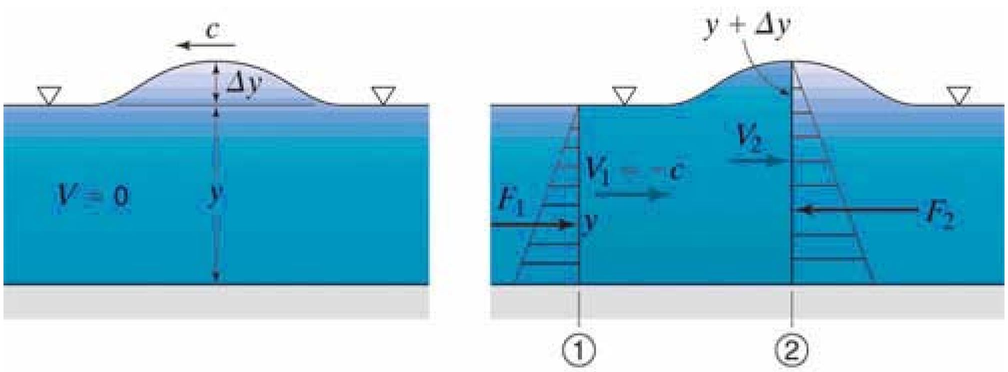
OPEN CHANNEL FLOW

Open channel flow is characterized by a surface in contact with a gas phase, allowing the fluid to take on shapes and undergo behavior that is impossible in a pipe or other filled conduit. Examples include not only "channels" as the word is usually used, but also flow across flat surfaces such as parking lots or streets, open ocean flow, flow exiting dams, etc. Like pipe flow, open channel flow can be laminar or turbulent, and steady or unsteady; it can also be uniform (constant depth along the channel) or non-uniform.

In one-dimensional open channel flow, the distance axis is conventionally labeled *x*, and the depth *y*. The Reynolds number is often defined as $\mathrm{Re} = r_h V \rho / \mu$, where r_h is the hydraulic radius. With this definition, flow is typically laminar if $\mathrm{Re} < 500$ and turbulent if $\mathrm{Re} > 12{,}500$; the wide gap reflects both a more gradual transition in open channels than in pipes, and also variations among channels in terms of geometry. Flow in river-sized systems is typically turbulent, and that in thin-layer flow along the ground is often laminar.

A key feature of open channel flow is the presence of waves on the liquid surface. The velocity of such waves is of particular interest. To understand the features that control the velocity of waves, imagine a single wave that is moving right-to-left across the surface of a water body at a velocity *c*, as shown in the figure on the left below. Note that the wave is just the disturbance of the water surface shape; the water underneath the wave is essentially stationary, other than the nearly circular rotation that water near the surface undergoes as the wave passes.



(*a*) As seen by a stationary observer

(*b*) As seen by an observer moving with the wave

Now imagine that the water to the left (with depth *y*) is caused to move with velocity *c* to the right. This causes the wave to become stationary relative to an observer who is outside the system and is not moving. (Note that this same scenario could be established by having the observer move with velocity *c* to the left, without movement of the underlying water.) As the water goes under the wave, it slows down in accord with the continuity equation, as follows.

$$\dot{m} = \rho Q$$

$$\rho V_1 y_1 b = \rho V_2 y_2 b = \rho V_2 \left(y_1 + \Delta y \right) b$$

$$V_1 y_1 = V_2 \left(y_1 + \Delta y \right) \qquad (1)$$

Defining the +*x* direction as movement to the right, the water loses *x*-directed momentum as is passes through a CV defined as the space between points 1 and 2 on the *x* axis, the depth of the water column, and a width *b*. Therefore, a force must be applied in the –*x* direction. The only forces on the water are the pressure forces, so the momentum equation applied to the CV yields:

$$F = \dot{m} \Delta V$$

$$p_1 A_1 - p_2 A_2 = \dot{m} \left(V_2 - V_1 \right) \qquad (2)$$

$$\left(\frac{1}{2} \gamma y_1 \right) \left(y_1 b \right) - \left(\frac{1}{2} \gamma \left[y_1 + \Delta y \right] \right) \left(\left[y_1 + \Delta y \right] b \right) = \rho Q \left(V_2 - V_1 \right)$$

$$\frac{1}{2} \gamma b \left(y_1^2 - \left[y_1 + \Delta y \right]^2 \right) = \frac{\gamma}{g} \left(V_1 y_1 b \right) \left(V_2 - V_1 \right)$$

$$\frac{1}{2} \left(-2 y_1 \Delta y - \left[\Delta y \right]^2 \right) = \frac{1}{g} \left(V_1 y_1 \right) \left(V_2 - V_1 \right)$$

$$-\Delta y \left(y_1 + \frac{1}{2} \Delta y \right) = \frac{1}{g} \left(V_1 y_1 \right) \left(V_2 - V_1 \right)$$

