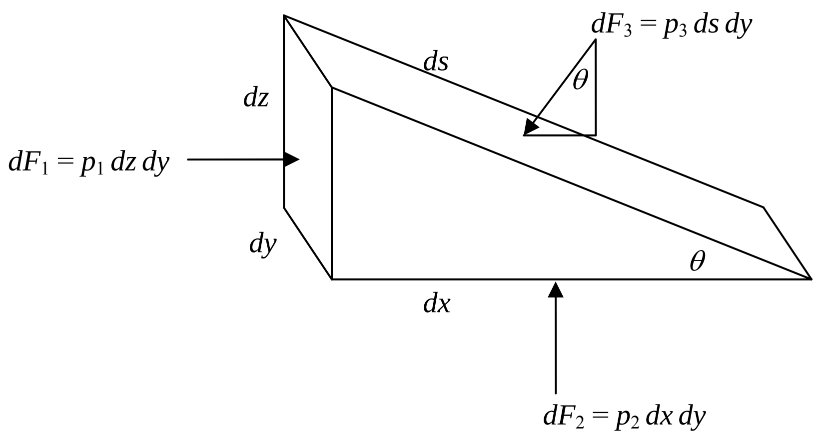
PRESSURE VARIATION IN FLUIDS AT REST; PIEZOMETERS

Direction of the pressure-induced forces in a fluid at rest

Because fluids deform continuously when exposed to any shear stress at all, the only force that can be applied to a surface in a fluid at rest must be normal to that surface. Thus, as a (real or imaginary) surface is moved in any direction in a fluid, the force on it shifts to always remain perpendicular to every part of the surface.

Variation of pressure with direction at a point in a fluid at rest

Consider a force balance on an arbitrary, but differential volume of a fluid at rest. To keep the analysis mathematically simple, we choose the shape to be a wedge with one vertical and two horizontal sides, although in principle any shape would be acceptable. A diagram defining key terms and parameters for such a wedge is shown below.



A balance on the forces in the x direction yields:

$$\sum F_x = p_1 dydz - p_3\left(ds\,dy\right)\sin\theta$$

where p_1 and p_3 are treated as constant along the surface on which they act because the dimensions of the wedge are differential. In the preceding equation, the $\sin\theta$ is included to account for the fact that the force is directed into the ds x dy surface, but only a portion of that force is in the x direction. By geometry, however, we see that $ds \sin\theta$ is dy, so we can write:

$$0 \quad = p_1\,\cancel{dydz} - p_3\,\cancel{dzdy}$$

$$p_1 = p_3$$

A similar balance on the forces in the z direction must take into account the weight of the wedge of fluid, in addition to the pressure-induced forces, yielding:

$$\sum F_z = p_2 dxdy - \rho g dV - p_3 \left(ds\, dy\right)\cos\theta$$

$$0 \quad = p_2 \cancel{dxdy} - \rho g \left(\frac{1}{2}\cancel{dxdy}\, dz\right) - p_3 \cancel{dxdy}$$

$$p_2 - p_3 = \frac{1}{2}\rho g dz$$

If dz shrinks to zero, so that wedge becomes a point, $p_2 = p_3$. Thus, at a point, $p_1 = p_2 = p_3$, *i.e.*, pressure is identical in all directions. This derivation is for a fluid at rest, but a similar result is obtained in an ideal flowing fluid (*i.e.*, a fluid with zero viscosity, known as an *inviscid* fluid).

