Waves

What is a wave? It is a physical process whereby energy (or information about boundary conditions or forcing) is transported through a medium without any significant transport of the material in that medium. Each type of wave is characterized by a **restoring force** that leads to the oscillatory behavior. For example, for waves on the surface of the ocean, the restoring force is gravity…if you raise a water parcel from the sea surface and release it, gravity tends to pull it down again.

A simple wave form is the propagating (or progressive) sinusoidal wave in only one spatial dimension, described by the function for the displacement of a surface (say the sea surface) as a function of space and time:

$$\eta(x,t) = a\cos(kx - \omega t)$$

See Knauss, Figure 9.2 for a diagram of this function. Here are some of the basic quantities that describe a wave.

1. Amplitude, a : the magnitude of the oscillation. [*Wave height,* H = height from crest to trough, $H = 2a$.]

2. Wavelength of wave, Λ: distance from crest to crest. [*wavenumber* $k = 2\pi/\Lambda$.]

3. Period of wave, T: time for a single oscillation. [*frequency* $\omega = 2\pi/T$.] Units are radians per second. Alternate units are cycles/sec. 2π radians = 1 cycle.

4. Phase, $\varphi = kx-\omega t$: measure of the location within a wave cycle. At a given x and t, the phase will tell you whether or not you are at a wave crest, a trough or someplace in between. Phase varies from 0 to 2π and then repeats.

Crests occur when $\varphi = 0, 2\pi, 4\pi, \ldots$ (where cos $\varphi = 1$).
Troughs occur when $\varphi = \pi, 3\pi, 5\pi, \ldots$ (where cos $\varphi = -1$).
Surface displacement goes through zero at $\varphi = \pi/2, 3\pi/2, 5\pi/2, \ldots$

5. Phase velocity (phase speed if one spatial dimension) quantifies how fast crests (or troughs) travel. Following a crest, the phase is constant (same point in wave cycle) so $kx-\omega t$ = constant, and $\Delta x/\Delta t = \omega/k$. Thus we define phase speed (C for "celerity") as

$$C = \omega/k = \Lambda/T.$$

6. Dispersion relation: relationship between ω and k, $\omega = f(k)$, that characterizes a type of wave. Based on the dispersion relation waves are classified into two kinds:

(i) Nondispersive: $\omega = c*k$, where c is constant, so all wave numbers and frequencies have the same phase speed

(ii) Dispersive: $w = F(k)k$, so different wave numbers have different phase speeds.

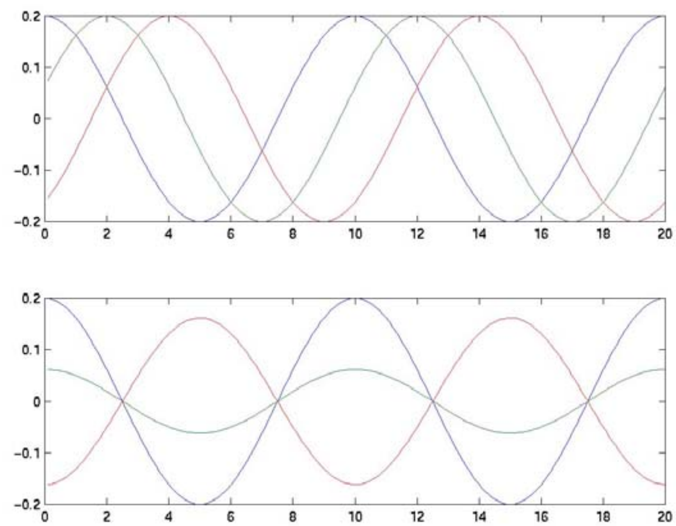
Standing waves

Some waves do not travel, but instead oscillate in place. A standing wave can be thought of as two waves traveling in opposite directions with equal amplitudes and the same wave number and frequency. That is

$$\eta(x,t) = \frac{a}{2}\cos(kx - \omega t) + \frac{a}{2}\cos(kx + \omega t)$$

Using trigonometric identities, we can write this as

$$\eta(x,t) = a\cos(kx)\cos(\omega t)$$



Shown in the next figure are two types of waves. The top picture is for a propagating wave at three different times (blue at t=0, green at t=2 and red at t=4 for a period of 10 units) as a function of x. The lower picture is for a standing wave at the same times.

7. *Group velocity*, $C_g = \frac{\partial \omega}{\partial k}$: velocity at which the wave propagates energy.

Why is this different from phase velocity? First, how do we define energy? It can be different for different waves, but generally it is proportional to the square of the amplitude, Energy ~ a^2. Any forcing will generate waves at several, often closely related frequencies, and since there is a relationship between frequency and wavenumber, the forced waves will also have several different wavelengths. If the waves are dispersive, these different wave components will travel at different speeds. Dispersive waves interfere constructively and destructively to produce a pattern of displacements that can be quite complicated. The group velocity is the rate at which the displacement (made up from the sum of all the different wave components) travels.