Substituting the expression from continuity from above and carrying out some algebra:

$$-\Delta y \left(y_1 + \frac{1}{2} \Delta y \right) = \frac{1}{g} \left(V_1 y_1 \right) \left(\frac{V_1 y_1}{y_1 + \Delta y} - V_1 \right)$$

$$-\Delta y \left(y_1 + \frac{1}{2} \Delta y \right) \left(y_1 + \Delta y \right) = \frac{1}{g} \left(V_1 y_1 \right) \left(V_1 y_1 - V_1 \left(y_1 + \Delta y \right) \right)$$

$$-\Delta y \left(y_1 + \frac{1}{2} \Delta y \right) \left(y_1 + \Delta y \right) = \frac{1}{g} \left(V_1 y_1 \right) \left(-V_1 \Delta y \right)$$

$$\left(y_1 + \frac{1}{2} \Delta y \right) \left(y_1 + \Delta y \right) = \frac{1}{g} \left(V_1^2 y_1 \right)$$

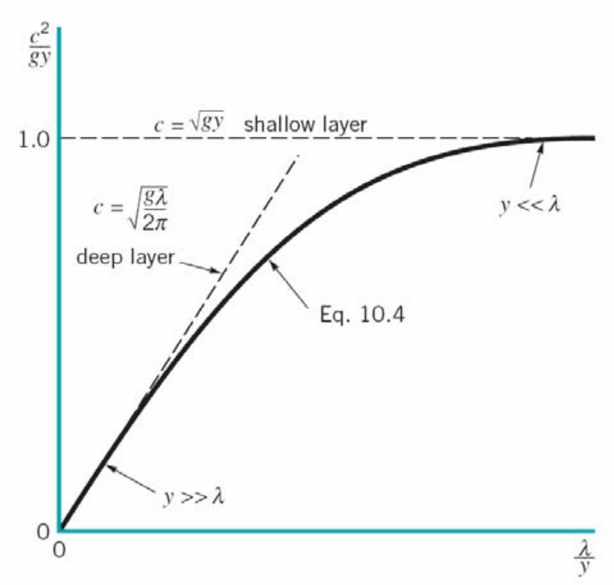
$$g\left(y_1+\frac{1}{2}\Delta y\right)\frac{y_1+\Delta y}{y_1}=V_1^2$$

Finally, assuming that $\Delta y << y_1$ and letting $V_1 = c$, we obtain:

$$gy_1 = c^2 \quad (3a)$$

$$c = \sqrt{gy_1} \quad (3b)$$

This result is perhaps surprising, in that is indicates that the velocity of the wave relative to the underlying, stationary water depends only on the total depth of the water and is independent of the properties of the fluid (in particular, ρ and μ) and of the amplitude of the wave (as long as the constraint that $\Delta y << y$ applies),. The derivation is also restricted to "shallow water waves," in which the effects of the wave motion are "sensed" at the bottom of the channel. For deep water waves (e.g., in the ocean), the velocity becomes much less than $\sqrt{gy_1}$ and is given by the following figure (Fig. 10.5 from Munson):



If we now imagine the same scenario as above but allow the water far to the left to move at a velocity anywhere from zero to values greater than *c*, we see that the net movement of the wave could be either to the left (for $V_1 < c$) or to the right (for $V_1 > c$). In the latter case, the existence of the wave (or any such disturbance) is never sensed by the fluid to the left. The distinction between the two situations in which disturbances can and cannot propagate upstream is an important one, so the two conditions are given distinctive names: when the fluid velocity is greater than the wave velocity, the conditions are said to be *supercritical*, and when the fluid velocity is less than the wave velocity, the conditions are *subcritical*.