Variations of pressure with location in a fluid at rest

Consider a small box (δx x δy x δz) of fluid at rest, with pressure p in the middle. As in the preceding analysis, because the fluid is static, the sum of forces in each direction is zero. Since p is the pressure in the middle of the box, the pressure on each face equals p plus or minus the product of the gradient in pressure plus the distance from the center to that face. Thus, for example, in the x direction:

$$\sum F_x = 0 = \left(p - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\left(\delta y\ \delta z\right) - \left(p + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\left(\delta y\ \delta z\right)$$

$$0 = \left(p - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\cancel{\delta y \delta z} - \left(p + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\cancel{\delta y \delta z}$$

$$0 = -2\left(\frac{\partial p}{\partial x}\frac{\delta x}{2}\right) = -\frac{\partial p}{\partial x}\delta x$$

$$\frac{\partial p}{\partial x} = 0$$

An essentially identical derivation for $\sum F_y$ indicates that $\frac{\partial p}{\partial y} = 0$. On the other hand, for the vertical forces:

$$\sum F_z = 0 = \left(p - \frac{\partial p}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y - \left(p + \frac{\partial p}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y - \gamma \delta x \delta y \delta z$$

$$0 = -\frac{\partial p}{\partial z}\cancel{\delta x \delta y \delta z} - \gamma\, \cancel{\delta x \delta y \delta z}$$

$$\frac{\partial p}{\partial z} = -\gamma$$

To sum up:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$ **pressure in a fluid at rest does not vary in a horizontal plane**

$$\frac{\partial p}{\partial z} = -\gamma = -\rho g$$ **pressure in a fluid at rest increases in direct proportion to depth and the specific weight (or, equivalently, the density) of the fluid**

Since we now see that *p* depends on location only in the *z* direction, we can convert the partial differential into an ordinary differential, yielding:

$$\int dp = -\int \gamma dz$$

Integration of this equation between any two elevations for a fluid of constant density yields: $\Delta p = -\gamma \Delta z = -\rho g \Delta z$. The minus signs in these equations reflect the fact that pressure increases with depth but, by convention, *z* increases in the upward direction. It is common to eliminate the minus signs by defining *h* as the depth (increasing downward) from the top of the fluid, in which case we can write $\Delta p = \gamma h = \rho g h$, where Δp is understood to be the pressure increase from the top of the fluid. If the top of the fluid is at atmospheric pressure, then its gage pressure is zero, so we can equate γh with the gage pressure at depth *h*.

These results justify expressing pressure differences in terms of heights (heads) of fluid, if the identity (specifically, the density) of the fluid is specified. That is, one can state that the pressure at a point is "5 m of water" or "6 inches of Hg," meaning that the pressure is the same as would be found at that depth of the specified fluid. The following are all (approximately) equivalent ways of reporting normal atmospheric pressure: 1 atm ≈ 10 m of H_2O ≈ 34 ft of H_2O ≈ 100 kPa ≈ 14.7 psi ≈ 2117 psf.

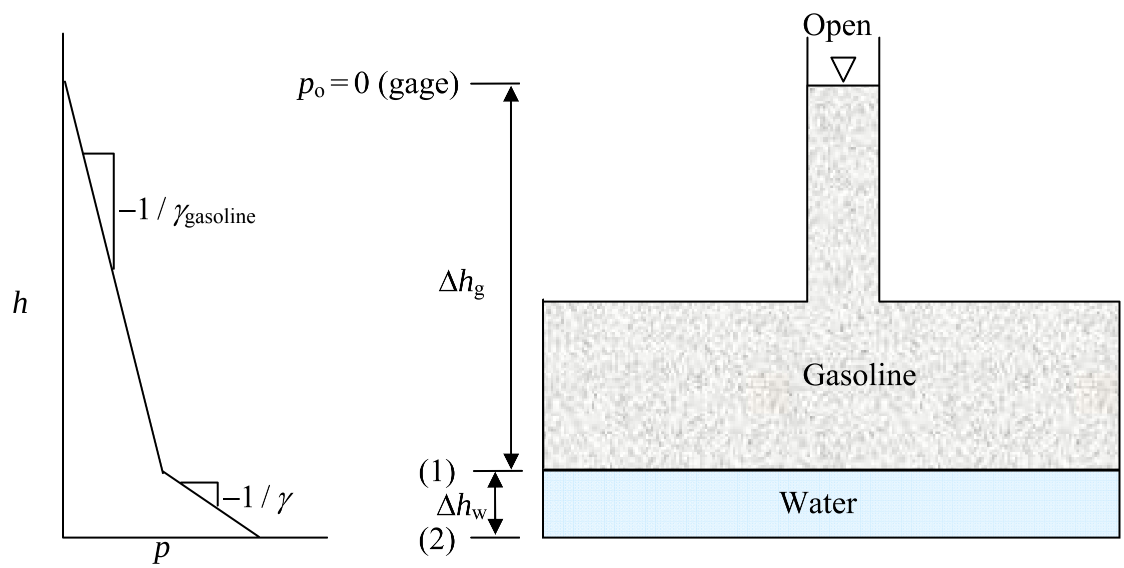
A corollary of the relationship derived above is that, for any arbitrary points 1 and 2 in a static fluid of uniform specific weight (i.e., density):

