PRESSURE FORCES ON PLANAR SURFACES

Objective: In this section, we determine the force that fluids at rest exert on flat (planar) surfaces and, correspondingly, the force that those surfaces must exert on the fluids to keep them static. Applications include enclosed and open vessels for containing water, such as fish tanks, standpipes, large water tanks, dams, etc.

<u>Key concept</u>: *Force balance: an object at rest experiences no net force.* The forces that act on any mass of fluid in a static system always include a gravitational force, which acts on the whole mass of fluid, and a pressure force, which acts on the boundaries of the fluid. Normally, when we think of the weight of an object, we consider the object in its entirety (*i.e.*, as a single item). In a more formal, mathematical sense, though, we could compute the gravitational force on the object by dividing its volume into differential elements, *dV*. The gravitational force on each such element would be $dF = \gamma dV$, and the total (resultant) gravitational force, we often wish to know the location at which that force acts, which we refer to as the *centroid of mass* or the *center of gravity*.

Pressure also acts over a distributed range, but it acts over an area, not a volume. Correspondingly, the product of the local pressure and differential area on which it acts yields the force (and its direction) at that locality, *i.e.*, dF = p dA. To determine the total pressure-induced force on an object, we can move around the whole surface of interest, determining the pressure and the direction in which it is exerted on each differential area, *i.e.*, the pressure-induced force is $\int p \, dA$. This calculation is different from that for gravity both because of the volume vs. area consideration, and because the pressure at different locations might act in different directions, whereas gravity always acts downward. As with gravitational forces, it is often convenient to compute the net pressure-induced force on the whole object and express those as a single, resultant force. Again, this resultant is slightly more complicated than for gravity, because in addition to its magnitude and location, we need to determine its direction. In essence, this portion of the course provides a few tools for determining the magnitude and direction of this resultant. Although at times the calculations can get a bit complex (just because of geometry, not conceptual complexity), it is important to keep in mind that we are always simply trying to carry out the integration of pressure over area (i.e., $\int p \, dA$) described above.

Overall Approach

Carrying out the integration of $\int p \, dA$ by characterizing the surface as a large number of δA -sized segments and adding up all the $p \, \delta A$ will always yield a correct result for F, but it is a tedious, brute force approach – similar to integrating a function by graphing the function and counting the squares underneath the curve. In fact, it is even more complicated than that, because each dF has a direction that we need to keep track of, as well as a magnitude.

If the surface is curved, we might have no alternative but to carry out such an integration. However, if the surface is planar, and especially if it is planar and has a common shape (rectangular, triangular, etc.), the integration can often be carried out

more easily using other techniques. These techniques work well for planar surfaces, because the forces on all parts of such surfaces are in the same direction (perpendicular to the surface), and the ratio between a coordinate along the surface and the increase in pressure is linear.

Before we analyze the system of interest, it is useful to review the idea of a center of gravity and, in general, the centroid of any distribution. The *x* value of the center of gravity represents the point where the *moment* of weight in the +*x* direction (integral of weight * *x*-distance from the reference point) exactly equals the moment of weight in the -x direction, so there is no tendency for the object to "tilt" in either the +*x* or -x direction at that point; the same is true for the *y* and *z* directions. More formally, we can say that at the centroid of mass, the *moment* of weight relative to any reference point equals the moment computed using the total weight and the location of the center of gravity, *i.e.*:

$$m_{tot} x_{cg} \equiv \int x dm;$$
 $m_{tot} y_{cg} \equiv \int y dm;$ $m_{tot} z_{cg} \equiv \int z dm$ (1)

$$x_{cg} = \frac{\int x dm}{m_{tot}}; \qquad \qquad y_{cg} = \frac{\int y dm}{m_{tot}}; \qquad \qquad z_{cg} = \frac{\int z dm}{m_{tot}}$$
(2)

In general, then, as shown by Equation (2), the centroid of mass along a given axis equals the moment of mass along that axis divided by the total mass. By analogy, the centroid of area equals the moment of area along an axis divided by the total area:

$$l_c = \frac{\int l \, dA}{A_{tot}} \tag{3}$$

where l_c is the location of the centroid of area along an axis l.

