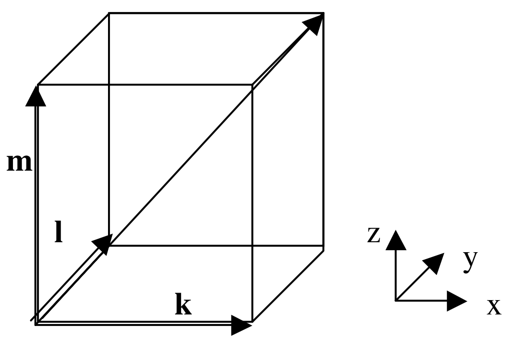
Two-dimensional wave propagation

So far we have talked about wave propagation in one-dimension. For two or three spatial dimensions, we vectorize our ideas of wavelength, wavenumber and propagation velocities. The basic waveform for surface displacement becomes:

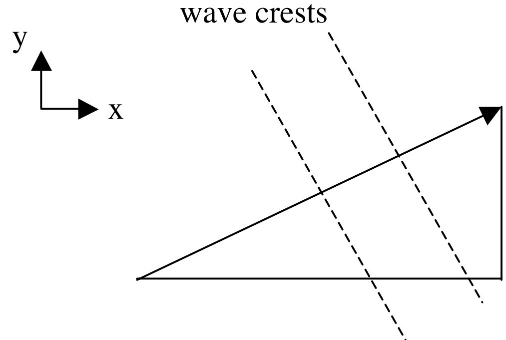
$$\eta(x,y,z,t) = a\cos(kx + ly + mz - \omega t)$$

The three-dimensional wavelength, is a vector drawn from crest to crest with components of distance between the wave crests in each of three dimensions, $\underline{\Lambda} = (\Lambda_x, \Lambda_y, \Lambda_z)$.

k = wavenumber in x-direction = $2\pi/\Lambda_x$
l = wavenumber in y-direction = $2\pi/\Lambda_y$
m = wavenumber in z-direction = $2\pi/\Lambda_z$



$\underline{k} = (k,l,m)$



wavenumber vector

We define a *wavenumber vector* oriented perpendicular to the wave crests, pointing in the direction of wave phase propagation. For surface waves, there is no vertical propagation, and we are only concerned with the two horizontal dimensions, as sketched above.

Phase and group velocity vectors are defined as,

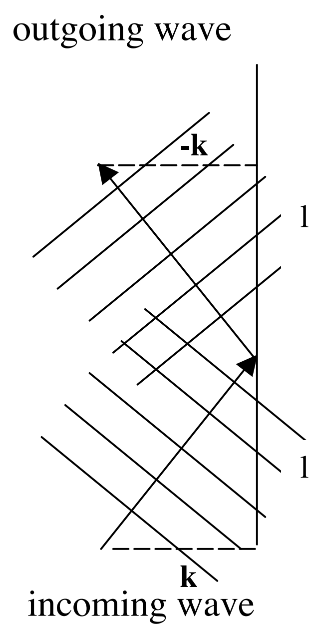
$$C = \omega\left(\frac{1}{k}, \frac{1}{l}, \frac{1}{m}\right)$$

$$C_g = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right)$$

Notice that the phase velocity does not satisfy ordinary vector composition rules. The phase speed in the direction of the advance of each crest is given by

$$C = \frac{\omega}{\left(k^2 + l^2 + m^2\right)^{1/2}}$$

As an example, consider a shallow water wave that is propagating towards a wall. What happens? The wave vector is given by (k,l). The wave will reflect from the wall with wave vector as shown below



Notice that the wave number in the y-direction (l) does not change, while the wave number in the x-direction (k) changes sign. We can write the *incoming* waveform as

$$\eta = a\cos(kx + ly - \omega t)$$

while the *outgoing* wave is written as

$$\eta = a\cos(-kx + ly - \omega t)$$

Note that the phase speed and group velocity have not changed in magnitude -- they have just changed directions. Also, the total wavelength has stayed the same.

In the open ocean all wind-generated waves are deep-water waves. As the waves approach the shore, *refraction* of waves tends to align the direction of propagation normal to the bathymetric contours (in-class exercise). Thus, crests are pretty much parallel to the shore when they break, regardless of the direction the deep water swell was traveling.

For a coastline of variable bathymetry, the effect will be to warp the incoming wave field along lines that are perpendicular everywhere to the bathymetry (Knauss, Fig. 9.18). For convex coastlines (headlands), wave rays are focused inwards, and we expect wave height there to become more intense as energy is concentrated into a smaller area. For concave coastlines (canyons), wave rays are focused outwards. The ray paths depend on the wavelength (or period) of a wave as well, longer wavelength and period waves will begin to be refracted in deeper water – energy from these waves may wrap around promontories. Sheltering of the coastline from wave energy can develop for shorter waves, which begin refraction relatively close to shore and have ray paths that are not bent around promontories (Knauss, Fig. 9.19).

