# Resonant Circuits by Time and Frequency

## 1. Objectives

- Measure the time response of a series RLC circuit to a square wave input.
- Measure the frequency response of a series RLC circuit to a sinusoidal input.
- Correlate the time and frequency parameters (overshoot, quality factor, damping) between the time and frequency measurements.
- Display the output of a frequency sweep on a Bode plot.
- Compare the measurements to PSpice simulation results.

## 2. Reference

The response of RLC circuits is covered in your textbook. Look over this material and in particular the definitions of natural oscillation frequency, and the criteria between under damped, critically damped, and over damped series RLC circuits. Make sure you understand how to calculate the step response of series RLC circuits.

Review the usage of the laboratory oscilloscope and function generator.

## 3. Circuits

Figure 1 shows a series RLC circuit driven by a voltage source. The voltage source V1 from the function generator will be either a square wave for time-domain measurements, or a sine wave for frequency-domain measurements. The two node voltages of interest are labeled V2 and V3 as shown below. The resistor R1 will be one of three possible values: 200 Ω, 2.0 kΩ, or 20 kΩ. Changing the value of R1 will vary the damping factor for this circuit and will cause significant changes in the time and frequency domain responses.

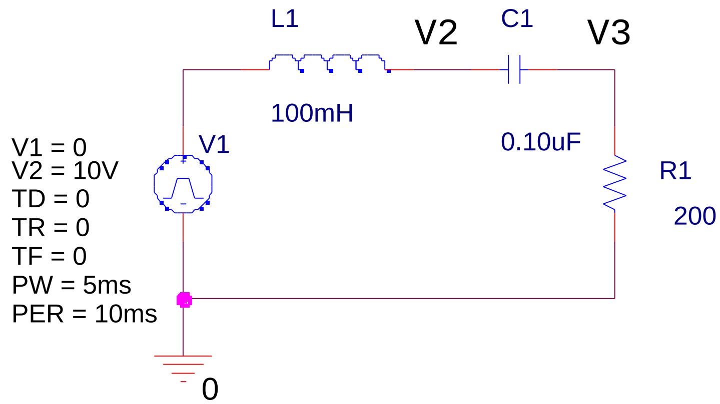


**Figure 1. Series RLC circuit driven by a voltage source.**

## 4. Components and specifications

| *Quantity* | *Description* | *Comments* |
|---|---|---|
| 1 | 100 mH inductor | |
| 1 | 0.10 μF capacitor | |
| 1 | 200 Ω resistor | |
| 1 | 2.0 kΩ resistor | |
| 1 | 20 kΩ resistor | |

## 5. Discussion

### *5.1 Second order circuits*

Second order circuits are those that contain at least two reactive elements, i.e. inductors or capacitors, and whose defining differential equation involves terms with up to a second derivative in time. For example, the series RLC circuit of Figure 1 is described by the second order differential equation:

$$L_1 \frac{d^2 I}{dt^2} + R_1 \frac{dI}{dt} + \frac{1}{C_1} I(t) = \frac{dV_1(t)}{dt}.$$

The characteristic equation whose roots define the natural response of the circuit is:

$$s^2 + \frac{R_1}{L_1} s + \frac{1}{L_1 C_1} = 0.$$

The roots of the characteristic equation can be expressed as

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$$

where the resonant frequency (in radians/second) is:

$$\omega_o = \frac{1}{\sqrt{L_1 C_1}}$$

and the damping rate (in nepers/second) is:

$$\alpha = \frac{R_1}{2L_1}.$$

### *5.2 Response of second order circuits*

The response of a second order circuit will be of the general form

$$v(t) = Ae^{s_1 t} + Be^{s_2 t},$$

where $s_1$ and $s_2$ are the roots to the characteristic equation above. Three distinct cases emerge depending upon the relative magnitude of $\alpha$ to $\omega_o$.

