TOPIC 11: DIMENSIONAL ANALYSIS AND HYDRAULIC MODELING

We now consider a topic that has a different “feel” to it than most other topics we cover in this course or other engineering courses. Normally, we seek explicit solutions to problems so that we can determine the value of desired quantities unambiguously. By contrast, in this section, we acknowledge that many fluids problems are unsolvable in their entirety, and we seek to gain information and insight from semi-quantitative analogies between the system of interest and other systems that we are familiar with or can study more conveniently. In many cases, the objective is to identify key parameters and ranges (sometimes only orders of magnitude) over which they imply one thing or another about the system. The applications of some of these concepts will become clearer as we proceed in the course. For now, we just have to understand that the objectives and outcomes of this section are somewhat more fuzzy than we are used to.

The behavior of a fluid can depend on a plethora of variables. We might wish to explore the individual effects of several of these variables on the system, but changing each of them in isolation (for each of several values of the others) would require an impractically large number of experiments. In many situations, not all of these variables are critical, and in other cases, the effects of different variables are closely tied to one another. In addition, the scale of the phenomena of interest are often very large or very small; in the former case, it is difficult or impossible to test the effects of changing certain parameters at full-scale, and in the latter, it is often impossible to probe the phenomena by available instrumentation, without disturbing the flow patterns that we are trying to study.

To address the practical problem of having many independent variables that can affect system behavior, it is useful to identify composite parameters that establish the relative effects that different parameters have. For example, if we believed that the behavior of the fluid passing through a pipe expansion depended in a crucial way on the diameters of the smaller and larger sections, we might want to study how the system behaves at 5-10 values of $d_1$ (e.g., 1, 2, 3, 4, 5, 6, and 7 cm) for each of 5-10 values of $d_2$ (e.g., 2, 4, 6, 8, 10, 12, and 14 cm), corresponding to, 25-100 experiments (49 for the example values). If, however, we can establish that the system behavior depends on the ratio $d_1/d_2$, independent of the absolute values of the diameters, then we can get the same information from many fewer experiments. For example, in that case, if we did an experiment at $d_1=3$ cm, $d_2=6$ cm, there would be no need to do experiments at (1 cm, 2 cm), (2 cm, 4 cm) . . . (7 cm, 14 cm), and so on. Note that this approach would not tell us the details of the relationship between the behavior of interest and the diameters (i.e., it would not allow us to predict the behavior of a 10-to-20 cm expansion based on the results of a 10-to-15 cm expansion), but it would nevertheless greatly reduce the number of experimental observations necessary to characterize the system of interest. The question is, how can we determine whether the system behavior actually does depend on $d_1/d_2$, short of doing all those experiments?

The same information (about how to identify useful composite parameters) can also help us with the second problem identified above – knowing how to construct test (“model”) systems at convenient scales that will accurately mimic the behavior of full-scale (“prototype”) systems that are the real target of investigation. For example, say we wanted to model a prototype system that had a 2-m- to 3-m-diameter pipe expansion. If we were confident that the $d_1/d_2$ ratio was critical, we might construct a model system in which the expansion is from 2 to 3 cm. However, that leaves us facing the question of what flow rate to use to cause the model system to behave like
the prototype – should we scale the flow so that it is only 1% of that in the prototype, just like the
diameters ($Q_1/Q_2 = d_1/d_2$)?; should we scale it so that it is 0.01% of that in the prototype, so that
the velocities are the same in the two systems ($Q_1/Q_2 = (d_1/d_2)^2$, so that $v_1 = v_2$)?; or should we use
some other scaling ratio? At this point, we cannot answer that question, but we can see that
developing some general rules for selecting the parameter ratios in model and prototype systems
would be valuable. Identifying critical combinations of parameters that control system behavior
will help us develop such rules.

**Dimensional Analysis: The Buckingham Pi Method**

One approach that can be used to identify useful parameter combinations is based on the
requirement for dimensional consistency in the equation governing a process of interest; the
approach is therefore known as **dimensional analysis**. Although the requirement for dimensional
consistency can be applied to equations that have dimensions in each term, it is invariably
applied in ways that convert all the terms to dimensionless groups. Dimensional analysis can be a
powerful tool in the following ways:

- It organizes data, making clear relationships that were not necessarily clear before
- It eliminates units from the results, so that the results are applicable in any system of
  units
- It facilitates modeling of systems where the underlying physical phenomena are not
  fully understood, in which case analytical modeling is very difficult or impossible
- It facilitates calculation of scale factors applicable to physical models

The specific technique of dimensional analysis that has been used most often in fluid
dynamics is known as the **Buckingham pi** method. This approach starts with assumptions about
what independent variables and dimensional constants are likely to have a significant effect on a
dependent variable of interest. An algorithm for carrying out dimensional analysis is presented
next, in the context of pipe flow as an example system.