We quantify the idea of wave refraction and its effect on wave height, by writing down equations for the conservation of energy flux.

First, what is energy?

We define the wave energy (density) as $E = 1/2\ \rho g a^2$. The units are J/m^2, which is the energy density per unit horizontal area of sea surface. A wave has *potential energy*, from moving parcels up and down in the gravity field, which is proportional to the amplitude squared. A wave also has *kinetic energy*, associated with the water parcel velocities and kinetic energy is also proportional to amplitude squared. In surface gravity waves, the contributions from potential and kinetic energies are equal, and we could derive the above expression for energy by averaging our expressions for sea surface amplitude (related to potential energy changes) and water parcel velocity over one wavelength. For instance, consider a wave with 1m amplitude, the energy of that wave will be

$$E = \frac{1}{2}\rho g a^2 = .5 * 1025 * 9.8 * 1 * 1 Jm^{-2} = 5022 Jm^{-2}$$

The energy is the total energy in a water column of a given area, where the area in the sketch below is s*dx and *s* is the perpendicular distance between two wave rays (wavenumber vectors). The energy flux is the *rate* of energy flow through the area

Energy flux (in Watts) = E * dx/dt * s
= energy * group velocity * s = E C_g s

For instance, for a shallow water wave traveling in 4000 m of water with amplitude of 1 m, we have

$$C_g = \sqrt{gH} = 198 m/s$$

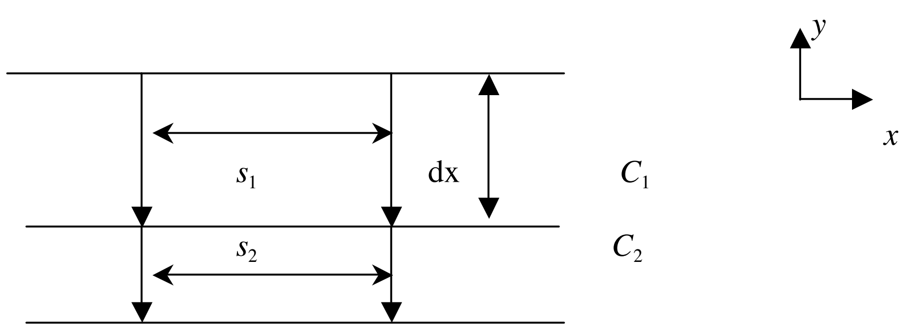
Thus, the energy flux 994000W/m, and for a 1m meter section of wave (measured along the wave crest, this gives about 1 MegaWatt.

For a Deep water wave, we have $C_g = \frac{1}{2}\frac{g}{\omega} = \frac{1*9.8}{2*2*\pi} m/s = 7.8 m/s$

Thus, the energy flux would be about 40,000W/m, quit a bit smaller than for the deep water wave.

Energy flux is conserved along a ray-path.

Example 1



For *shallow water waves* approaching a straight coastline head-on, there is no refraction. *s* is constant. However, wave speed decreases because *H* is decreasing

$$E_1 C_1 s_1 = E_2 C_2 s_2$$
$$\frac{E_2}{E_1} = \frac{C_1}{C_2} = \sqrt{\frac{gH_1}{gH_2}} = \sqrt{\frac{H_1}{H_2}}$$

Energy tends to increase as the waves move onshore – waves steepen (and break!)
For instance, if a shallow water wave with amplitude of 1m moved from 4000m of water to 10 m of water, what would happen to the amplitude of the wave?
$\frac{E_2}{E_1} = \frac{a_2^2}{a_1^2} = \sqrt{\frac{4000}{10}}$, thus, the amplitude of the wave would increase to 4.5 m

Example 2

For a more complicated example as in Knauss, Fig. 9.18, waves are refracted, changing cross-sectional area, and wave speed slows approaching shore. Let's consider a headland:

$$E_1 C_1 s_1 = E_2 C_2 s_2$$
$$\frac{E_2}{E_1} = \frac{C_1 s_1}{C_2 s_2} = \frac{s_1}{s_2}\sqrt{\frac{H_1}{H_2}}$$

Wave refraction by the headland induces another tendency for wave amplitude to increase moving onshore in this area, because $s_1 > s_2$.

Finally, the *rule of thumb* is that when the ratio of the wave height to water depth (a/H) increases to between 0.7 and 0.8, the wave becomes unstable and breaks.