A key dimensionless number for open channel flow is the Froude number, which can be interpreted in general as the ratio of the inertial force on the water to the gravitational force. In general, $\mathrm{Fr} = V/(gl)^{1/2}$, where *l* is a characteristic length. In open channel flow, *l* is taken as the depth of flow, so the Froude number expresses the ratio of the flow velocity to the velocity of a surface wave. In the special case where $\mathrm{Fr} = 1$, surface waves remain stationary and the conditions are said to be *critical*. The Froude number is equally or more important than Re for open channel flow.

To explore the behavior of flow in open channels quantitatively, we begin with (and often model more complex systems as) consideration of in one-dimensional flow through a simple rectangular cross-section of width *b*. We designate the flow rate per unit channel width (*Q*/*b*) as *q*, and the elevation of the bottom of the channel relative to a specified datum as z_{bot}. We assume that *b* is constant, so *q* is constant as well. However, the channel bottom might or might not be horizontal; therefore, for the general case z_{bot} is a function of *x*. The velocity can be related to *q* and *b* by:

$$V(x) = \frac{Q}{A(x)} = \frac{qb}{y(x)b} = \frac{q}{y(x)} \tag{4}$$

Because we are assuming that the fluid is ideal, the total energy per unit weight (i.e., the total head) must remain constant both with depth and along the flow path. Furthermore, if we consider a streamline along the water surface, the pressure head is zero everywhere. Therefore the total head at the water surface is just the sum of the elevation head (h_{elev}) and the velocity head (h_{vel}). The total head at any depth is the same as the total head at the surface, so we can write:

$$h_{tot} = h_{tot,surf} = h_{elev,surf} + \cancel{h_{pressure,surf}} + h_{vel,surf} \tag{5}$$

$$h_{tot} = z_{bot}(x) + y(x) + \frac{\left[V(x)\right]^2}{2g} = z_{bot}(x) + y(x) + \frac{q^2}{2g\left[y(x)\right]^2} \tag{6}$$

Or, equivalently, in terms of the conditions at the bottom of the water column (assuming that the pressure distribution is hydrostatic, equivalent to assuming that the flow is uniform):

$$h_{tot} = z_{bot}(x) + \frac{p_{bot}(x)}{\gamma} + \frac{q^2}{2g\left[y(x)\right]^2} \tag{7}$$

Note that, even though the terms on the right in the two preceding equations all vary with *x*, h_{tot} is independent of *x*. In the remainder of the discussion, we will drop the explicit indication that z_{bot} and *y* are functions of *x*.

The sum of the velocity head and the depth (or the velocity head and the pressure head at the bottom of the channel) at a given location is referred to as the *specific energy, E*. Thus, the specific energy at any location where the flow is uniform can be expressed as:

$$E = y + \frac{q^2}{2gy^2} = \frac{p_{bot}}{\gamma} + \frac{q^2}{2gy^2} = h_{tot} - z_{bot} \qquad (8)$$

As indicated by the final equality in Equation 8, for the given assumptions (fixed Q and b and steady, ideal flow), the specific energy is identical at all locations where z_{bot} is the same, but it differs at locations where the elevation of the channel bottom differs. Thus, the specific energy is the variable portion of the head at a given location; the first equality in Equation 8 indicates that this value depends only on the depth of the water at that location.[1]

For a given q in a given channel, the flow could have any depth whatsoever; the only constraint is that Equation 4 must be satisfied. According to Equation 8, for each depth, the water has a unique value of specific energy. A plot of E vs. y for a given q has a characteristic shape, approaching the asymptotes of $y = E$ and $y = 0$ as E gets large, and passing through a minimum value of E at some intermediate value of y. Conventionally, such plots are drawn with y as the vertical axis (to correspond to our intuitive way of thinking about depth), even though y is probably a more logical independent parameter. The plot is shown in both ways in Figure 1.