Say that we are interested in finding the pressure loss per unit length, $\Delta p_l$, accompanying
flow of fluid through a pipe. We expect this **dependent variable** to depend in some way on the
geometry of the system, the operational conditions, and the properties of the fluid. Specifically,
we might expect $\Delta p_l$ to depend on four **independent variables**: pipe diameter ($d$), fluid velocity
($V$), and fluid density ($\rho$) and viscosity ($\mu$). Temperature might also play a role, but its influence
would be exerted primarily through $\mu$ and $\rho$, so its effect is probably already taken into account
by listing those variables as important ones. We might consider flow rate to be an important
variable, but its effect is already built into the existing variables, via $d$ and $V$. Note that we have
incorporated at least one variable associated with each of three “domains” of the problem: $d$
relates to geometry, $V$ relates to the operational characteristics of the system, and both $\rho$ and $\mu$
relate to the fluid properties. Other variables might be suggested, but for now, let’s assume that
we have captured the most important ones, so we can write:

$$\Delta p_l = \phi(d, V, \rho, \mu)$$

(1)
where $\phi$ is some as yet undefined function.¹

Since we are ultimately going to draw some conclusions about the function $\phi$ based on the dimensions of the terms that appear in it, we also need to identify and list any constants that have dimensions (such as $g$) that we think might appear in the relationship. In our example system, no such constants are likely candidates, so we will proceed on the assumption that the variables listed in Equation 1 are the only ones needed for the dimensional analysis. (Later, we will consider examples in which $g$ or other constants must be included.)

The function $\phi$ might have any form whatsoever – it could have summations, products, exponentials, trigonometric functions, etc. To make the point in an exaggerated way, we might imagine that the correct expression was something very bizarre and complicated, like the following:

$$
\phi = f_1\{d,v,\rho,\mu\} + f_2\{d,v,\rho,\mu\} \cdot \sin\left(\hat{j}_1\{d,v,\rho,\mu\}\right)^{\left(f_2\{d,v,\rho,\mu\}\right)}
+ f_3\{d,v,\rho,\mu\} \cdot \exp\left(\hat{j}_3\{d,v,\rho,\mu\}\right)
$$

(2)

However, no matter how complex the function $\phi$ is, each additive term in it (i.e., each additive term on the right-hand side of Equation 2) must have dimensions that correspond to pressure loss per unit length (force per length cubed) for the equation to be dimensionally consistent. Furthermore, since all exponential, logarithmic, trigonometric, and other such functions are always dimensionless,² those functions do not affect the dimensions of terms in which they appear. We therefore conclude that the dimensions of $f_1$, $f_2$, and $f_3$ in Equation 2 must correspond to pressure loss per unit length.

Generalizing the above discussion, the requirement for dimensional consistency in any relationship limits the ways in which the independent variables and dimensional constants can be combined to form the terms that appear in the relationship. Dimensional analysis represents an approach for taking advantage of those constraints to gain insights into the relationship.

By separating the algebraic from the non-algebraic terms, any relationship can be written in a way that is analogous to Equation 2. That is, it can be written such that each term on the right-hand side is the product of one function that determines the dimensions of the term ($f_1$, $f_2$, or $f_3$) and a second function that is dimensionless. When the function is written in this way, $f_1$, $f_2$, and $f_3$ are each of the form: $d^a v^b \rho^c \mu^d$. If we divide all the terms on both sides of the equation

¹ The squiggly brackets { } are used here to emphasize that $\phi$ is a function, and the parameters inside the brackets are the arguments of the function. This approach avoids confusion with the designation $\phi()$, which might look like a product, especially when the function has only one argument.

² The functions in Equation 2 that are dimensionless are designated by $\hat{j}$. 
by $f_1$, $f_2$, or $f_3$, then all the terms become dimensionless. In that case, the functional relationship in Equation 1 becomes one between the non-dimensionalized dependent variable (a “dimensionless pressure”) on the left, and a group of as yet unspecified, but dimensionless combinations of the dependent variables on the right, e.g.:

$$\frac{\Delta p_i}{f_1} = \phi\left\{\frac{f_2}{f_1}, \frac{f_3}{f_1}, \ldots\right\}$$  \hspace{1cm} (3)

where all the fractions are dimensionless.