1. If $\alpha > \omega_o$, then $\alpha^2 > \omega_o^2$ and the square root will return two real values, making the two values of s both real, negative numbers. This results in a purely decaying time response, which is termed over-damped.
2. If $\alpha = \omega_o$, then the square root term vanishes, creating two equal, real, negative roots for s. This situation is termed critically damped.
3. Finally, if $\alpha < \omega_o$, then $\alpha^2 - \omega_o^2 < 0$, and the square root will return two imaginary components which will create a complex conjugate pair of roots for s. This is termed under-damped, and the time response will have an oscillatory component with an angular frequency of

$$\omega_{osc} = \sqrt{\omega_o^2 - \alpha^2} .$$

   These oscillations will decay with a time constant equal to $\tau = 1/\alpha$ seconds.

Two other parameters can be used to classify the damping of a second order system: the damping factor $\zeta$, defined by $\zeta = \alpha/\omega_o$, and the quality factor Q, defined by $Q = 1/(2\zeta)$.

1. $\zeta > 1$ or $Q < 0.5$: the system is over-damped.
2. $\zeta = 1$ or $Q = 0.5$: the system is critically damped.
3. $\zeta < 1$ or $Q > 0.5$: the system is under-damped.

The quality factor Q and damping factor $\zeta$ can be directly used to predict some key parameters of the time and frequency response characteristics for different input signals.

### 5.2.1 Step response

If a step function excitation is applied to the second order system, it may ring if it is under-damped. This produces a decaying oscillation, which initially overshoots its final steady-state value. Only for the under-damped case, the step response overshoot is related to Q and $\zeta$ by

$$OS = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) = \exp\left(\frac{-\pi}{\sqrt{4Q^2-1}}\right).$$

### 5.2.2 Sinusoidal steady-state response

If a sinusoidal steady-state signal is applied to the second order system, under-damping will lead to a peaking in the frequency response at a frequency near to $\omega_{osc}$. Only for the under-damped case, the gain peaking factor is related to Q and $\zeta$ by

$$GP = \frac{\frac{1}{2\zeta}}{\sqrt{1-\zeta^2}} = \frac{2Q^2}{\sqrt{4Q^2-1}} .$$

For highly under-damped circuits, i.e. those with a large Q, the gain peaking is approximately equal to Q, which then represents the ratio of the amplitude at this peaking frequency to the amplitude of the excitation.

### *5.3 Resonance*

The series RLC circuit can be better understood by sinusoidal steady-state analysis. The equivalent impedance of the series RLC combination is

$$Z_{eq} = j\omega L_1 + \frac{1}{j\omega C_1} + R_1 .$$

At the resonant frequency of $\omega = \omega_o = (L_1 C_1)^{-1/2}$, the inductive and capacitive reactances cancel each other, leaving only the pure, real-valued resistance of $R_1$. At frequencies below resonance, the capacitive component dominates over the inductive, and the net impedance is that of a resistance with a capacitive reactance and thus a negative impedance phase angle. At frequencies above resonance, the inductive component dominates over the capacitive, and net impedance is that of a resistance with an inductive reactance and thus a positive impedance phase angle. The minimum impedance magnitude thus occurs at resonance, at which point the series RLC combination admits the maximum amount of current flow.

## 6. Prelab

### *6.1 Calculations*

The series RLC circuit will be constructed and measured with each of three sets of component values. The inductor and capacitor will remain fixed, but the resistor will be varied over the values of 200 Ω, 2.0 kΩ, and 20 kΩ. The different component value sets are tabulated below:

| Component Values | Inductor L1 | Capacitor C1 | Resistor R1 |
|---|---|---|---|
| Value Set 1 | 100 mH | 0.10 μF | 200 Ω |
| Value Set 2 | 100 mH | 0.10 μF | 2.0 kΩ |
| Value Set 3 | 100 mH | 0.10 μF | 20 kΩ |

For each of the component value sets, calculate the resonant frequency $\omega_o$ in rad/sec, the resonant frequency $f_o$ in Hz, the values of $\alpha$ in nepers/sec, and the decay time constant $\tau = 1/\alpha$ in seconds. From these values, then calculate the damping factor $\zeta$ and the quality factor Q, both of which are dimensionless. Classify each of the three component value sets as under-damped, critically damped, or over-damped.

### *6.2 PSpice simulation of the time-domain response*

Capture the circuit schematic of Figure 1 and use a pulse source for V1 and R1 = 200 Ω.