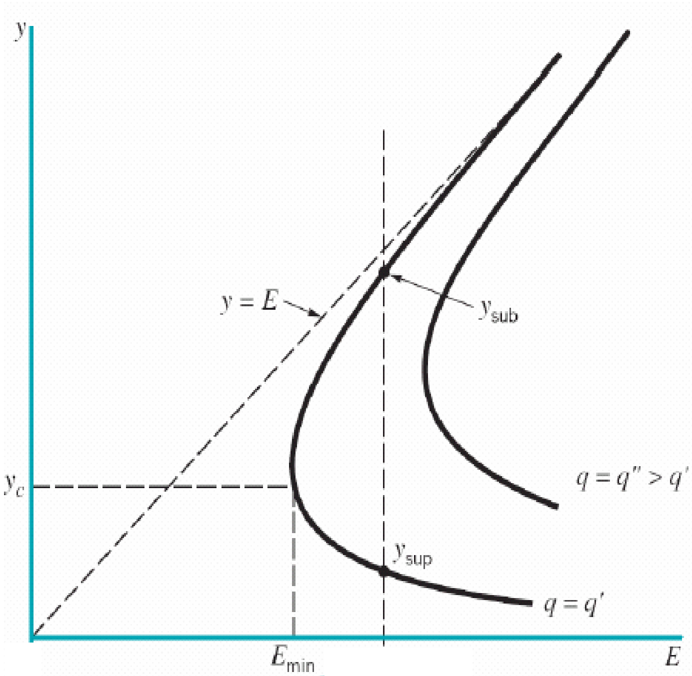


Figure 1. A typical specific energy diagram.

[1] The analogous parameter for confined flow with a fixed Q and fixed system geometry would be the pressure head, since that is the only type of head that can change at a given location in such a system. Because of this, in confined systems, there is no point in defining a separate term analogous to specific energy.

The value of y at which the specific energy passes through its minimum can be found by differentiating Equation 8 with respect to y and setting the result to zero. Designating this value of y as y_c, we find:

$$\frac{dE}{dy} = 1 - \frac{q^2}{gy^3} = 0$$

$$q^2 = gy_c^3$$

$$y_c = \left(\frac{q^2}{g}\right)^{1/3} \qquad (9)$$

The corresponding values of E_{min} and the fluid velocity are:

$$E_{min} = y_c + \frac{q^2}{2gy_c^2} = y_c + \frac{1}{2}\frac{q^2}{g}\frac{1}{y_c^2} = y_c + \frac{1}{2}y_c^3\frac{1}{y_c^2} = \frac{3}{2}y_c$$

$$V_c = \frac{q}{y_c} = \frac{\left(y_c^{3/2}g^{1/2}\right)}{y_c} = \sqrt{gy_c} \qquad (10)$$

Comparison of the final expression in Equation 10 with that in Equation 3b indicates that the velocity corresponding to the minimum specific energy is the velocity of a surface wave in the system. As a result, on the portion of the specific energy curve where $V > V_c$, the flow is supercritical, and on the portion where $V < V_c$, flow is subcritical.

To summarize, to sustain any specified flow rate per unit width (Q/A, or q) of an ideal fluid in an open channel, the specific energy of the fluid must exceed a minimum value, E_{min}. If E ever decreases below this value, the value of q passing the location will be less than the value of q approaching it, and the fluid will back up. This transient event increases both y and E just upstream, a process that continues until the fluid has attained depth y_c and specific energy E_{min}, at which point steady flow is again achieved. When the depth is exactly y_c and the specific energy is E_{min}, the velocity is $\sqrt{gy_c}$, corresponding to a Froude number of 1.

If the fluid has any specific energy that is greater than E_{min}, then uniform flow can be sustained at two different velocities: one subcritical and one supercritical. In a general sense, when the velocity is subcritical, the elevation head is more significant than the velocity head, and when the flow is supercritical, the reverse is true, although the transition does not correspond exactly to the point where the two heads are equal.

Now consider an open channel with a fixed width but a region in which the bottom has a slope between two regions where it is horizontal. Initially, we will consider four different scenarios: either downward and upward sloping bottom, and with the upstream flow being either sub- or super-critical. Then we will consider a fifth scenario in which the water encounters a bump that consists of an upward slope followed by an equivalent downward one. In all cases, the combination of the continuity and energy equations relating upstream and downstream conditions can be written as:

$$z_{bot,1} + y_1 + \frac{V_1^2}{2g} = z_{bot,2} + y_2 + \frac{V_2^2}{2g}$$

$$E_1 = E_2 + z_{bot,2} - z_{bot,1} = E_2 + \Delta z_{bot} \qquad (11)$$

Scenario 1: *Upward slope, sub-critical upstream flow*. In this scenario, the upstream condition is characterized by a point on the "large y" portion of the specific energy diagram which asymptotically approaches the line $y = E$. The downstream location has less specific energy, so the shift is away from the $y = E$ line, in the direction of decreasing depth and therefore increasing velocity. More specifically, at any value of Δz_{bot}, the system is characterized by the point on the sub-critical part of the E curve where $E = E_1 - \Delta z_{bot}$. Because dy/dE is >1 on this leg of the curve, and the magnitude of the change in E equals the magnitude of the change in z_{bot}, the decrease in y is greater than the increase in z_{bot}. Thus, perhaps counter-intuitively, as the bottom of the channel gets higher, the surface of the water gets lower. This process continues until y declines to y_c. If the bottom continues to slope upward (i.e., if the channel bottom has not become horizontal by the point where $y = y_c$), it still must be the case that $E = E_1 - \Delta z_{bot}$. However, E cannot decline any more at the given value of q, because it has reached E_{min}. Therefore, the assumption that was used to develop the y vs. E curve (constant q) is transiently violated, and q passing the critical point becomes less than q approaching it. This causes water to back up upstream, increasing y_1 and E_1, until the steady flow of q is re-established. This process continues as long as the channel bottom continues to rise, so that y and E always reach y_c and E_{min}, respectively, just at the point where the bottom flattens out; from there downstream, the flow remains critical as long as z_{bot} remains constant. In essence, the system satisfies Equation 11 by allowing E_2 to decrease until it reaches E_{min}, and satisfies the equation thereafter by causing E_1 to increase.

Scenario 2: *Upward slope, super-critical upstream flow*. In this scenario, the upstream condition is characterized by a point on the "low y" leg of the specific energy diagram, along which the curve approaches $y = 0$ asymptotically. As in Scenario 1, the downstream location has less specific energy than the upstream location (by Δz_{bot}), so the shift is away from the $y = 0$ asymptote, in the direction of decreasing velocity and increasing depth. Thus, in this case, the water surface gains elevation as the channel bottom gets higher, opposite from the case in Scenario 1. Once again, if the bottom elevation increases enough, the system eventually reaches the limiting condition of $E = E_{min}$, and the water backs up. As in Scenario 1, backup continues until critical conditions are reached right at the location where the bottom flattens out.

Scenarios 3 and 4: *Downward slope, sub- or super-critical upstream flow*. These scenarios are easier to analyze than the scenarios described above, because with a downward sloping bottom, the shift is toward increasing E, away from E_{min}. As a result, the fluid remains on whichever leg of the specific energy diagram characterized the upstream flow, just moving to the right. In the case of super-critical flow, the velocity increases in the downstream direction, and the depth decreases; since both the channel bottom and the depth of water decrease, the water surface goes downhill. By contrast, and again perhaps counter-intuitively, if the upstream flow is sub-critical, the depth increases more than the bottom elevation drops, and the water surface increases; i.e., the water flows uphill! This occurs because the gain of potential energy is compensated by the decrease in kinetic energy.

Scenario 5: *Bump in the channel bottom, modeled as a section with an upward slope, a flat section, and then a section with a downward slope, back to the original elevation*. If a sub-critical flow approaches a bump, its velocity increases and its depth decreases as it climbs the uphill slope. If the flow never becomes critical, then the exact reverse process occurs on the other side of the bump, and the downstream flow characteristics are identical to those of the upstream flow. However, if the critical condition is reached, then the flow can return to E_1 via either the sub-critical or super-critical leg of the specific energy curve. Empirically, whenever E increases from a critical flow condition, the super-critical path is taken preferentially; some obstruction or other impediment to flow is required for the flow to follow the sub-critical path.

