CIVE 342 Aut 2005 Exam #2 Solutions

1. (a) The idea underlying the continuity equation is conservation of mass. Specifically, at steady state, the rate that mass enters a control volume must equal the rate at which it leaves.

(b) The principle of conservation of mass does not depend on the presence or absence of friction, the compressibility of the mass, or the energy content of the mass. Furthermore, it is intuitive that the idea "at steady state, mass in equals mass out" has to apply regardless of what other processes occur in the CV, given that no process can create or destroy mass. Therefore, the continuity equation applies independently of any of the listed possibilities.

(c) Continuity requires that $\dot{m}_A = \dot{m}_B$, where $\dot{m}$ is the mass flow rate, which can be expressed as ρAV. Thus:

$$\rho_A A_A V_A = \rho_B A_B V_B$$

$$V_B = \frac{\rho_A}{\rho_B}\frac{A_A}{A_B}V_A = \left(\frac{1}{2}\right)(1)\left(3\frac{\text{m}}{\text{s}}\right) = 1.5 \text{ m/s}$$

The velocity head is $V^2/2g$, so:

$$\text{Velocity head at } B = \frac{V_B^2}{2g} = \frac{(1.5 \text{ m/s})^2}{2\left(9.81 \text{ m/s}^2\right)} = 0.115 \text{ m}$$

2. (a) The term i_{mom} represents the momentum (mV) of the fluid per unit mass (m), so it is just the velocity, V. The dot product $\vec{\mathbf{V}} \bullet \vec{\mathbf{n}}$ is a scalar with magnitude equal to $Vn\cos\theta$, where n is the magnitude of the unit vector and is therefore equal to 1.0, and θ is the angle between $\vec{\mathbf{V}}$ and $\vec{\mathbf{n}}$. The direction of $\vec{\mathbf{n}}$ is perpendicular to the surface and out of the CV, so it forms an angle of 150$^\text{o}$ with $\vec{\mathbf{V}}$. Therefore:

$$\int i_{mom}\rho\vec{\mathbf{V}} \bullet \vec{\mathbf{n}}dA = V\rho V(1)\left(\cos 150^\text{o}\right)A = \rho V^2 A\cos 150^\text{o}$$

$$= \left(1000\frac{\text{kg}}{\text{m}^3}\right)\left(1.6\frac{\text{m}}{\text{s}}\right)^2\left(6 \text{ m}^2\right)(-0.866) = -13{,}300\frac{\text{kg-m}}{\text{s}^2} = -13.3 \text{ kN}$$

(b) The term $\dot{E}_{en}$ represents the rate of energy addition to the CV per unit mass of fluid in the CV. According to the problem statement, this value is –80 MW. The inefficiency of the turbine does not affect this value.

The term $\frac{d}{dt}\int_{\text{CV}} i_{en}\rho d\mathcal{V}$ represents the time rate of change of the amount of energy stored in the fluid in the CV. At steady state, the amount of energy stored in the fluid in the CV is not changing, so the value of this term is zero.

3. The average velocity (V_{avg}) of fluid in any flow channel is defined as the volumetric flow rate of the fluid divided by the cross-sectional area. In any layer of thickness dh, the volumetric flow rate is the product of the velocity (v) and the area (Wdh), so:

$$V_{avg} = \frac{\int v dA}{A} = \frac{\int vWdh}{WH} = \frac{1}{H}\int vdh$$

In general, the integration would have to be carried out over the whole cross-section of the flow path. However, because the flow is the same from the midline to either edge, the average velocity is the same above and below the midline, so we can just integrate over one of those cross-sections. Choosing to integrate from the midline to the top, and substituting the given expression for v, we find:

$$V_{avg} = \frac{1}{H}\int_0^H v_c\left(1-\frac{h^2}{H^2}\right)dh = \frac{v_c}{H}\int_0^H\left(dh-\frac{h^2dh}{H^2}\right) = \frac{v_c}{H}\left[h-\frac{1}{3}\frac{h^3}{H^2}\right]_0^H$$

$$= \frac{v_c}{H}\left(H-\frac{1}{3}\frac{H^3}{H^2}\right) = \frac{2}{3}v_c = \frac{2}{3}\left(3\frac{\text{m}}{\text{s}}\right) = 2\frac{\text{m}}{\text{s}}$$

4. The horizontal and vertical components of the velocity at $(x, y) = (2\text{ m}, 4\text{ m})$ are $u = 8$ m/min, $v = -16$ m/min. The magnitude of the overall velocity is therefore:

$$V = \sqrt{u^2+v^2} = \sqrt{(8\text{ m/min})^2+(16\text{ m/min})^2} = 17.9\text{ m/min}$$

The direction of the velocity is arctan (16/8), or 63.4$^{\text{o}}$ (1.11 radians).