Now, consider a flat surface in a fluid, making an angle θ with the horizontal plane. Although the actual surface of interest might be at any depth in the fluid, we can (conceptually) extrapolate the plane in which the surface lies all the way to the top of the fluid, and define a coordinate *l* that originates at that surface and is directed downward along the surface of the submerged object. The object of interest therefore spans a length *L* between coordinates l_{top} and l_{bot} . At each value of *l* between l_{top} and l_{bot} , the object has a width *w*, which might vary from one value of *l* to the next. Define *dA* as the differential area corresponding to a differential distance *dl* along the flat plate and the width of the plate, *w*, at that value of *l*.

The force on the surface has aspects that might be of interest: magnitude, direction, and location of action. Since the surface is flat, the direction of the force is identical everywhere (perpendicular to the surface), so the unknowns are the magnitude and location of action. In the following analyses, three approaches are presented for calculating these unknowns.



The Magnitude of the Pressure-Based Force on the Surface

The Integration Approach

The total pressure-based force on the surface of interest can be written as follows:

$$F_{p} = \int p \, dA = \int \gamma h \, dA = \int \gamma \left(l \sin \theta \right) dA = \sin \theta \int \gamma l \, dA \tag{4a}$$

$$=\gamma\sin\theta \int l\,dA\tag{4b}$$

where the integration is carried out over all differential areas on the surface of the object of interest. The expressions in Equation (4a) apply even if the surface is in contact with multiple fluids with different γ values, and Equation (4b) applies in the common situation where all the fluid in contact with the surface has the same γ . Based on these results, one surfire approach for calculating F_p is to insert an expression for dA into Equation (4a) or (4b) and carry out the integration, either analytically (if possible) or numerically.

The Formula Approach

The integral $\int l dA$ in Equation (4) is the moment of the area along the *l* axis. According to Equation (3), this integral can be written as $A_{tot}l_c$. Making that substitution, we obtain:

$$F_{p} = \gamma l_{c} A_{tot} \sin \theta = \gamma h_{c} A_{tot}$$
(5)

Equation (5) indicates that F_p can be computed from knowledge of the specific gravity of the fluid, the total surface area of the object of interest, and either the depth of the centroid of the area (h_c) or the value of l at the centroid (l_c) and θ . Using this formulation to compute F_p is convenient, because l_c is tabulated for many common geometric shapes (see, for example, Munson Fig. 2.18, p.61). Therefore, if the surface of interest has one of those shapes, we can determine F_p using the tabulated information without carrying out the integration that is required if we use a form of Equation (4). (Even if the centroid is not tabulated for the shape of interest, we could use Equation (3) to compute the location of the centroid and then use Equation (4) to compute the force on the plate. That, however, would require integration of Equation (3) and would be equivalent to using the integration method.) Note that Equation (5) applies only to an area that is in contact with a constant γ . If a surface is in contact with multiple fluids, a separate value of F_p must be computed for each section of the surface; the overall force can then be computed as the sum of the forces on the various sections.

The "Pressure Prism" Approach

Yet another, equivalent way to analyze F_p is to imagine a prism with dimensions equal to the length and width of the surface of interest and, in the third coordinate direction (perpendicular to the plate), the pressure. The "volume" of that prism then corresponds to the integral $\int p \, dA$, *i.e.*, it equals F_p . If the "pressure prism" has a shape whose volume is easy to calculate, F_p can be computed using simple principles of geometry. In such a case, the pressure prism approach is even easier to use than the formula approach. In theory, the pressure prism approach could be used for many geometries of the flat plate. However, as a practical matter, it is useful only for plates that have a constant width, W, because only in that case does the prism have a shape (triangular or rectangular) whose volume is easily computed. Like the formula approach, the pressure prism approach can be used only for parts of the surface exposed to a single fluid (*i.e.*, single γ). If a surface is in contact with multiple fluids, the force must be computed separately for each area in contact with a given fluid.