$$\gamma\left(z_2 - z_1\right) = p_1 - p_2$$

$$\gamma z_2 + p_2 = \gamma z_1 + p_1$$

Thus, **the sum $\gamma z + p$ is identical everywhere in a static fluid of uniform density.** Note that this statement applies even if we have to take a circuitous route to get from point 1 to point 2. Note also that these equalities explain why we can treat the pressure as being uniform in a gas phase of "human dimensions": γ_{gas} is so small that $\gamma(z_2 - z_1)$ is negligible for most Δz of interest. However, if we are interested in very large Δz, then we do have to take that term into account.

We can use the above relationship to determine Δp between any two points in a uniform, continuous fluid at rest. If we reach an interface, where one fluid terminates and a different fluid with a different density begins, the relationship applies to each fluid individually.



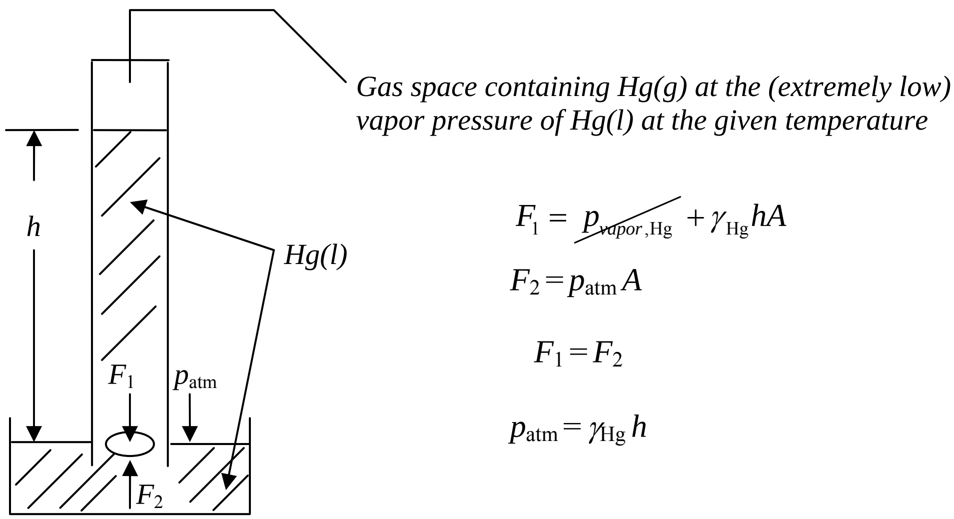
(Note: Difference in $\gamma_{gasoline}$ and γ_w is greatly exaggerated in above schematic.)

When this relationship is combined with the fact that the pressure must be the same on both fluids right at the interface (because the pressure is identical in all directions at a single point), we can determine Δp between two points connected by a sequence of different fluids. That is, as we move from one fluid to the next, the relationship between z and p changes (because γ changes), but we can nevertheless apply the principle to each layer individually and then sum the Δp's to determine the overall Δp. This idea is central to the design of at least one set of tools to measure pressure differences, as is shown next.