1. Set the properties of the V1 source for zero rise, fall, and delay time, a low voltage of zero and a high voltage of +5 V, a pulse width of 5 ms, and a period of 10 ms.
2. Create a new simulation profile for transient analysis over a time of 15 ms with a maximum step size of 10 μs.
3. Run the PSpice simulation and create a Probe graphical display of the voltages V1, V2 and V3 versus time.
4. Save a copy of the graphic display into another document.
5. Calculate, from this PSpice output, the degree of overshoot present in V2, and note whether the response is underdamped, critically damped, or overdamped.

Repeat the above procedure for R1 = 2.0 kΩ and R1 = 20 kΩ.

### *6.3 PSpice simulation of the frequency-domain response*

Return to the captured circuit schematic and modify the properties of the V1 source by setting VAC to 5 V. This will create a 5 V AC source for the AC analysis, which will ignore the other transient parameters of V1 that were entered previously.

Simulate the circuit for each of these R1 values: 200 Ω, 2.0 kΩ, and 20 kΩ.

1. Create a new simulation profile for AC analysis that sweeps the frequency from 100 Hz to 10 kHz in 2 logarithmic decades with 50 points per decade.
2. Run the PSpice simulation and create a Probe graphical display of the voltages V1, V2 and V3 versus frequency.
3. Measure the amount of peaking present in V2, and save a copy of the graphical display to a document.
4. Experiment with other ways to display the frequency-domain simulation results. Plot the magnitude of the voltage in decibels versus frequency (magnitude Bode plot). Next, plot the phase of the voltage in degrees versus frequency (phase Bode plot). Practice with Probe to display a magnitude and phase Bode plot in the same window (two different vertical axes) and save these to a document.

## 7. Experimental procedures

### *7.1 Instruments needed for this experiment*

The instruments needed for this experiment are a function generator and an oscilloscope.

### *7.2 Time domain measurements*

1. Construct the circuit of Figure 1, initially using a value of 200 Ω for R1. Drive the circuit from the function generator and configure the generator to produce a square wave with a period of 10 ms (frequency of 100 Hz), a low voltage level of zero and a high voltage level of +5.0 Volts. Monitor the voltage V1 with Channel 1 of the oscilloscope, which should be used to verify the

above configuration of the function generator settings. Use Channel 2 of the oscilloscope to measure either voltage V2 or V3.

2. Measure the step response overshoot OS of the voltage V2 by finding its maximum minus minimum and then dividing this by the amplitude of the step that caused it and finally subtracting one from the result. For example, if when V1 rises from 0 to 5 V, V2 rises from 0 to a peak value of 7.5 V, the overshoot is (7.5 V / 5.0 V) – 1 = 1.50 – 1.00 = 0.50, or an overshoot of 50 percent.
3. Change the value of R1 to 2.0 kΩ and repeat the above measurements.
4. Change the value of R1 to 20 kΩ and repeat the above measurements. If the voltage V2 does not exhibit any overshoot for these new values of R1, measure the decay time constant instead of the overshoot. The decay time constant is the time required for the transient to fall to 1/e of its initial value. The decay time constant may in some cases be more readily measured from V3 instead of V2. Examine both V2 and V3 waveforms and choose the one which provides the best signal for these measurements.