Let's look at the simplest possible case - two waves traveling in the same direction at slightly different wavenumbers and frequencies. At any given time, there will be points in space where the waves constructively interfere to produce an oscillation twice as strong -- and there will be points in space where the waves destructively interfere to produce a small amplitude. See Knauss, Figures 9.7 and 9.8. The packets of constructive interference are termed wave "groups". Both waves are moving off to the right in Fig.

9.8 and the wave groups will also propagate to the right -- but at a much slower rate than the individual wave crests.

For Figure 9.8, the group velocity is the rate at which the sum

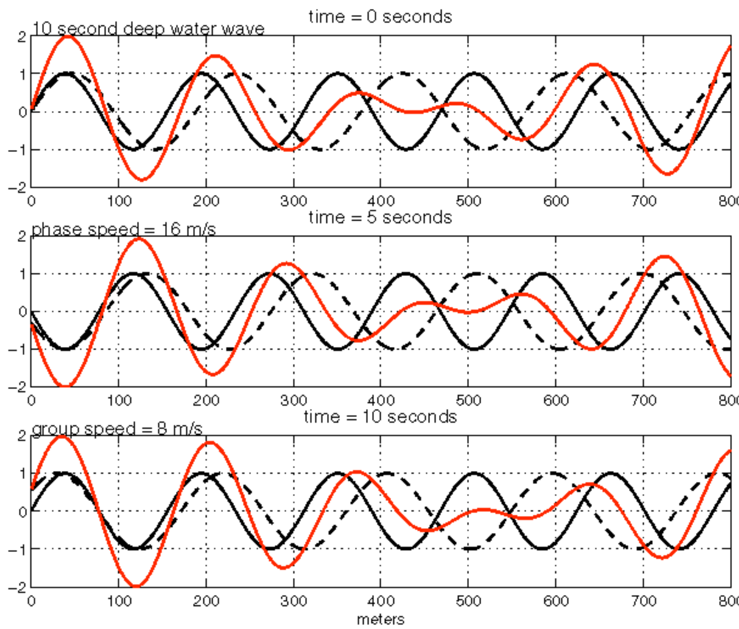
$$\eta = a\sin(k_1 x - \omega_1 t) + a\sin(k_2 x - \omega_2 t)$$

translates to the right. This occurs at a speed = $\dfrac{\omega_2 - \omega_1}{k_2 - k_1} = \dfrac{\Delta\omega}{\Delta k}$. See Knauss, Box 9.2 for the derivation (note: there is a typo in the 2nd edition, correct result is below). The sum of the two waves is an oscillation with the average wavenumber and average frequency of the two waves, modulated by a wave of wavenumber Δk and frequency $\Delta\omega$

$$\eta = 2a\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\sin\left[\frac{(k_1 + k_2)}{2}x - \frac{(\omega_1 + \omega_2)}{2}t\right]$$

Real forcing often generates waves over a small range of frequencies and wavenumbers, so that Δk and $\Delta\omega$ are small. The result is that the modulation has a much longer wavelength and longer period than the individual waves that make it up. Since we often have a theoretical functional relationship, the dispersion relationship, between ω and k, the group velocity is defined in a continuous sense as $C_g = \dfrac{\partial\omega}{\partial k}$.

[Notation alert: Knauss uses a script V for group velocity, rather than the C_g used here.]



A difference between phase and group velocities only occurs when the phase speed is not a constant.

Nondispersive waves: $\omega = ck$, where c is a constant. Then, $\omega/k = \partial\omega/\partial k = c$ = constant. group velocity = phase velocity.

Dispersive waves: $\omega = F(k)k$, so $C_g = \dfrac{\partial F}{\partial k}$, and group and phase velocities are different.

For the remainder of the course, we will be considering the various kinds of waves that are found in the ocean. These various waves are distinguished by:

(1) Frequency distribution. See Knauss, Figure 9.1. The *spectrum* is a plot of the distribution of wave energy as a function of frequency. How would we measure a wave spectrum? Suppose we have a time series at one location, say from a pressure gauge on the bottom of the ocean. Through a process called Fourier decomposition, we can separate out the sines and cosines at different frequencies that add up together to make the observed signal. The amplitude (squared) of the wave component for each frequency is proportional to the energy at that frequency.

(2) Dynamics. (What are the important terms in $F = ma$?)

(3) Dispersion relation, $\omega = F(k)$. How the wave moves energy and information around in the ocean.

(4) The form of the wave. What the wave looks like, in terms of variables we might be interested in measuring such as η (surface displacement), p (pressure), $\underline{u}$ (particle velocity), etc.

Generally we will not be concerned with solving the wave equations to get the solutions. We will focus on the terms balances, the form of the solutions and dispersion relation, and the role of these waves in ocean processes.