The Center of Pressure

The preceding analysis yields a value for the total (resultant) force on a flat plate of arbitrary shape. We also know that the direction of that force is perpendicular to the plate. However, we do not yet know the location of the resultant. This location is known as the *center (centroid) of pressure*, although *center (centroid) of force* might be a better name. The idea is that the total force on a plate is simply the sum of the forces on the various parts of that plate. In the analysis of that force, the location of a particular force is irrelevant; a force near the top of the plate "counts" as much as one near the bottom, and both such forces must be resisted to hold the plate in place. On the other hand, the <u>location</u> of the resultant force reflects a balancing of the moments of force, *i.e.*, it is at the centroid of force. This centroid is at a location nearer to the larger forces and farther from the smaller ones.

Like the total force, the location of the centroid of force can be computed using the integration method, the formula method, or the pressure prism method. These approaches are described next.

The Integration Approach for Computing the Center of Pressure

The location of the center of pressure (or center of force) can be computed following exactly the same approach as shown above for identifying the centroid of area (Equation (3)); the only difference is that in this case we take the first moment of force rather than the first moment of area, *i.e.*:

$$l_{ctrof \ pressure} \equiv l_p = \frac{\int_{l_{op}}^{l_{bot}} l \ dF_p}{F_p}$$
(6)

Equation (6) is universally applicable for finding the center of pressure. However, just like the center of area, if the surface of interest is flat surface and has a common geometric shape, the formula approach or the pressure prism approach might be easier.

The "Formula" Approach for Computing the Center of Pressure

For a surface in contact with a single fluid, we can differentiate Equation (4b) to obtain: $dF_p = d(\gamma \sin \theta \int l \, dA) = \gamma l \sin \theta \, dA$. Also, by Equation (5), $F_p = \gamma l_c A_{tot} \sin \theta$. Substituting these expressions into Equation (6), we obtain:

$$l_{p} = \frac{\int l \, dF_{p}}{F_{p}} = \frac{\int l_{top}}{\not/l_{c}A_{tot}} \sin\theta \, dA = \frac{1}{l_{c}A_{tot}} \int l^{2} \, dA = \frac{1}{l_{c}A_{tot}} I_{o}$$
(7)

where I_0 is the moment of inertia of the surface about an axis where the plane of the plate intersects the top of the fluid (l=0). If we wish to express l_p with respect to the centroid of the object rather than the top of the fluid, we can use the parallel axis theorem, which tells us that:

$$I_{o} = A_{tot} l_{c}^{2} + I_{c}$$

$$\tag{8}$$

Combining Equations (7) and (8), we obtain:

$$l_{p} = \frac{1}{l_{c}A_{tot}} \left(A_{tot} l_{c}^{2} + I_{c} \right) = l_{c} + \frac{I_{c}}{l_{c}A_{tot}}$$
(9)

Like Equation (5), the convenient feature of Equation (9) is that I_c values for many commonly shaped objects are tabulated. Therefore, for such objects, we can evaluate F_p from Equation (5) and l_p from Equation (9), knowing only the geometry of the system

and the specific weight of the fluid. Equation (9) indicates that l_p is always lower (for our coordinate system, larger l) than l_c , which is consistent with what we would expect intuitively.

The "Pressure Prism" Approach for Computing the Center of Pressure

As noted above, for any submerged surface, a pressure prism can be drawn whose volume represents the total force on the surface. If the surface has a constant width, then the shape of the prism is simple, making this approach a convenient one for analyzing F_p . Correspondingly, since the pressure prism shows the distribution of forces in a pseudo-3-D manner, the centroid of the volume of the prism indicates the location of action of the force. Since the centroid of the volume of rectangular and triangular prisms is easily identified, this approach is easy to use for identifying l_p whenever it can be used for identifying F_p .

Example

In the following section, the three approaches presented above are all used to solve the same example problem.

Problem Statement. Calculate the minimum force F_G necessary to prevent the 12-ft x 12-ft gate shown in the following schematic from opening. The gate weighs 500 lb, and the tank contains water under air at a pressure of 10 psi. Draw a free body diagram of the gate.



The pressure force acts perpendicular to the gate and tends to push it open. The forces holding the gate closed include its weight, the force exerted by the hinge, and the force F_{gate} applied at point R, while the force tending to open the gate is due to the pressure on its underside. At equilibrium, the sum of the forces perpendicular to the gate and the moment around the hinge must both be zero.

