II. Internal waves in continuous stratification

The real ocean, of course, is continuously stratified. For continuous stratification, $\rho = \rho(z)$, internal waves can propagate not only horizontally (as they did along the interface between layers) but *vertically* as well, **k** = (k,l,m). When these waves break, they generate vertical mixing, and breaking internal waves are one of the important mechanisms that contribute to vertical turbulent eddy diffusivity.

Before we can discuss internal waves in a continuously stratified fluid, we need to understand the *restoring* force. Recall that we previously discussed what would happen to a fluid parcel depended on the background density profile (see figure 2.4 in Knauss) and on the *compressibility of the fluid*. We know that density depends on temperature, salinity and pressure. As the parcel moves down in the water column, it is compressed by the water above it and the density increases. To examine this effect further, we need to look at the difference between **in situ** temperature and **potential temperature**. The in situ temperature is simply the observed temperature.

To understand the potential temperature, consider a fluid parcel encased in an insulated balloon at a depth of 5000m with in situ temperature of 1.00 degree C and salinity of 35. If this water then rises toward the surface, but does not mix with the surrounding water, it will expand and cool. When it reaches the surface, its temperature is 0.58 degrees C, which is the **potential temperature** θ**.** See figures 2.3 and 2.4 in Knauss for plots of temperature and potential temperature in the ocean.

Note: the compressibility of air is 1000 times greater than the compressibility of water.

To characterize a stratified water column, we can use *stability*, the (proportionate) rate of increase of the density with depth

$$stability = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial z}\right)$$

But now we also need to consider how density will change with pressure. Imagine the water parcel encased in a balloon forced upward from its equilibrium position where it is heavier than the surrounding water. The buoyancy force will tend to push the parcel downward to its equilibrium position. Similarly if the parcel is moved downward. The stronger the density gradient, the stronger the restoring force. However, it would be wrong to use in situ density for stability since the density of the water parcel will change as we move a it upward or downward in the water column. It will expand as it is moved up (density decreases) and contract (density increases) as it moves down. Thus, the in situ stability will tend to *overestimate* the buoyancy forces on the fluid parcel. Thus, the proper density gradient for determining stability is

$$stability = \frac{1}{\rho}\left(\frac{\partial \rho_\theta}{\partial z}\right)$$

where *potential density is calculated using the potential temperature.*

Now, what would be the oscillation period of the balloon? That frequency is given in terms of the Brunt-Väisälä frequency N, where

$$N = \left(-\frac{g}{\rho} \frac{\partial \rho_\theta}{\partial z} \right)^{1/2}$$

The Brunt-Väisälä or buoyancy frequency is a measure of the strength of the stratification.

Derivation. A spring oscillator with mass m and spring constant k has an equation of motion that looks like

$$\frac{\partial^2 x}{\partial t^2} + \frac{k}{m} x = 0$$

and a solution $x(t) = a\cos(\omega t), \quad \omega = \sqrt{\frac{k}{m}}$,

The equation of motion for buoyancy oscillation looks similar to the equation of motion for the spring

$$\frac{F}{m} = a$$

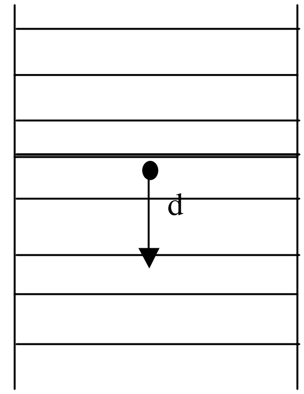
$$\frac{g\Delta\rho}{\rho} = \frac{\partial^2 z}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial t^2} + \frac{-g}{\rho} \frac{\partial \rho_\theta}{\partial z} z = 0$$

so, by analogy to the spring problem $\omega = N$ for internal waves.

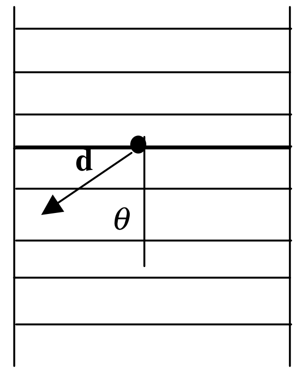
In-class exercise

Consider a column of water that is density stratified. You displace a parcel of fluid by some small distance d *and release it. The parcel eventually returns to rest in its neutrally buoyant position in the stratification. However, before it does so, it oscillates up and down.*



a) *What is the restoring force for the oscillations? How does it depend on* d*?*

b) *Now imagine that you could displace the water parcel by the same distance, but at an angle* θ *to the vertical. Now what is the restoring force? Do you expect the period of the oscillation to be any different? At 90°? At 45°?*



More strongly stratified fluids have higher frequency oscillations, because the restoring force for vertical displacements is larger. Consider the internal waves propagating at an angle to the vertical (with density surfaces horizontal). The restoring force is proportional to the number of density contours the fluid parcel crosses

$$\Delta\rho = \frac{\partial\rho}{\partial z}\Delta z = \frac{\partial\rho}{\partial z} d\cos\theta$$

As the angle decreases, it will start to pick up more and more restoring force per unit distance displacement. The largest restoring force and the *highest frequency* internal waves occur at a frequency $\omega = N$, and correspond to situations where water parcels are oscillating nearly vertically, perpendicular to the density surfaces. So the frequency of internal waves is always $\omega \leq N$. The period for internal waves in realistic stratification is typically *hours*, in many cases, long enough that the waves are affected by *rotation*.