The fact that each term in Equation 3 is dimensionless is just another way of saying that each term in Equation 2 had to have dimensions of pressure per unit length. As in Equation 2, the requirement for dimensional consistency in Equation 3 places some constraints on the ways that $d$, $V$, $\rho$, and $\mu$ can be combined to form the various terms in that equation, and thereby provides insights into the relationship. One benefit of writing the expression in terms of dimensionless parameters is that it facilitates the development of a standard approach that can be generalized and applied to any relationship of interest. Each dimensionless term is traditionally represented as a “$\Pi$” term ($\Pi_1$ for the dependent variable, and $\Pi_2$, $\Pi_3$, etc., for the independent variables). That is, the ultimate relationship that is derived is represented as:

$$\Pi_{\text{dependent variable}} \equiv \Pi_1 = \phi\{\Pi_2, \Pi_3, \Pi_4, \ldots\}$$  \hspace{1cm} (4)

The steps in this approach can be summarized as follows:

1. Identify all the variables and dimensional constants expected to be important in the relationship of interest. In the pipe flow example, we have five such variables; in others, the number might be larger or smaller. Define the number of variables plus dimensional constants as $k$.

2. Write out the dimensions of each of the variables and constants identified in step 1. Normally, we think of four fundamental dimensions: Length (L), Mass (M), and time (T). It is acceptable to substitute any of the primary dimensions with a different one that is a combination of the others, as long as we end up with an independent set of dimensions. For problems of interest to us, the choice is often made to delete Mass from the list and replace it with Force (F). We will adopt that choice for the current example. However, note that the result of the analysis does not depend on this decision. Expressing the variables in our example pipe system in terms of L, F, and T, we find:

$$\Delta p_i [=] F^1L^{-3}; \hspace{1cm} d [=] L^1; \hspace{1cm} \mu [=] F^1L^{-2}T^1;$$

$$v [=] L^1T^{-1}; \hspace{1cm} \rho [=] F^1L^{-4}T^2$$

where the equal sign in brackets [=] means “has dimensions of.”
3. Determine the number of dimensionless terms (\(\Pi_i\)) that are needed to describe the functional relationship. To do this formally, we need to determine the maximum number of the variables listed in step 2 that can be used to form a dimensionally complete and independent set, in the following sense: when these variables are combined in the form \((\text{var}_1)^a (\text{var}_2)^b (\text{var}_3)^c \ldots\), it is impossible to cause the given product to be dimensionless, no matter what values are assigned to \(a\), \(b\), \(c\),….. By identifying the maximum number of variables that meets the criterion of being independent, we assure that every dimension that is in any of the variables identified in step 2 appears in at least one of the variables in the set. We designate the number of variables in such a set \(r\). Fortunately, despite the complexity of the criterion stated above, there is usually a very easy way to determine \(r\): it is almost invariably equal to the number of dimensions, \(n_D\), that appear in the variables identified in step 2. In rare cases, \(r < n_D\), but \(r\) can never be \(> n_D\).

In the example pipe system, \(n_D = 3\) (F, L, and T), so \(r\) is likely to be 3. To show that we can in fact identify three of the variables that form a dimensionally independent set, consider the variables \(d\), \(\rho\), and \(\mu\). To eliminate the dimension F from any multiplicative combination of \(\mu\), \(\rho\), and \(d\), the exponent on \(\mu\) would have to be opposite that on \(\rho\) (since F does not appear in the dimensions of \(d\)). However, if \(\mu\) and \(\rho\) had opposite exponents, then the product of those two terms would have the dimension T to some finite exponent, and this dimension could not be eliminated by combining the result with \(d\) raised to any exponent, since \(d\) does not have T as a dimension. Thus, there is no way in which we could combine those three variables to generate a dimensionless product, and we conclude that \(r = 3\).^3

The number of dimensionless ("\(\Pi\)"") terms that will appear in Equation 4 equals \(k - r\). In the example system, \(k\) is 5 and \(r\) is 3, so the relationship of interest will have only two dimensionless terms – the dimensionless pressure on the left, and one dimensionless parameter (as yet undetermined) on the right.

