CEE 342 Aut 2005 HW#5 Solutions

4.6. The problem statement indicates that the streamlines are consistent with the equation $x^2 y - y^3/3 = c$. By definition, the velocity vector at any point (x, y) in a flow field is the slope of the streamline at that point, i.e., it is dy/dx. Differentiating the expression for the streamline, we find:

$$d\left(x^2 y - y^3/3\right) = dc$$

$$2xy\,dx + x^2 dy - y^2 dy = 0$$

$$2xy\,dx + \left(x^2 - y^2\right)dy = 0$$

$$\frac{dy}{dx} = -\frac{2xy}{x^2 - y^2} \qquad (1)$$

The velocity vector is also given by v/u, and expressions for u and v are given. Therefore:

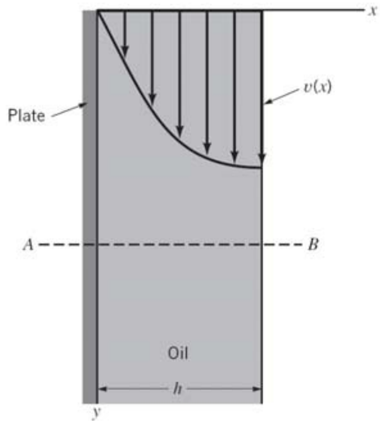
$$\frac{v}{u} = \frac{-2cxy}{c\left(x^2 - y^2\right)} = -\frac{2xy}{x^2 - y^2}$$

This is the same equation as we derived for dy/dx. Thus, the given expressions for u and v are consistent with the equation given for the streamlines.

For the flow to be parallel to the y axis, the streamline has to have an infinite slope. Based on Equation 1, this occurs at any point where $x^2 = y^2$, i.e., at $y = \pm x$.

For the flow to be stationary, u and v must both be zero. For this to occur, x^2 must equal y^2 and either x or y must equal zero. Both these conditions are met only at $x = y = 0$.

4.55.



(a) Flow is only in the y direction (the direction of the $\vec{j}$ unit vector), and the magnitude of the velocity is given by:

$$V = \frac{V_0}{h^2}\left(2hx - x^2\right)$$

At the plate, $x = 0$, so $V = 0$ (showing that, as the text says, the fluid “sticks” to the plate). For laminar flow, the shear stress is given by $\tau = \mu \frac{dV}{dx}$, where x is the direction perpendicular to the velocity. Thus:

$$\tau = \mu \frac{dV}{dx} = \frac{V_0}{h^2}\left(2h - 2x\right)$$

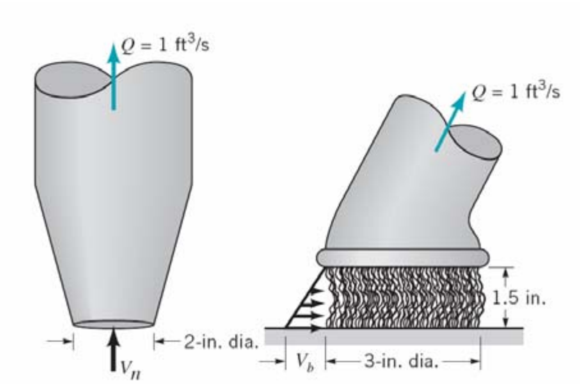
Clearly, at $h = x$, $\tau = 0$.

(b) Defining b as the width of the plate, the flowrate across section AB is given by:

$$Q = \int VdA = \int_0^h \frac{V_0}{h^2}\left(2hx - x^2\right)bdx = \frac{V_0}{h^2}b\int_0^h \left(2hx - x^2\right)dx$$

$$Q = \frac{V_0}{h^2}b\left(hx^2 - \frac{x^3}{3}\right)\Bigg|_0^h = \frac{V_0}{h^2}b\left(h^3 - \frac{h^3}{3}\right) = \frac{2}{3}V_0 bh$$

5.2.