Flow constrictions. The preceding discussion focuses on changes in the elevation of the channel bottom (z_{bot}) in systems with constant width (b). We next consider the inverse case of changes in b in systems with constant z_{bot}, corresponding to constrictions or expansions in the flow path. For this analysis, we focus on a constriction in which $b_1 > b_2$. Assuming, as before, that Q is constant, we conclude that in this case $q_1 < q_2$. However, if the flow is ideal, $E_1 = E_2$ (because Equation 11 applies with $\Delta z_{bot} = 0$). Therefore, the situation is described not by one, but two y vs. E curves, one for each q. Since the value of q in the constriction is larger than that upstream, the E_2 curve is to the right of the E_1 curve. As the flow enters the constriction, the fact that E remains constant means that the system must move vertically from the q_1 curve to the q_2 curve. Given the relative locations of these curves, it is clear that the shift is to shallower flow if the upstream flow is sub-critical and to deeper flow if the upstream flow is super-critical. As the constriction gets tighter, b decreases and q continues to increase, so the y vs. E curve for the constriction moves farther to the right. Eventually, it moves so far that the flow in the constriction becomes critical. Analogous to the scenarios analyzed above, if the constriction is made even tighter, it cannot sustain the flow Q, and the water backs up upstream. As a result, E_1 increases, until it equals E_{min} in the constriction, and steady flow is once again established.

The same situation can be characterized on a single graph by plotting y vs. q for the given E, as opposed to y vs. E for a given q. The plot can be prepared by solving Equation 11 for q, which yields:

$$q = \sqrt{2g\left(y^2 E - y^3\right)}$$

A typical plot of this equation is shown in Figure 2. In this case, the conditions characterizing flow along the constricting path correspond to those that move along the curve from left to right, along the upper leg if the upstream flow is sub-critical and the lower leg if the flow is super-critical.

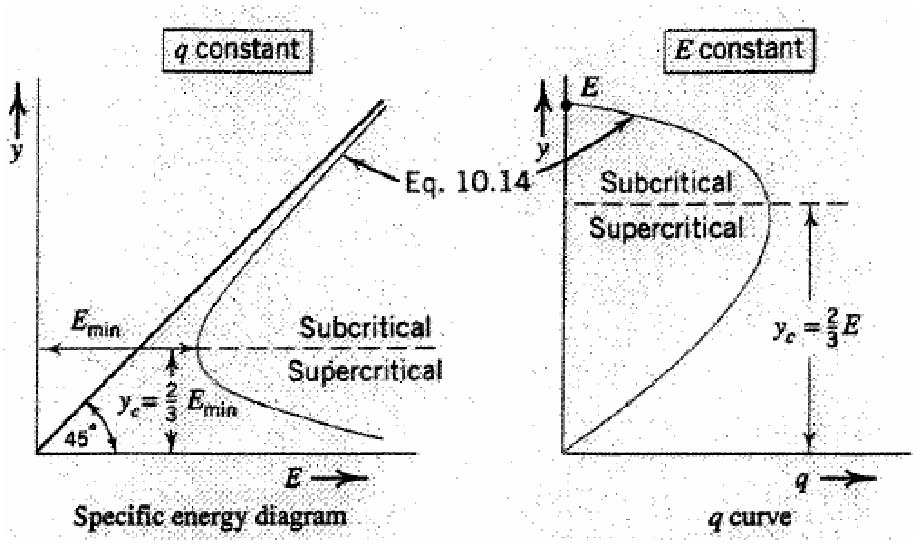
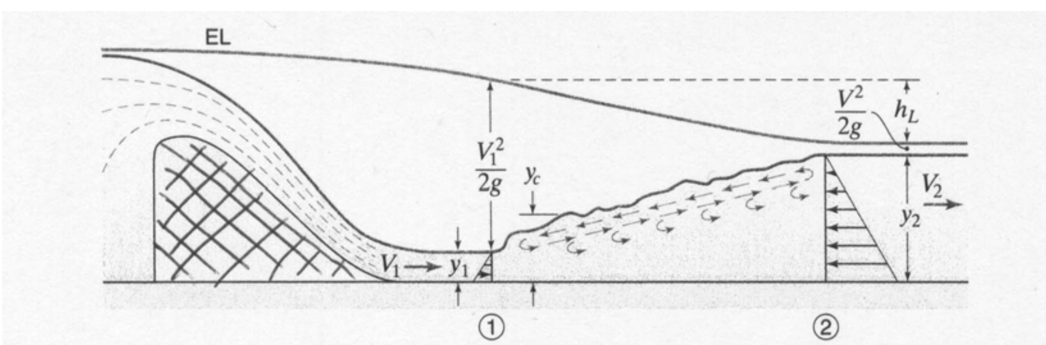
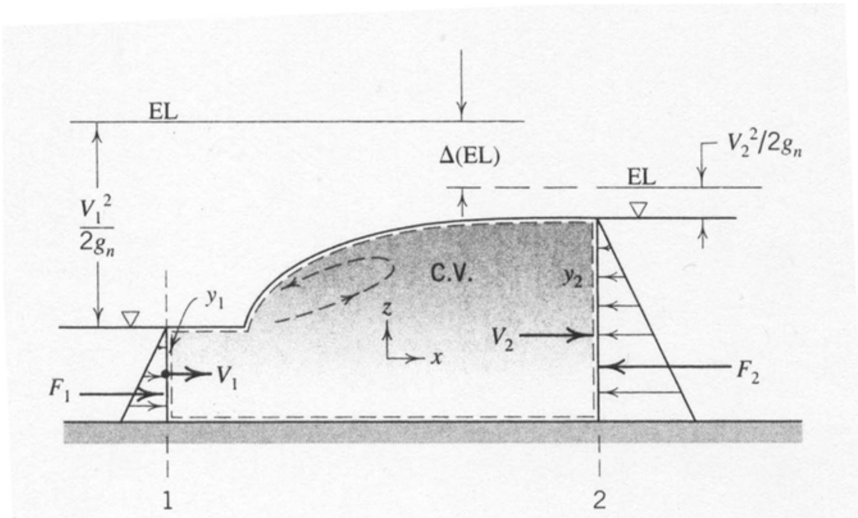