5. If we define the control volume to include all the water in pipe/ tank/ nozzle system, we see that the water has no horizontal velocity when it enters the CV, but it does have a horizontal velocity when it leaves. Therefore, a force to the right must be exerted on it by the pump and other parts of the boat, and it exerts an equal reaction force to the left on the boat. This force on the boat must, in turn, be resisted by the cable. Applying the energy equation between the location where the gage connects to the tank (A) and the outlet of the nozzle (B), and assuming that the fluid velocity at A is negligible, we can find the velocity exiting the nozzle and then the flow rate. Noting that $z_A = z_B$ and that the p_B is the hydrostatic pressure at a depth of 5' of water, we obtain:

$$\frac{p_A}{\gamma} + \frac{v_A^2}{2g} + \cancel{z_A} = \frac{p_B}{\gamma} + \frac{v_B^2}{2g} + \cancel{z_B}$$

$$\frac{(10 \text{ psi})(144 \text{ in}^2/\text{ft}^2)}{62.4 \text{ lb/ft}^3} + 0 = \frac{(5 \text{ ft})\gamma}{\gamma} + \frac{v_B^2}{2(32.2 \text{ ft/s}^2)}$$

$$2(32.2 \text{ ft/s}^2)\left[\frac{(10 \text{ psi})(144 \text{ in}^2/\text{ft}^2)}{62.4 \text{ lb/ft}^3} - 5 \text{ ft}\right] = v_B^2$$

$$v_B = 34.1 \text{ ft/s}$$

$$Q = v_B A_B = (34.1 \text{ ft/s})\left(\frac{\pi(0.5 \text{ ft})^2}{4}\right) = 6.70\frac{\text{ft}^3}{\text{s}}$$

The horizontal force on the water is given by:

$$F_{ext,x} = Q\rho\Delta V_x = \left(6.70\frac{\text{ft}^3}{\text{s}}\right)\left(1.94\frac{\text{slug}}{\text{ft}^3}\right)\left([34.1-0]\frac{\text{ft}}{\text{s}}\right)\left(\frac{1 \text{ lb}}{\text{slug-ft/s}^2}\right) = 443 \text{ lb}$$

As noted above, this horizontal force to the left is exerted on the water by the pump and other structures (the pipes) that are attached to the boat. Therefore, the water exerts an equal force to the right, and that force must be resisted by the cable to hold the boat in place. Thus, the cable experiences a tension of 443 lb.

6. (a) The power supplied to the water equals $Q\gamma\Delta h_p$. We can find Q from the given velocity exiting the nozzle and the nozzle diameter. Δh_p can be found in a few ways, but the easiest is to apply the energy equation between two points where p, z, and v are all known: (1) the surface of the lower reservoir and (2) the nozzle outlet. Because friction is negligible between these locations, the difference in head between them equals Δh_p.

$$Q = v_2 A_2 = \left(53.1\frac{\text{ft}}{\text{s}}\right)\left(\frac{\pi(0.333 \text{ ft})^2}{4}\right) = 4.62\frac{\text{ft}^3}{\text{s}}$$

$$\cancel{\frac{p_1}{\gamma}} + \cancel{\frac{v_1^2}{2g}} + z_1 + \Delta h_p = \cancel{\frac{p_2}{\gamma}} + \frac{v_2^2}{2g} + z_2$$