(a) The average velocity is given by:

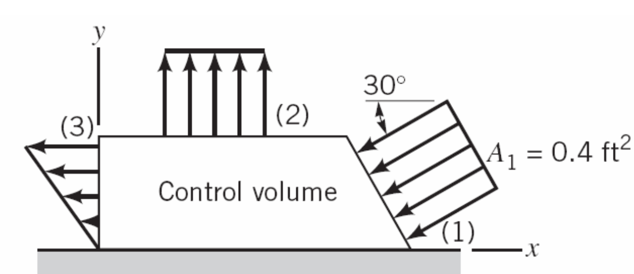
$$V_n = \frac{Q}{A} = \frac{1\text{ ft}^3/\text{s}}{(\pi/4)\left(\frac{2}{12}\text{ ft}\right)^2} = 45.8\text{ ft/s}$$

(b) The *average* radial velocity around the brush is sufficient to provide an overall volumetric flow rate of 1 ft^3/s. Therefore:

$$\bar{V}_r = \frac{Q}{A} = \frac{1\text{ ft}^3/\text{s}}{\pi\left(\frac{3}{12}\text{ ft}\right)\left(\frac{1.5}{12}\text{ ft}\right)} = 10.2\text{ ft/s}$$

Because the velocity profile is linear, the average velocity is one-half of the maximum value (V_b), so $V_b = 20.4$ ft/s.

5.17.



(a) We are to find the value of $\int_{(1)} \gamma \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} dA$. The product $\vec{\mathbf{V}} \bullet \vec{\mathbf{n}}$ is the component of the fluid velocity that is normal to and directed outward from the control volume. Correspondingly, $\vec{\mathbf{V}} \bullet \vec{\mathbf{n}} dA$ is the volumetric flow rate leaving the control volume across the control surface *dA*, and

$\gamma \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} dA$ is the weight flow rate (weight per unit time) leaving the CV across dA. $\int_{(1)} \gamma \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} dA$ is therefore the weight flow rate leaving the CV across area A_1.

The system has steady flow so, by continuity:

$$\dot{m}_1 = \dot{m}_2 + \dot{m}_3$$

where all the mass flow rates are defined to be positive in the direction of flow. The mass flow rates across sections 2 and 3 can be computed from the given information, so the mass flow rate across section 1 is given by:

$$\dot{m}_1 = \dot{m}_2 + \rho_3 Q_3 = 3\frac{\text{slug}}{\text{s}} + \left(1.94\frac{\text{slug}}{\text{ft}^3}\right)\left(2\frac{\text{ft}^3}{\text{s}}\right) = 6.88\frac{\text{slug}}{\text{s}}$$

As noted above, we are trying to determine the weight flow rate leaving the CV across area A_1. $\dot{m}_1$ is the mass flow rate entering the CV across this area, so:

$$\int_{(1)} \gamma \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} dA = -\dot{m}_1 g = -\left(6.88\frac{\text{slug}}{\text{s}}\right)\left(32.2\frac{\text{ft}}{\text{s}^2}\right)\left(\frac{1\text{ lb}}{\text{slug-ft/s}^2}\right) = -222\frac{\text{lb}}{\text{s}}$$

(b) $\int_{(1)} \vec{\mathbf{V}} \rho \vec{\mathbf{V}} \bullet \vec{\mathbf{n}} dA$ is the rate at which momentum is advected into the CV across area A_1. We can find this rate in a number of ways, one of which is as follows.

We found in part *a* that the mass flow rate crossing surface 1 ($\dot{m}_1$) is 6.88 slug/s. The ratio $\dot{m}_1/\rho$ is therefore the volumetric flow rate crossing this area, and $\dot{m}_1/(\rho A_1)$ is the velocity perpendicular to the area. The flow is perpendicular to A_1, so the magnitude of the total velocity at that surface is the same as the component that is perpendicular to the surface. Further, A_1 is given as 0.4 ft^2, so:

$$V_1 = \frac{\dot{m}_1}{\rho A_1} = \frac{6.88\text{ slug/s}}{\left(1.94\frac{\text{slug}}{\text{ft}^3}\right)\left(0.4\text{ ft}^2\right)} = 8.87\frac{\text{ft}}{\text{s}}$$