4. Select a group of \(r\) variables that form a dimensionally independent set and that include all the dimensions expected to appear in the governing equation. These variables are referred to as repeating variables (hence the designation \(r\)). Although this is not required, it is convenient to choose only independent variables as repeating variables. For the example system, \(r = 3\). If we follow the guideline that the dependent variable (\(\Delta p\)) should not be chosen as a repeating variable, we could choose any three of the four variables \(d\), \(v\), \(\rho\), and \(\mu\) as repeating variables. Confirm for yourself that any of

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^3 If we could find a way to combine \(\mu\), \(\rho\), and \(d\) to form a dimensionless group, we would have to try other combinations of three variables to see if we could find a combination of those variables to form a dimensionless group. Only if every possible combination of three variables could be combined to form a dimensionless group, we would conclude that \(r\) was less than 3. This will very rarely be the case; it occurs only when two dimensions have the same relationship to one another in all the variables under consideration (e.g., it would occur if, every time \(F\) appeared as a dimension in one of the variables, \(T^{-1}\) also appeared.
these combinations meet the criteria specified above. For now, we will choose \( d, V, \) and \( \rho \) as our repeating variables, leaving \( \mu \) and \( \Delta p_i \) as non-repeating variables.

5. Each of the \( \Pi_i \) terms can now be formulated as a combination of all the repeating variables and one of the non-repeating variables, in the following way:

\[
\Pi_i = (\text{non-rptg. var})^i (\text{rptg. var}_1)^{a_1} (\text{rptg. var}_2)^{a_2} (\text{rptg. var}_i)^{a_i} \ldots
\]  

or, in our example system:

\[
\Pi_1 = (\Delta p_i)^i (d)^{a_1} (V)^{b_1} (\rho)^{c_1}
\]

\[
\Pi_2 = (\mu)^i (d)^{a_2} (V)^{b_2} (\rho)^{c_2}
\]

Because we have specified that each \( \Pi \) term is dimensionless, we can write the dimensions of each such term as \( F_0L_0T_0 \). Then, equating these dimensions to those on the right-hand side of each of the preceding two equations, we obtain:

\[
F_0^0L_0T_0 = \left[ (F_1L^{-3})^i (L)^{a_1} (L^1T^{-1})^{b_1} (F_1L^{-4}T^2)^{c_1} \right]
\]

\[
F_0^0L_0T_0 = \left[ (F_1L^{-2}T) ^i (L)^{a_2} (L^1T^{-1})^{b_2} (F_1L^{-4}T^2)^{c_2} \right]
\]

Each of the above expressions yields three simultaneous equations that can be solved for \( a_i, b_i, \) and \( c_i, \) as follows. For \( \Pi_1: \)

\[
\begin{align*}
\text{F:} & \quad 0 = 1 + c_1 \\
\text{L:} & \quad 0 = -3 + a_i + b_i - 4c_i \quad \begin{cases} a_i = 1; & b_i = -2; & c_i = -1 \\
\text{T:} & \quad 0 = -b_i + 2c_i
\end{cases}
\end{align*}
\]

and for \( \Pi_2: \)

\[
\begin{align*}
\text{F:} & \quad 0 = 1 + c_2 \\
\text{L:} & \quad 0 = -2 + a_2 + b_2 - 4c_2 \quad \begin{cases} a_2 = -1; & b_2 = -1; & c_2 = -1 \\
\text{T:} & \quad 0 = 1 - b_2 + 2c_2
\end{cases}
\end{align*}
\]

We therefore can write:

\[
\Pi_1 = (\Delta p_i)(d)^i (V)^{-2} (\rho)^{-1} = \frac{\Delta p_i d}{V^2 \rho}
\]

\[
\Pi_2 = (\mu)^i (d)^{-1} (V)^{-1} (\rho)^{-1} = \frac{\mu}{dV \rho}
\]
or, finally:

\[ \Pi_1 = \phi(\Pi_2) \]

\[ \frac{\Delta p d}{V^2 \rho} = \phi \left\{ \frac{\mu}{d V \rho} \right\} \]  

(6)

The result shown in Equation 6 is useful in a number of ways. Among other things, it establishes that the combination of variables on the left will have the same value whenever the combination on the right has a given value. This allows us to extrapolate from one geometry, fluid, and/or operational condition to another. For example, the result tells us that doubling the fluid velocity will have exactly the same effect on the dimensionless pressure drop as doubling the pipe diameter, if the other variables are held constant. From that knowledge, if we determine the effect of doubling the velocity, we can predict the effect of doubling the diameter without carrying out the experiment. Thus, the result of our dimensional analysis can dramatically reduce the number of experiments we need to carry out to fully characterize a system’s performance.

