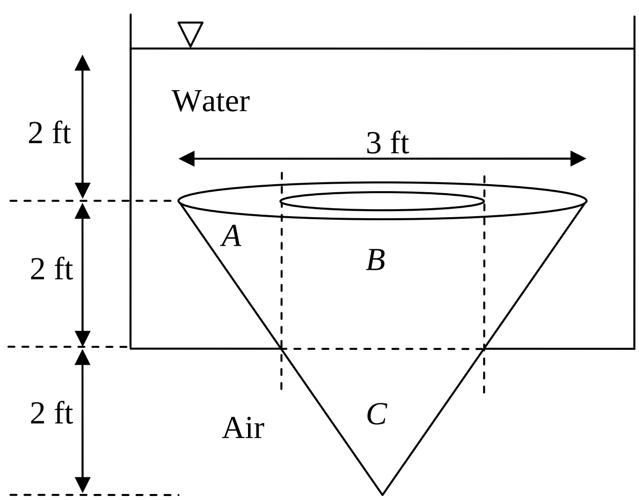
$$= W_{\substack{above \\ bottom}} - W_{\substack{above \\ top}} = W_{\substack{displaced \\ volume}} \qquad (6)$$

Thus, for an object that is completely submerged in a single fluid, the buoyant force equals the weight of the fluid displaced by the object's volume. This is the proof of Archimedes' first law.

The net horizontal force on such an object would be zero, since its projected areas to the left and right would be the same. However, there would be a net horizontal force on the object if it had different fluids on its two sides (*e.g.*, if it were a plug, so that it had water on one side and air on the other).

If an object is only partially submerged in a liquid (i.e., if it sticks out of the liquid at any location), the analysis follows the same outline. Since the system is at rest, the downward force of the object (typically, its weight plus the pressure-based forces on its upper surfaces) must be balanced by the upward force on it (any support structure plus the pressure-based force on its lower surfaces). As above, we can compute the buoyant force on the object as the difference between the upward, pressure-based force on its lower surface and the downward, pressure-based force on its upper surface. These forces can be computed by applying Equation 3 to the lower and upper surfaces, but we must take care to include in W_{above} only the projection of those parts of the surface that are actually in contact with the liquid.

Example. In the system shown schematically below, the solid cone fits into the 1.5-ft diameter opening in the bottom of the tank to prevent water from draining from the tank. The volumes of the cylindrical portion of the cone above the opening (*B*), the conical section below the opening (*C*) and the remaining, semi-annular portion of the cone (*A*) are shown in the figure. What is the minimum weight of the cone that will prevent it from lifting away from the opening?



$$V_A = \frac{3}{2}\pi \text{ ft}^3$$

$$V_B = \frac{9}{8}\pi \text{ ft}^3$$

$$V_C = \frac{3}{8}\pi \text{ ft}^3$$

Solution. Section *A* of the cone is completely submerged in the water (*i.e.,* it has water both above and below it), so the buoyant force on that section equals the weight of water that it displaces, i.e., $F_{buoyant,A} = V_A\gamma$. Section *B*, on the other hand, is a cylinder with water above it and air below it. The air pressure on the bottom and the projected area of this section are identical to the air pressure on the top and the top projected area, respectively, so the two forces on section *B* associated with air pressure cancel one another. However, the section also experiences a force downward from the water pressure on its upper surface, equal to $\gamma h_{B,top} Area_B$. The distance $h_{B,top}$ is given as 2 ft, and, by geometry, the diameter of the *B* cylinder is 1.5 ft. We can therefore calculate the force on section *B*. Section *C* is not in the water and therefore experiences no buoyant or pressure-based force from the water; its only effect is to contribute to the weight of the cone, which is what we will solve for. We can therefore carry out a force analysis on the whole cone. Defining up as the direction of positive force, we have:

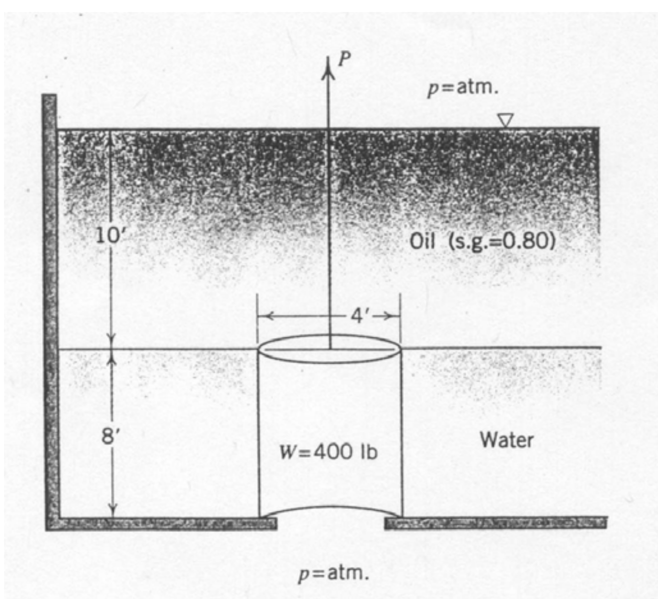
$$F_{tot} = 0 = F_A - F_B - W_{cone}$$

$$= \gamma\left(V_A - h_{B,top} Area_B\right) - W_{cone}$$

$$W_{cone} = 62.4\left(\frac{3}{2}\pi - (2)\left(\frac{\pi[1.5]^2}{4}\right)\right)\text{lb} = 73.6 \text{ lb}$$

The cone must weigh at least 73.6 lb to keep the water from leaking out.

Example. Calculate the force *P* necessary to lift the 4-ft diameter, 400 lb container off the bottom in the scenario shown in the schematic. Draw a free body diagram of the container to assist in the solution.



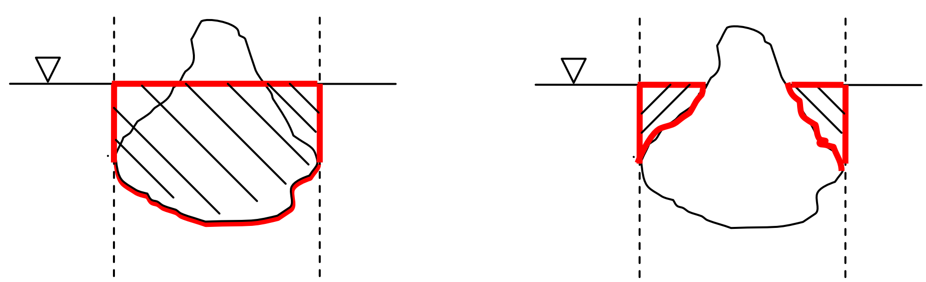
Solution. The water exerts a compressive force on the container, but no vertical force, so it has no effect on the force required to lift the container. Also, the atmospheric pressure exerts a force downward on the top surface of the container that is exactly equaled by the upward force it exerts on the bottom, so it also has no effect on the required force P. The pressure at a depth of 10 ft of oil is $\gamma_{oil}h$, and this pressure is exerted on the upper surface of the container. The total downward force is therefore given by:

$$F_{down} = W + \gamma_{oil} h A$$

$$= 400 \text{ lb} + 0.8\left(62.4\frac{\text{lb}}{\text{ft}^3}\right)(10 \text{ ft})\left(\pi\frac{[4 \text{ ft}]^2}{4}\right) = 6673 \text{ lb}$$

The force P must exceed 6673 lbs to lift the container.

If an object is floating in a fluid, the analysis hardly changes: the buoyant force on the object must still balance the downward force of the object's weight and any other downward force on the object. The only difference is that, in this case, the volume of displaced fluid is less than the full volume of the object. The upward force on the bottom of such an object equals the weight of water that would be contained above the bottom surface, plus the force associated with any pressure of the overlying gas. A similar statement applies to any upper surface that is submerged. The effects of the gas pressure cancel out, and the net result is that the buoyant force on the object equals the weight of liquid that it displaces. Thus, a floating object displaces a volume of water with weight equal to the weight of the object. This result is known as Archimedes' second law.



Weight of liquid in the volumes outlined in bold equals the upward force (on left) and downward force (on right) on the semi-submerged object. The difference between these two volumes equals the submerged volume of the object. Correspondingly, the net upward force equals the weight of a volume of water equal to the submerged volume of the object. Conclusion: a floating object displaces a volume of liquid that has the same weight as the object does.

The *center of buoyancy* of an object, i.e., the location of the resultant buoyant force, can be defined analogously to a center of gravity or of pressure. The center of buoyancy is always at the center of gravity of the liquid that has been displaced. Thus, the net vertical force on a partially or fully submerged object is the vector sum of the (downward) gravitational force acting at the center of gravity and the (upward) buoyant force acting at the center of buoyancy.