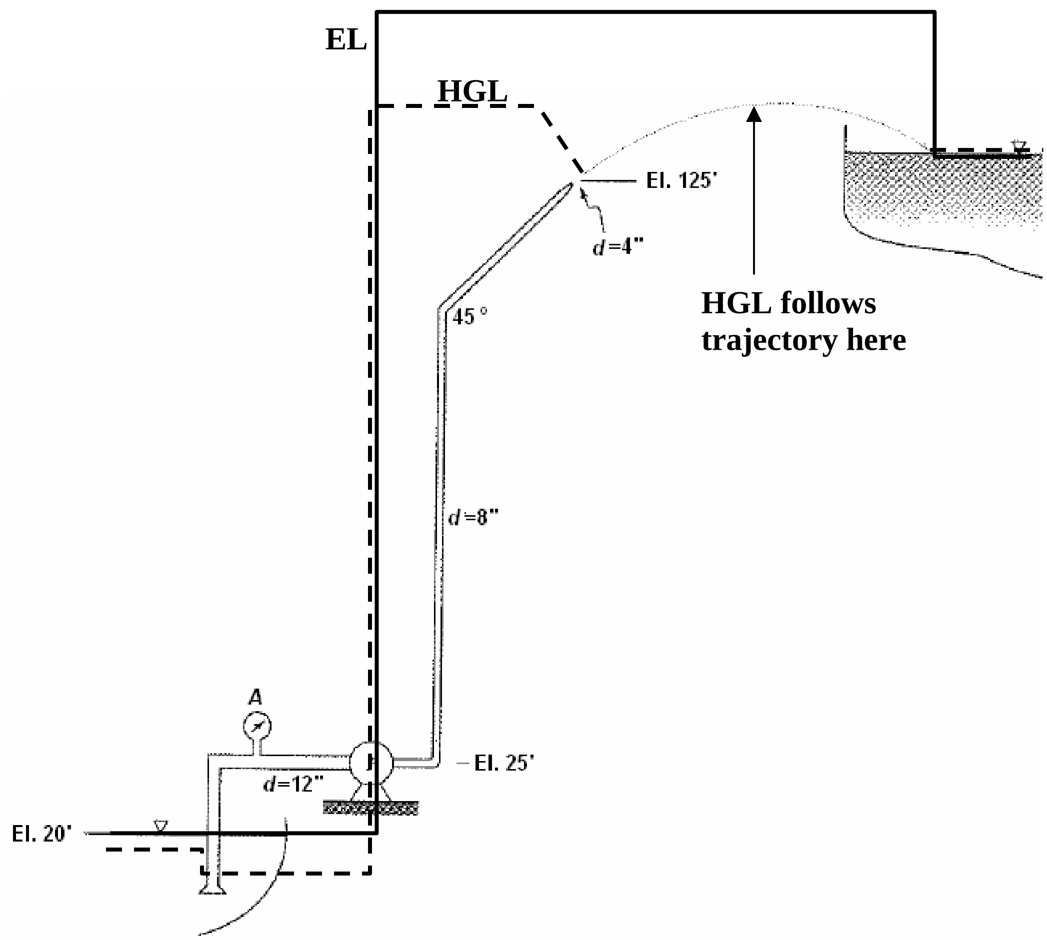
$$\Delta h_p = \frac{v_2^2}{2g} + z_2 - z_1 = \frac{(53.1\text{ ft/s})^2}{2(32.2\text{ ft/s}^2)} + 125\text{ ft} - 20\text{ ft} = 148.8\text{ ft}$$

$$P = Q\gamma\Delta h_p = \left(4.62\frac{\text{ft}^3}{\text{s}}\right)\left(62.4\frac{\text{lb}}{\text{ft}^3}\right)(148.8\text{ ft})\left(\frac{1\text{ HP}}{550\text{ ft-lb/s}}\right) = 78.0\text{ HP}$$

(b) Since v and p are both zero at the surface of the lower reservoir, the EL at that location equals z, i.e., it is at the surface. The only way that the total energy of the water changes is if energy is added or removed mechanically, or if it is lost due to friction. In this system, energy is added by the pump, but is not gained or lost anywhere else in the system except in the upper reservoir. Therefore, the level of the EL line remains constant until the pump, undergoes a step increase at that location, then remains constant again until the upper reservoir, where it drops. For the same reason as given above ($v = p = 0$), once in the upper reservoir, the EL coincides with the surface of the reservoir. Between the pump and the upper reservoir, the EL must be above the top of the jet, because it is at a level equal to $p/\gamma + z + v^2/2g$. At the top of the jet, $p = 0$, z equals the elevation of the top of the jet, and v^2 must be >0, so the sum of the three terms is >z.

At all locations, the HGL is below the EL by an amount equal to $v^2/2g$. Thus, wherever $v = 0$, the HGL and EL coincide, and wherever v is finite, the HGL is below the EL. Thus, the HGL is below the EL everywhere other than at the surfaces of the two reservoirs. Furthermore, the height of the HGL equals $p/\gamma + z$, so wherever $p = 0$, it coincides with z; this is the case everywhere that the water is exposed to the atmospheric, such as in the free jet. Finally, we note that when the pipe diameter decreases, the fluid velocity increases, so the gap between the HGL and EL must increase.

These ideas are all incorporated into the sketches of the EL and HGL shown in the figure below.


EL
HGL
El. 125'
d=4"
HGL follows
trajectory here
45°
d=8"
A
d=12"
El. 25'
El. 20'