<u>Piezometers</u>

Piezometers are instruments in which the difference in pressure between two locations is measured based on the difference in elevation of a fluid that is exposed to those two pressures. The elevation difference is directly proportional to Δp, as we have seen. If the difference in pressures is large, it is convenient to use a dense liquid as the *gage fluid*, so that the elevation of that fluid is not too large. Also, we might want to take precautions to prevent the gage fluid from entering the reservoirs having the two pressures of interest, even if the pressure difference became rather small. These considerations lead to the design of a typical U-tube manometer.

A classic example of use of a piezometer to measure pressure is the mercury barometer. In this instrument, a tube is filled with mercury (Hg), which is a very dense liquid ($s.g. = 13.56$). This tube is then inverted into a pool of mercury. The gravitational force pulls the liquid column down, generating a gas space above it. However, the only material available to fill this space is Hg gas. Hg(l) has a very low vapor pressure at normal temperatures ($1.7\text{x}10^{-4}$ kPa, $2.5\text{x}10^{-5}$ psi), so the pressure at the top of the liquid column is essentially zero. Because the column is at rest, the net force on any portion of it is zero. If we carry out a force analysis on a differential element of the liquid at the bottom of the column, at the same elevation as the top of the pool of Hg, we find that the force down on that element is $\gamma_{\text{Hg}}h$ (where h is the height of the column above the pool), and the force up is the atmospheric pressure. Therefore, since we know γ_{Hg} and h, we can determine p_{atm}.



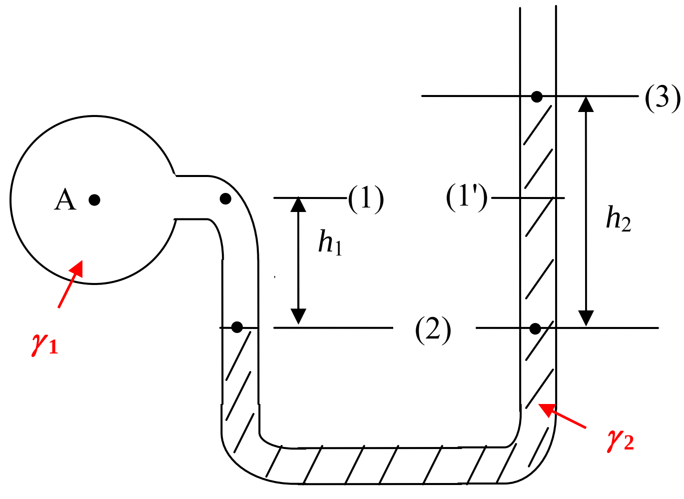
$$F_1 = \cancel{p_{vapor,\text{Hg}}} + \gamma_{\text{Hg}} hA$$