### *7.3 Frequency domain measurements*

1. After completing the above time-domain measurements, re-configure the function generator to produce a sinewave output with a peak value of 5 Volts and zero DC offset. Monitor the voltage V1 with Channel 1 of the oscilloscope to insure that you have configured the function generator properly for this. The frequency of the function generator will be varied from 100 Hz to 10 kHz, so you should first try manually scrolling the frequency up and down to insure that the amplitude of the sinewave does not vary over this frequency range.
2. Starting with a value of R1 of 200 Ω, monitor the node voltage V2 with Channel 2 of the oscilloscope. Sweep the frequency upwards from 100 Hz to 10 kHz and observe the amplitude of the V2 signal. You should see the signal amplitude first grow with increasing frequency and then decrease after a maximum was attained. Sweep the frequency up and down and determine the frequency which causes the voltage V2 to be maximized. Record this frequency as $f_o$. Measure the gain peaking GP of the circuit by dividing the maximum amplitude of V2 by the amplitude of V1 at this frequency.
3. After finding the frequency $f_o$ which maximizes the voltage V2, next find the bandwidth of the circuit. This is accomplished by sweeping the frequency away from the previously located maximum until the amplitude of V2 has fallen to a factor of 0.707 of its peak value. These frequencies, one below and one above the frequency of the maximum, are termed the half-power points, $f_1$ and $f_2$. The bandwidth of the circuit is the difference between the half-power points, $BW = f_2 - f_1$. Calculate the quality factor Q for the circuit as the peak frequency divided by the bandwidth, $Q = f_o/BW$.
4. Repeat the above measurements for an R1 value of 2.0 kΩ.
5. Repeat the above measurements for an R1 value of 20 kΩ. If the response does not exhibit a visible peaking in the frequency of V2, the node voltage V3 can be measured instead. If the gain peaking or Q is too small to accurately measure, just state this in your write-up.

# 8. Data analysis

### *8.1 Comparison to hand calculations*

1. Compare the measured resonant frequency $f_o$ to the value that was calculated in the prelab. If both the inductor and capacitor can vary from their nominal (printed) values by no more than 20 percent, what is the worst case variation in the resonant frequency that can be attributed to this component variation? Suggest any other possible origins of difference between the measured and calculated resonant frequencies.

2. Compare the measured quality factor Q or decay time constant $\tau = 1/\alpha$ to the values that were calculated in the prelab. How much variation is possible in the values for Q and $\tau$ if the component variations are each no more than 20 percent? Suggest other origins of difference between the measured and calculated values.

### *8.2 Comparison to PSpice time-domain analysis*

1. Compare the measured step response overshoot OS to that predicted by the PSpice simulations for each of the three values of R1. Calculate the predicted value of overshoot from the damping factor $\zeta$ that was calculated in the prelab. Which two of these values of overshoot best agree? Suggest the primary sources of disagreement in the third.

### *8.3 Comparison to PSpice frequency-domain analysis*

1. Compare the measured values of gain peaking GP to those predicted by the PSpice simulations for each of the three values of R1. Calculate the predicted value of gain peaking from the quality factor Q that was calculated in the prelab. Which two of these values of gain peaking best agree? Suggest the primary sources of disagreement in the third.

### *8.4 Construction of Bode plots*

1. Create a semi-logarithmic (Bode) plot of amplitude versus frequency for the node voltage V2. Plot the frequency axis from 100 Hz to 10 kHz, and on the vertical axis, plot the magnitude of the voltage in decibels (dB). Sketch three curves on the axes, one for each of the three values of R1. You can confirm your sketch by re-running the PSpice AC sweep and plotting the magnitude of V2 in dB.

# 9. Further research

You do not have to turn this section in.

1. Network analyzers are instruments which automate the process of frequency-domain measurements, like the simple frequency sweep that was performed in this laboratory. Find a recent Hewlett-Packard or Tektronix catalog and examine the specifications for an audio frequency range network analyzer (1 Hz to 100 kHz range). Try to learn from the catalog what this instrument can measure and to what accuracy it can acquire data. Also examine the purchase price of these instruments! From your experience in the laboratory, try to deduce what key components or subsystems must be part of the internal system of a network analyzer.

2. The frequency-domain measurements in the lab involved only the magnitude of the signals. If phase information were also measured, what additional parameters could be obtained? More expensive network analyzers are capable of resolving both magnitude and phase information. Consult a Hewlett-Packard or Tektronix catalog to find the performance differences between a scalar network analyzer and a vector network analyzer. Note also the difference in price!

## 10. Self-test

1. This laboratory should have convinced you that the time and frequency behavior of a circuit are very interrelated. For example, peaking the frequency domain is directly connected to overshooting in the time domain. Is there any time-domain or frequency-domain performance measure that cannot be obtained from measurements in the complementary domain?
2. What frequency-domain measurements would you make to determine step response overshoot?
3. What time-domain measurements would you make to determine gain peaking?
4. Which domain is the easiest for making measurements of resonant frequency?
5. Which domain is the easiest for making measurements of decay time?