The Integration Approach. We first solve the problem by computing the force of the fluid on the plate via formal integration of $p \, dA$. The force of water on the plate is perpendicular to the gate, directed upward and to the left at a 45° angle. We can define the *l* axis to run along the plate from the hinge to point R, so that *l* has possible values from 0 to 12 ft. The width of the gate is also 12 ft. Consider a differentially thin strip of the gate, with length *dl* and width equal to the full width of the plate (12 ft). The force of the water on such a strip is:

$$dF = p \, dA = p \left(12 \, \text{ft} \right) dl \tag{Ex1}$$

The pressure at the top of the water is 10 psi, so the pressure at any value of l equals that pressure plus the pressure contributed by the water depth:

$$p = \left(10 \text{ psi}\right) \left(144 \frac{\text{in}^2}{\text{ft}^2}\right) + \gamma h = 1440 \text{ psf} + \gamma l \sin 45^\circ$$
(Ex2)

where h is the vertical distance from the top of the water to l. Substituting from Equation Ex2 into Equation Ex1, we obtain:

$$dF = (1440 \text{ psf} + \gamma l \sin 45^\circ)(12 \text{ ft})dl$$
 (Ex3)

Inserting $\gamma = 62.4 \text{ lb/ft}^3$ and $\sin 45^\circ = 0.707$:

$$dF = \left(17,280\frac{\mathrm{lb}}{\mathrm{ft}} + 529.5\frac{\mathrm{lb}}{\mathrm{ft}^2}l\right)dl \tag{Ex4}$$

Integrating, we find the total force exerted by the water on the gate to be:

$$F = \int_{0}^{L} \left(17,280 \frac{\text{lb}}{\text{ft}} + 529.5 \frac{\text{lb}}{\text{ft}^{2}} l \right) dl$$
$$= \left[\left(17,280 \frac{\text{lb}}{\text{ft}} \right) l + \frac{1}{2} 529.5 \frac{\text{lb}}{\text{ft}^{2}} l^{2} \right]_{0 \text{ ft}}^{12 \text{ ft}}$$
$$= 245,482 \text{ lb}$$
(Ex5)

To find the location where the pressure acts, we use Equation (6):

$$l_p = \frac{\int_0^L l \, dF_p}{F_p} \tag{Ex6}$$

We can evaluate the integral on the right side of this equation by substituting for dF from Equation Ex4:

$$\int_{0}^{L} ldF_{p} = \int_{0}^{L} l(17280 + 529.5l) dl = \int_{0}^{L} (17280l + 529.5l^{2}) dl$$
$$= \left[(17280) \frac{l^{2}}{2} + (529.5) \frac{l^{3}}{3} \right]_{0}^{12 \text{ ft}}$$
$$= 17280 \frac{12^{2} - 0^{2}}{2} + 529.5 \frac{12^{3} - 0^{2}}{3} = 1.244e6 + 3.05e5 = 1.549e6$$

Inserting that result and the value of F_{tot} (from Equation Ex5) into Equation Ex6:

$$l_p = \frac{\int_{0}^{L} ldF}{F_{tot}} = \frac{1.549e6}{2.45e5} = 6.31 \text{ ft}$$
(Ex7)

A free-body diagram showing the forces on the gate is provided below. The force required to keep the gate closed can be determined by setting the moment around the hinge to zero. This calculation, considering the resultant pressure force, the weight of the gate (500 lb), and the force exerted at the bottom of the gate, is as follows:

$$0 = -(500 \text{ lb})(6 \text{ ft x } \sin 45^\circ) - (F_{gate})(12 \text{ ft}) + (245482 \text{ lb})(6.31 \text{ ft})$$

$$F_{gate} = 128906 \text{ lb}$$



The "Formula" Approach. In the formula approach, we use Equation (5) to compute the force on the gate exerted by the water, F_{water} . To use that equation, we need to determine

the water depth at the centroid of the area. The gate is rectangular, so its centroid of area is in the middle of the rectangle. Thus, $l_c = 0.5$ (12 ft), or 6 ft. The depth at this value of lis represented as h_c and equals $l_c \sin 45^\circ$. Therefore:

$$F_{water} = \gamma h_c A$$

= (62.4 lb/ft³)(6 ft (sin 45°))(12 ft x 12 ft) = 38122 lb

This force acts at the centroid of pressure, which is related to the centroid of area by the parallel axis theorem (Equation (8)):

$$I_{o} = A_{tot} l_c^2 + I_c$$

To use the above equation, we need to evaluate I_c , which is given for a rectangle as $bh^3/12$, where b and h are the base and height of the rectangle. Plugging values into this equation, we find:

$$l_p = l_c + \frac{I_c}{l_c A} = l_c + \frac{bh^3 / 12}{l_c A} = 6.0 \text{ ft} + \frac{(12 \text{ ft})(12 \text{ ft})^3 / 12}{(6 \text{ ft})(12 \text{ ft x } 12 \text{ ft})} = 8.0 \text{ ft}$$

We now know both the magnitude and location of the resultant force attributable to the water. We need to add to this the resultant force from the overlying air, F_{air} . In the current scenario, this force is the product of the gas pressure and the area of the plate:

$$F_{air} = p_{air}A_{tot} = (1440 \text{ lb/ft}^2)(12 \text{ ft x } 12 \text{ ft}) = 207360 \text{ lb}$$

Because this pressure acts uniformly on all parts of the plate, the resultant is in the center of the plate, at l = 6 ft.

The two force vectors, F_{water} and F_{air} , can now be added.

Resultant:
$$F_{tot} = F_1 + F_2 = (207360 + 38122)$$
lb = 245482 lb

$$l_p = \frac{F_1 l_1 + F_2 l_2}{F_1 + F_2} = \frac{(207360 \text{ lb})(6 \text{ ft}) + (38122 \text{ lb})(8 \text{ ft})}{207360 \text{ lb} + 38122 \text{ lb}} = 6.31 \text{ ft}$$

These values are the same as we obtained using the integration method, as they must be. The free body diagram is, of course, also the same, leading to the same result for the required magnitude of force F_{gate} ; the diagram and subsequent calculations are not repeated here.

The "Pressure Prism" Approach. In the pressure prism approach, the contributions of the overlying air and the water to the pressure on the gate are represented by separate prisms. Each prism has two dimensions that correspond to the dimensions of the flat plate

(12 ft x 12 ft). The third dimension is shown as an arrow whose length is proportional the pressure at that location, as shown below.



The "volume" of each prism represents the force of the corresponding fluid on the gate, and the location at which that component of the force acts is the center of volume of the prism. For the system of interest, the magnitudes of the forces are:

$$F_{air} = LWp_{air} = (12 \text{ ft})(12 \text{ ft})(1440 \text{ psf}) = 207,360 \text{ lb}$$

$$F_{water} = \text{Area of triangle *thickness of prism} = \left(\frac{1}{2}Lp\right)W = \frac{1}{2}L(L\gamma \sin 45^{\circ})W$$

$$= \frac{1}{2}(12 \text{ ft})[(12 \text{ ft})(62.4 \text{ lb/ft}^{3})\sin 45^{\circ}](12 \text{ ft})$$

$$= 38,122 \text{ lb}$$

The resultant force associated with the water acts at a value of l corresponding to the center of area of the triangle. The center of area of a triangular is at one-third of the distance up from its base. In the system of interest, l is measured down from the apex (where l=0), so the center of area is two-thirds of the way down from that point, i.e., it is at l = (2/3)L. Note that, although this point is l_c for the geometric shape, it corresponds to l_p for the force represented by the pressure prism.

The air pressure acts uniformly on the whole gate, so the location of action of the resultant from that force is in the center of the gate, at l = (1/2)L. Thus:

$$l_{p,water} = 2/3*12 \text{ ft} = 8 \text{ ft}$$
 $l_{p,air} = 1/2*12 \text{ ft} = 6 \text{ ft}$

Resultant: $F_{tot} = F_1 + F_2 = (207360 + 38122)$ lb = 245482 lb

$$l_p = \frac{(207360 \text{ lb} * 6 \text{ ft}) + (38122 \text{ lb} * 8 \text{ ft})}{(207360 + 38122) \text{ lb}} = 6.31 \text{ ft}$$

Thus, once again, we obtain the same result as when we used the other analysis methods.