$$F_2 = p_{\text{atm}} A$$

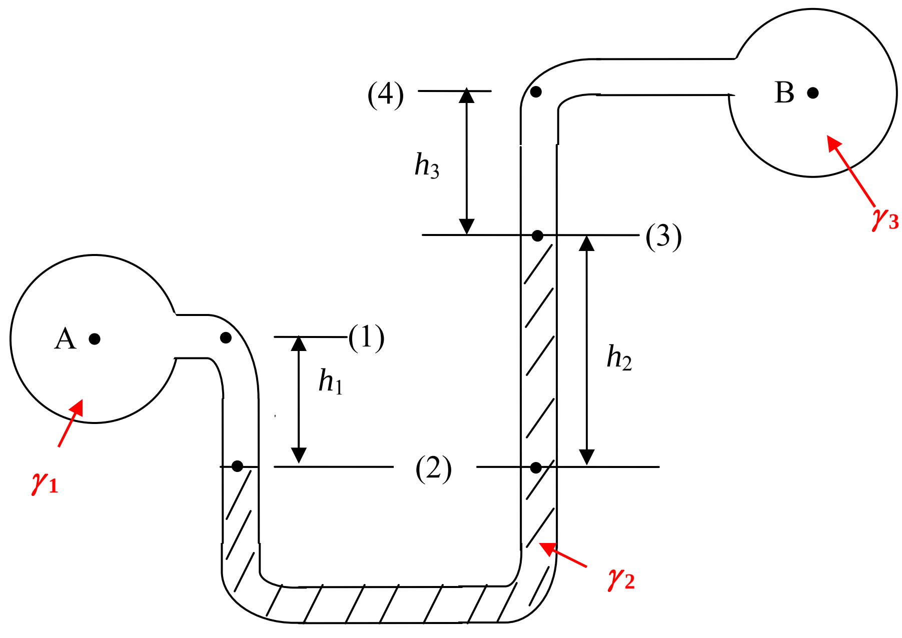
$$F_1 = F_2$$

$$p_{\text{atm}} = \gamma_{\text{Hg}} h$$

Manometers can also be used to determine the pressure in a closed container, the pressure difference between two containers, or the pressure difference between two locations in a pipe, as follows:

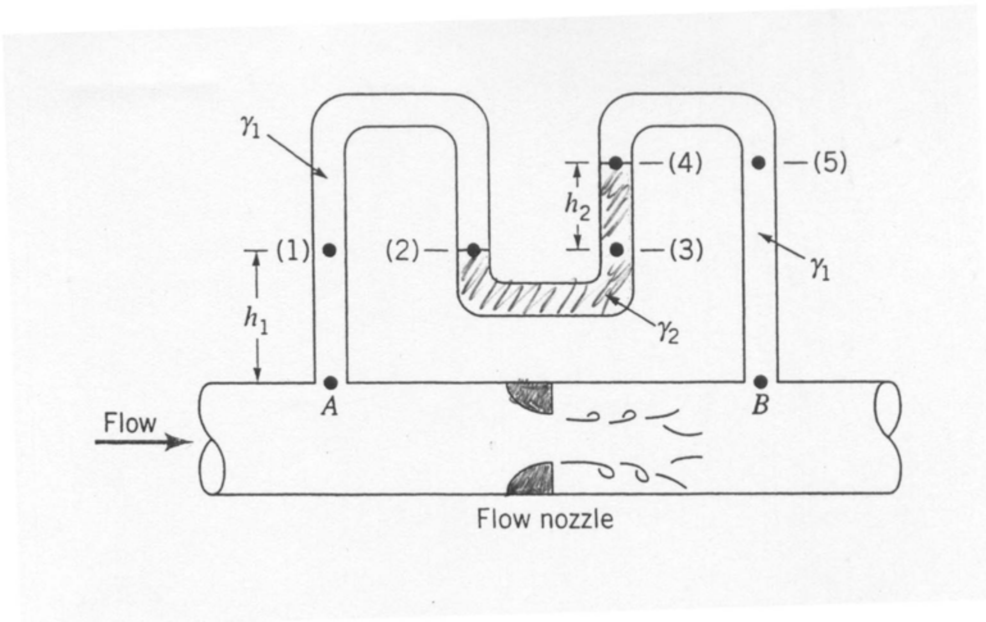


$$p_{A,gage} = 0\left(\text{at point } 3\right) + h_2\gamma_2 - h_1\gamma_1$$



$$p_B = p_A + h_1\gamma_1 - h_2\gamma_2 - h_3\gamma_3$$

$$p_B - p_A = h_1\gamma_1 - h_2\gamma_2 - h_3\gamma_3$$



$$p_B = p_A - \cancel{h_1}\gamma_1 - h_2\gamma_2 + \left(\cancel{h_1} + h_2\right)\gamma_1 = p_A + h_2\left(\gamma_1 - \gamma_2\right)$$

(Note that the Δp's between points (1) and (2), between points (2) and (3), and between points (4) and (5) are zero, so they need not be considered in the analysis.)

$$p_B - p_A = h_2\left(\gamma_1 - \gamma_2\right)$$