Ocean 420 Physical Processes in the Ocean
Project 6: Waves
Answers

3. **Capillary waves**
The phase speed for capillary waves is given by

$$C^2 = \frac{g}{k} + \frac{k}{\rho_0}\varsigma$$

Here ς is the surface tension and is $0.074\, kg/s^2$.

a) Derive a formula for the group velocity as a function of wave number for capillary waves.

$$C_g = \frac{\partial \omega}{\partial k}, \; C = \frac{\omega}{k}$$

Plugging the second equation into the phase speed equation above, we get a relationship between ω and k.

$$C^2 = \frac{\omega^2}{k^2} = \frac{g}{k} + \frac{k}{\rho_0}\varsigma$$

Solving for ω,

$$\omega^2 = gk + \frac{k^3\varsigma}{\rho_0}$$

$$\omega = \sqrt{(gk + \frac{k^3\varsigma}{\rho_0})}$$

Take the derivative using the chain rule,

$$\frac{\partial \omega}{\partial k} = \frac{1}{2}(gk + \frac{k^3\varsigma}{\rho_0})^{-\frac{1}{2}} \cdot (g + \frac{3k^2\varsigma}{\rho_0})$$

$$C_g = \frac{g + \frac{3k^2\varsigma}{\rho_0}}{2\sqrt{gk + \frac{k^3\varsigma}{\rho_0}}}$$

b) Now calculate the phase and group velocities for waves with wavelength 25 cm and 1 mm.

First, $\Lambda = 0.25m$.

$$k_{.25} = \frac{2\pi}{\Lambda} = 25.1m^{-1}$$

$$C_{g,.25} = \frac{9.8m/s^2 + \frac{3\cdot(25.1m^{-1})^2 \cdot 0.074kg/s^2}{1025kg/m^3}}{2\cdot\sqrt{9.8m/s^2 \cdot 25.1m^{-1} + \frac{(25.1m^{-1})^3 \cdot 0.074kg/s^2}{1025kg/m^3}}} = 0.316m/s$$

$$C_{.25} = \sqrt{\frac{9.8m/s^2}{25.1m^{-1}} + \frac{25.1m^{-1} \cdot 0.074kg/s^2}{1025kg/m^3}} = 0.626m/s$$

Next, $\Lambda = 0.001m$.

$$k_{.001} = \frac{2\pi}{\Lambda} = 6283m^{-1}$$

$$C_{g,.001} = \frac{9.8m/s^2 + \frac{3\cdot(6283m^{-1})^2 \cdot 0.074kg/s^2}{1025kg/m^3}}{2\cdot\sqrt{9.8m/s^2 \cdot 6283m^{-1} + \frac{(6283m^{-1})^3 \cdot 0.074kg/s^2}{1025kg/m^3}}} = 1.01m/s$$

$$C_{.001} = \sqrt{\frac{9.8m/s^2}{6283m^{-1}} + \frac{6283m^{-1}\cdot 0.074kg/s^2}{1025kg/m^3}} = 0.675m/s$$

c) What is the ratio of the group velocity to the phase velocity for the two wavelengths in (b)?

For $\Lambda = 0.25m$, $\frac{C_g}{C} = \frac{0.316m/s}{0.626m/s} = 0.5$

For $\Lambda = 0.001m$, $\frac{C_g}{C} = \frac{1.01m/s}{0.675m/s} = 1.5$

4. You are observing the sea level at two stakes by a pier. Answer the following questions about the wave that is seen in the observations. The data is in an excel file. Time is the first column, the measurements at stake 1 in the 2nd, and stake 2 in the 3rd. The stakes are 2 meters apart.

a) What is the period of the wave?

Looking at one stake, this is the distance between two crests, 3.6s.

b) What is the phase speed of the wave?

Speed is distance over time. We know the two stakes are 2m apart, giving a distance. We can figure out the time by seeing how long it takes a crest to get from one stake to the next. This gives us all the required information to find its speed.

$$C = \frac{\Delta x}{\Delta t} = \frac{2m}{0.7s} = 2.85m/s$$

c) What is the wavelength of the wave?

We use the period and the phase speed to find the wavelength.

$$\Lambda = CT = 2.85m/s \cdot 3.6s = 10.26m$$

d) What is the amplitude of the wave?

This can be read straight from the graph, 0.2m.

e) Does the wave satisfy the deep water wave dispersion relation?

$$\omega^2 = gk$$

$$\omega^2 = (\frac{2\pi}{T})^2 = (\frac{2\pi}{3.6s})^2 = 3.04$$

$$gk = \frac{2\pi g}{\Lambda} = \frac{9.8m/s^2 \cdot 2\pi}{10.26m} = 6.00$$

No, this wave does not satisfy the deep water dispersion relation.

f) If the wave is a shallow water wave, what must the depth of the water be by the pier?

To be a shallow water wave, it needs to satisfy $\Lambda > 20H$.

Solving for H, we have $H < \frac{\Lambda}{20}$.

$$H < \frac{10.26m}{20} = 0.51m$$

Alternately, plug the given information into the shallow water dispersion relation and solve for H.

$$\omega = \sqrt{gH} \cdot k$$

$$H = \frac{\omega^2}{gk^2} = \frac{(2\pi/T)^2}{g(2\pi/\Lambda)^2} = \frac{\Lambda^2}{gT^2} = \frac{(10.26m)^2}{9.8m/s^2 \cdot (3.6s)^2} = 0.829m$$