Note that, if we multiplied through the equations used to determine \( a \), \( b \), and \( c \) by \(-1\), the values of these coefficients would all be opposite the values we identified above, but the result would still satisfy all the equations. This procedure would cause the resulting \( \Pi \) group to be inverted. Thus, we can replace any of the \( \Pi \) groups by its inverse and still maintain dimensional consistency.

The dimensional analysis allows us to understand how to scale certain results from one system to another. For instance, say that we are interested in predicting the head loss in a 3-m diameter pipeline that is not yet built, for a given design flow rate. We might carry out experiments on a model pipe with a 30-cm diameter, in a laboratory setting. According to Equation 6, for a given temperature (and therefore given values of \( \mu \) and \( \rho \)), the term on the right in can be held constant by increasing \( V \) by the same proportion that we decrease \( d \). Thus, since the value of \( d \) in the model system is one-tenth of that in the prototype system, the dimensionless term will remain constant if \( v \) is increased by a factor of ten. Then, if we carry out an experiment in the model system with that higher velocity, the term on the left will be the same in the two systems. Accordingly, designating the prototype as system 1 and the model as system 2, we can write:

\[
\frac{\mu}{d V \rho}_2 = \frac{\mu}{d V \rho}_1, \quad \text{then} \quad \frac{\Delta p d}{V^2 \rho}_2 = \frac{\Delta p d}{V^2 \rho}_1
\]

\[
\frac{\Delta p_1}{(V^2 \rho / d)_2} = \frac{\rho_2}{\rho_1} \frac{d_2}{d_1} \left( \frac{V_1}{V_2} \right)^2 = 1 \ast \frac{1}{10} \ast \left( \frac{1}{10} \right)^2 = \frac{1}{1000}
\]

The interpretation is that the pressure drop per unit length in the model system will be 1000 times that in the prototype, so we can predict the behavior of the prototype based on the laboratory results.
It is important to understand both what we have, and what we have not, accomplished by this analysis. What we have accomplished is to use the constraints imposed by the requirement for dimensional consistency to convert a problem that originally had four independent variables into one that has only one. Because the relationship has only one independent variable, we know that when that independent variable is fixed, the dependent variable will also be fixed. Thus, if we carry out small-scale experiments in which we fix the non-dimensionalized independent variable at a value that we expect it to have in the prototype system, we can be confident that the non-dimensionalized dependent variable will also have the same value in the prototype as in the model system. From that information, and the value of $\Delta p_i$ in the model system, we can predict the value of $\Delta p_i$ in the prototype.

On the other hand, while we have established that $\frac{\Delta p_i d}{V^2 \rho}$ depends only on $\frac{\mu}{dV \rho}$, the analysis has not told us anything about how $\frac{\Delta p_i d}{V^2 \rho}$ changes when $\frac{\mu}{dV \rho}$ changes. It is for that reason that our model experiments have to be carried out at the same $\frac{\mu}{dV \rho}$ value as is expected in the prototype, for those experiments to be of use to us. If we carry out experiments at one value of $\frac{\mu}{dV \rho}$ and want to predict system behavior at any other value, we are in no better position than in the absence of the dimensional analysis. Understanding how the system responds to changes in $\frac{\mu}{dV \rho}$ is addressed in the next section of the course.

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Example. If flow in an open channel is sufficiently rapid and the channel discharges into a zone of lower velocity, a **hydraulic jump** occurs, in which the elevation of the water surface undergoes a dramatic increase over a short distance, accompanied by a great deal of turbulence and air entrainment. As a result, a standing wave is established in which the velocity head of the water flowing rapidly downstream is partially converted to elevation head; we will explore such a jump in a laboratory exercise later this term. Carry out a dimensional analysis to find how the height of water downstream of a hydraulic jump depends on other variables in the system. Use the Buckingham Pi approach.