Figure 2. **Plots the depth of flow as a function of (a) specific energy and (b) flow rate per unit width, in systems with open channel flow.**

Hydraulic Jumps. If flow in an open channel is sufficiently rapid and the channel discharges into a zone of lower velocity, a *hydraulic jump* occurs, in which the elevation of the water surface undergoes a dramatic increase over a short distance, accompanied by a great deal of turbulence and air entrainment. Despite the inherent non-ideality of the situation, analyzing a CV that includes the jump with the impulse-momentum equation, assuming ideal fluid behavior, provides some useful information. A definition sketch of the system is shown below.





Applying the impulse-momentum equation between points 1 and 2, the only external forces acting on the fluid in the CV are the pressure-based forces on its ends, so:

$$\sum F_{ext,x} = F_1 - F_2 = \frac{\gamma y_1^2 b}{2} - \frac{\gamma y_2^2 b}{2} = Q\rho\left(V_2 - V_1\right) \tag{12}$$

Substituting Q/yb for the V terms and γ/g for ρ, and rearranging:

$$\frac{\gamma y_1^2 b}{2} - \frac{\gamma y_2^2 b}{2} = Q\frac{\gamma}{g}\left(\frac{Q}{y_2 b} - \frac{Q}{y_1 b}\right) \tag{13}$$

$$\frac{y_1^2}{2} - \frac{y_2^2}{2} = \frac{1}{g}\left(\frac{Q^2}{y_2 b^2} - \frac{Q^2}{y_1 b^2}\right) = \frac{1}{g}\left(\frac{q^2}{y_2} - \frac{q^2}{y_1}\right) \tag{14}$$

$$\frac{q^2}{gy_1} + \frac{y_1^2}{2} = \frac{q^2}{gy_2} + \frac{y_2^2}{2} \tag{15}$$

Alternatively, by factoring a common factor $y_1 - y_2$ from both sides of Equation 14, the equation can be written as a quadratic in y_2/y_1 and solved via the quadratic equation to yield:

$$\frac{y_2}{y_1} = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8q^2}{gy_1^3}}\right] = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{8V_1^2}{gy_1}}\right] \tag{16}$$

The solution to Equation 16 is such that y_2 / y_1 is less than 1, equal to 1, or greater than 1 whenever V_1^2 / gy_1 falls into the same range. A situation in which y_2 / y_1 is less than 1 is impossible, since that would correspond to a spontaneous increase in the EL (i.e., a negative head loss across the jump). The situation where y_2 / y_1 equals 1 corresponds to stable flow and no jump. Thus, the only region of interest is where both y_2 / y_1 and V_1^2 / gy_1 are greater than 1; put another way, V_1^2 / gy_1 must be >1 for a hydraulic jump to be possible. Because $\sqrt{V^2 / gy}$ is the Froude number (Fr), we conclude that a hydraulic jump is possible only if the Froude number is >1 (i.e., if the flow is super-critical).