The rate of momentum advection across surface 1 is therefore:

$$\dot{m}_1 V_1 = \left(6.88\frac{\text{slug}}{\text{s}}\right)\left(8.87\frac{\text{ft}}{\text{s}}\right) = 61.0\frac{\text{slug-ft}}{\text{s}^2}$$

The preceding value is a vector, with a direction corresponding to the direction of the flow. The *x* and *y* components of this quantity are:

$$\left(\dot{m}_1 V_1\right)_x = \left(61.0\frac{\text{slug-ft}}{\text{s}^2}\right)\cos 30^{\text{o}} = 52.8\frac{\text{slug-ft}}{\text{s}^2}$$

$$\left(\dot{m}_1 V_1\right)_y = \left(61.0\frac{\text{slug-ft}}{\text{s}^2}\right)\sin 30^{\text{o}} = 30.5\frac{\text{slug-ft}}{\text{s}^2}$$

5.18. The mass flow rate of air through any annular section of width *dr* equals the product of the air's density (ρ), the velocity in that section ($V(r)$), and the area of the section ($2\pi r\, dr$). The total mass flow rate is therefore:

$$\dot{m} = \int_{r_1}^{r_2} \rho V\left(r\right)\left(2\pi r\right)dr$$

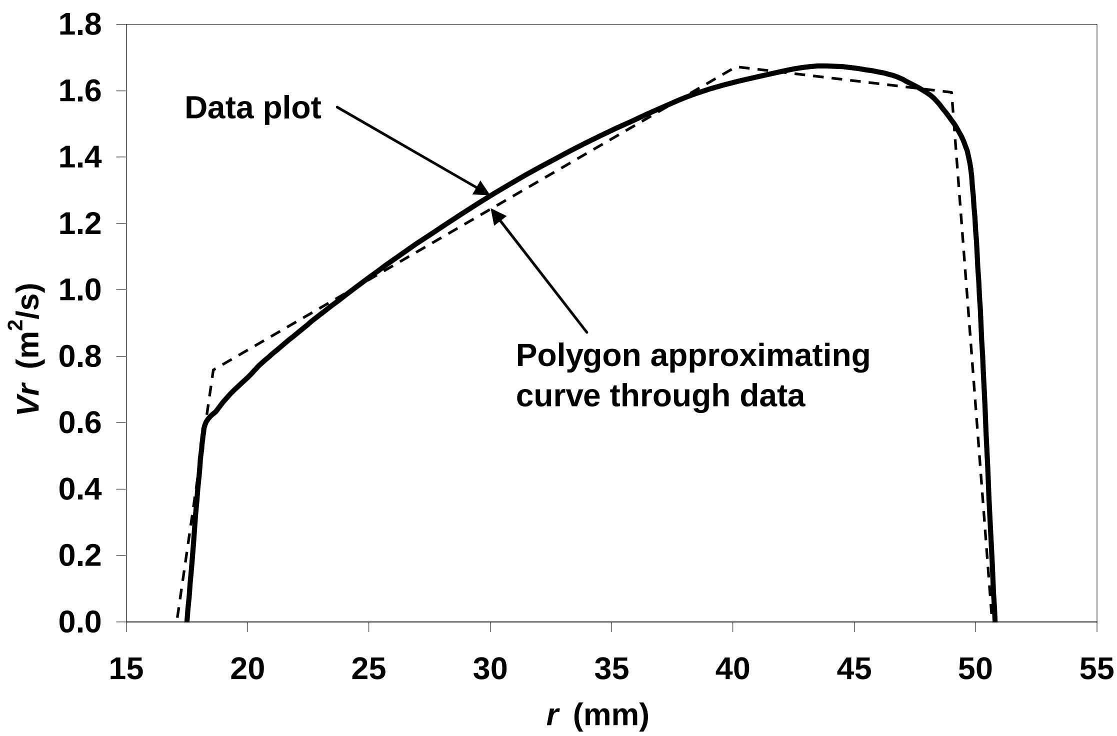
We have data for V vs. r, which we can use to carry out the above integration numerically. To carry out that integration, we should use values of r and $V(r)$ that are in the middle of each Δr segment. We can do this by setting up a spreadsheet, as follows:

r	***V***	**Δ*r***	***r*avg**	***V*avg**	***V*avg*(2π*r*avg Δ*r*)**
(m)	**(m/s)**	**(m)**		**(m/s)**	**(m³/s)**
0.0175	0				
0.0182	32	0.0007	0.01785	16	0.0012561
0.0187	33.9	0.0005	0.01845	32.95	0.0019099
0.0193	35.6	0.0006	0.019	34.75	0.0024891
0.02	36.8	0.0007	0.01965	36.2	0.0031286
0.0206	37.9	0.0006	0.0203	37.35	0.0028584
etc.	etc.	etc.	etc.	etc.	etc.
etc.	etc.	etc.	etc.	etc.	etc.
0.0508	0.0	0.001	0.0503	13.7	0.0043298
				SUM	**0.2699921**

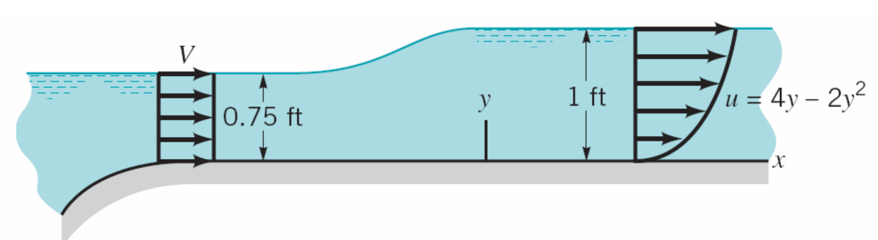
The values in the first two columns are the given data for r and V, with r converted to mm. The values in the third column are Δr, and correspond to the r value in that row minus the r value in the row above it (for this reason, the first row has no entry for Δr). The fourth and fifth columns indicate the average values of r and V in the given Δr segment; each of these is computed as the average of the corresponding value in that row and the row above it. Finally, the sixth column shows the volumetric flow rate through the Δr section, corresponding to the product of V_{avg} and $2\pi r_{\text{avg}}\,\Delta r$ in that section. The numerical integration involves taking the sum of all the values in the final column, which yields the volumetric flow rate through the entire annular cross-section. The mass flow rate is then computed as:

$$\dot{m} \approx \rho \sum_{r_1}^{r_2} V(r)(2\pi r)\Delta r = \left(1.23\frac{\text{kg}}{\text{m}^3}\right)\left(0.270\frac{\text{m}^3}{\text{s}}\right) = 0.332\frac{\text{kg}}{\text{s}}$$

As an alternative, one could integrate $\dot{m} = \int_{r_1}^{r_2} \rho V(r)(2\pi r)\,dr$ graphically. To do that, one would plot the function in the integrand against r and compute the area under the curve. I have taken the constants out of the integrand and plotted Vr vs. r in the graph below. The area under the curve is $\int_{r_1}^{r_2} V(r)\, r\, dr$ and could be estimated by a polygon, as is shown. The area would then be multiplied by $2\pi\rho$ to estimate $\dot{m}$, yielding a value that would presumably be as good as the estimate from numerical integration described above.



5.19.



By continuity, the mass flow rate must be the same at the two sections, and because the fluid is incompressible, the volumetric flow rate must also be equal at the two locations. The volumetric flow rate at the downstream section is given by:

$$Q = \int_0^{y_{\max}} uW dy = \int_0^{1\text{ ft}} \left(4y - 2y^2\right)\left(3\text{ ft}\right) dy = \left(3\text{ ft}\right)\left(2y^2 - \frac{2}{3}y^3\right)\Bigg|_0^{1\text{ ft}} = 4\frac{\text{ft}^3}{\text{s}}$$

The same volumetric flow rate applies at the upstream point, so:

$$4\frac{\text{ft}^3}{\text{s}} = V\left(3\text{ ft}\right)\left(0.75\text{ ft}\right)$$

$$V = 1.78\frac{\text{ft}}{\text{s}}$$