Solution. As water passes through the jump, its height (and therefore the cross-section of flow) increases from $y_1$ to $y_2$, and its velocity decreases. The variables likely to play a role in controlling $y_2$ include the height upstream of the jump, ($y_1$, a geometric variable), the viscosity ($\mu$) and density ($\rho$) of the fluid (fluid variables), and the flow rate per unit width ($q$, an operational variable); in addition, we expect the constant $g$ to be important, since the gravitational force will be greater on the larger column of water after the jump than on the smaller column upstream of it. Thus, we have six variables of interest, and we expect the relationship to be:

$$y_2 = \phi(y_1, \rho, \mu, q, g)$$
To carry out the dimensional analysis, we first choose an acceptable set of dimensions, and then write out the dimensions of each variable. Choosing $M$, $L$, and $T$ as the dimensions, we have:

$$y_1 \equiv y_2 \equiv L; \quad \rho \equiv ML^3 \quad \mu \equiv M^{-1}T^{-1}$$

$$q \equiv \frac{L^3T^{-1}}{L} = L^2T^{-1} \quad g \equiv L^2T^{-1}$$

We note that three dimensions appear in the variables of interest, and they appear independently (i.e., the ratios of exponents for any two dimensions are not the same in all terms). Therefore, $r = 3$, and we can expect to have $6 - 3 = 3$ dimensionless terms in our final relationship.

For the repeating variables, we can choose any three variables that as long as they span the space of dimensions. Several combinations of the variables meet this criterion. Remembering that we want to use only independent variables as our repeating variables, we choose $q$, $y_1$, and $\rho$ for this calculation. (Others would serve the purpose just as well.) Therefore, our three dimensionless terms can be written as:

$$\Pi_1 = y_1^aq\, y_1^b\, \rho^c$$

$$\Pi_2 = g^aq\, y_1^b\, \rho^c$$

$$\Pi_3 = \mu^aq\, y_1^b\, \rho^c$$

Dimensional analysis of the above equations yields:

$$M^0L^0T^0 \equiv (L^1\left(L^2T^{-1}\right)^{a_1} \left(L^3\right)^{b_1} \left(ML^{-3}\right)^{c_1})$$

$$M^0L^0T^0 \equiv (LT^{-1})^1\left(L^2T^{-1}\right)^{a_2} \left(L^3\right)^{b_2} \left(ML^{-3}\right)^{c_2}$$

$$M^0L^0T^0 \equiv (ML^{-1}T^{-1})^1\left(L^2T^{-1}\right)^{a_3} \left(L^3\right)^{b_3} \left(ML^{-3}\right)^{c_3}$$

Each of the above dimensional equalities requires three equations to be satisfied, viz.:

For $\Pi_1$:

$$M: \quad 0 = c_i$$

$$L: \quad 0 = 1 + 2a_i + b_i - 3c_i \quad a_i = 0; \ b_i = -1; \ c_i = 0$$

$$T: \quad 0 = -a_i$$
for $\Pi_2$:
\[
\begin{align*}
M &: \quad 0 = c_2 \\
L &: \quad 0 = 1 + 2a_2 + b_2 - 3c_2 \quad \left\{ \begin{array}{l}
a_2 = -2; \quad b_2 = 3; \quad c_2 = 0 \\
T &: \quad 0 = -2 - a_2
\end{array} \right.
\end{align*}
\]
and for $\Pi_3$:
\[
\begin{align*}
M &: \quad 0 = 1 - c_3 \\
L &: \quad 0 = -1 + 2a_3 + b_3 - 3c_3 \quad \left\{ \begin{array}{l}
a_3 = -1; \quad b_3 = 0; \quad c_3 = -1 \\
T &: \quad 0 = -1 - a_3
\end{array} \right.
\end{align*}
\]
The results indicate that:
\[
\begin{align*}
\Pi_1 &= y_2^1 q_0^0 y_1^{-1} \rho_0^0 = \frac{y_2}{y_1} \\
\Pi_2 &= g^1 q^{-2} y_1^3 \rho_0^0 = \frac{g y_1^3}{q^2} \\
\Pi_3 &= \mu q^{-1} y_1^0 \rho^{-1} = \frac{\mu}{q \rho}
\end{align*}
\]
We conclude that the relationship of interest, written as a relationship among dimensionless parameters, is:
\[
\frac{y_2}{y_1} = \phi \left\{ \frac{g y_1^3}{q^2}, \frac{\mu}{q \rho} \right\}
\]
(7)

We have thus reduced the complexity of the problem: a function that we initially represented as having a dependent variable that depended on five independent variables has been written in a form that has a non-dimensionalized independent variable and only two other non-dimensional variables. Although we have not identified the complete relationship of interest (i.e., we cannot predict the height of the jump from Equation 7 alone), we could use the results of this relatively simple analysis to direct experiments so that we maximize the information we gain from the experiments. We could also use Equation 7 in conjunction with some experiments to predict the dimensions of a hydraulic jump in a different system from the results of a system we have studied.