Internal wave dynamical description

First, we will look at the equation of motion for internal waves in two spatial dimensions, the vertical direction (z) and a horizontal direction aligned with wave propagation (x).

The momentum equations are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Note that internal waves are *not hydrostatic*, that is the vertical acceleration is important in the vertical momentum balance. We also need continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

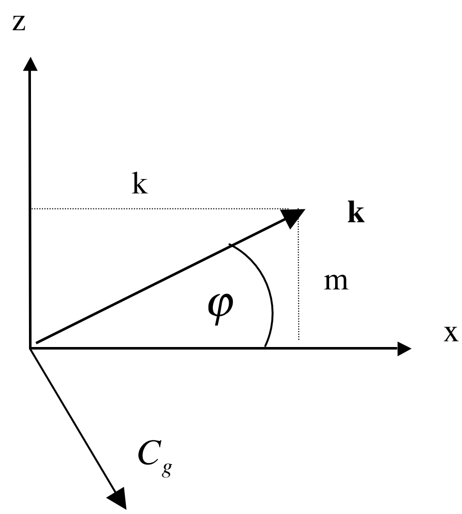
Finally, we need an equation that governs the density perturbations

$$\frac{\partial \rho}{\partial t} + w\frac{\partial \rho}{\partial z} = 0$$

The density balance says that changes in density come about because of vertical *advection* of the background density field.

Internal wave theory is a little complicated, and we will only look at some simple results.

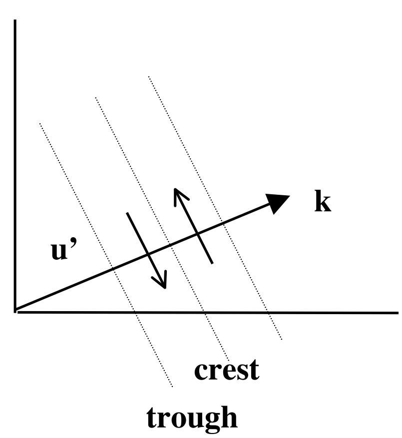
Internal waves have some counterintuitive properties. The wavenumber vector **k** points in the direction of the phase velocity, *C*, the direction in which wave crests propagate. However, energy (carried by the **group velocity** C_g) propagates at 90° to this angle. We define φ as the angle between the horizontal axis and the wavenumber vector. Wave energy travels at right angles to the direction that wave crests propagate (see Figure 10.9 in Knauss to visualize this). The angle is given by



$$\varphi = \cos^{-1}\frac{k}{(k^2+m^2)^{1/2}}$$

[Note that $\varphi = 90 - \theta$ where θ is the angle defined by Knauss.]

Water parcels move back and forth along wave crests and troughs, perpendicular to the wavenumber vector. Thus water parcels move parallel to the group velocity. Also, the vertical component of the phase and the group velocities are *opposite* to each other. Thus, if the group velocity has a positive vertical component and energy is propagating upwards, the phase velocity will have a negative vertical component, and phases propagate downward.



The **dispersion relation** for internal waves in a continuously stratified ocean is:

$$\omega^2 = N^2 \cos^2 \varphi$$

One immediate consequence of this expression is that internal waves have frequencies less than *N*.

Let's look at two extreme limits of this dispersion relationship.

(1) Let the wave number vector be horizontal ($m = 0$). Then energy propagates nearly vertically, and the water parcels oscillate nearly up and down. From the dispersion relationship, $\varphi = 0°$, $\cos\varphi = 1$ and $\omega \sim N$. This is consistent with our qualitative idea that oscillations with highest frequency will have water parcels that move most nearly vertical and in this limit they look like buoyancy oscillations.

(2) On the other hand, if the wavenumber vector is nearly vertical, the water parcels oscillate nearly horizontally (energy propagates horizontally as well) $\varphi \sim 90°$. From our thought experiment, you wouldn't expect any restoring force, but as the direction of water parcel oscillation tilts away from the vertical, frequency decreases (the oscillation period becomes longer) and rotation becomes important. The restoring force on purely horizontal water parcel displacements is the *Coriolis force* alone and internal waves in this limit approach **inertial oscillations**, which we'll consider next. This leads to the conclusion that the minimum frequency for internal waves is not actually 0, but rather is *f*, also known as the *inertial frequency*.

Sources of internal wave

What would happen if in a stratified fluid we had an energy source at a fixed frequency within a stratified fluid? Depending on the frequency, the response will look quite different. For low frequency, the response will be in beams that extend at a low angle away from the source with respect to the horizontal, while at higher frequencies, the beams of energy point more and more vertically. This can be understood in terms of the direction of energy propagation relative to the frequency. Note that in the limit that the source oscillates at very high frequency (higher than N), the response would not propagate. (see the figure below from Pond and Pickard).

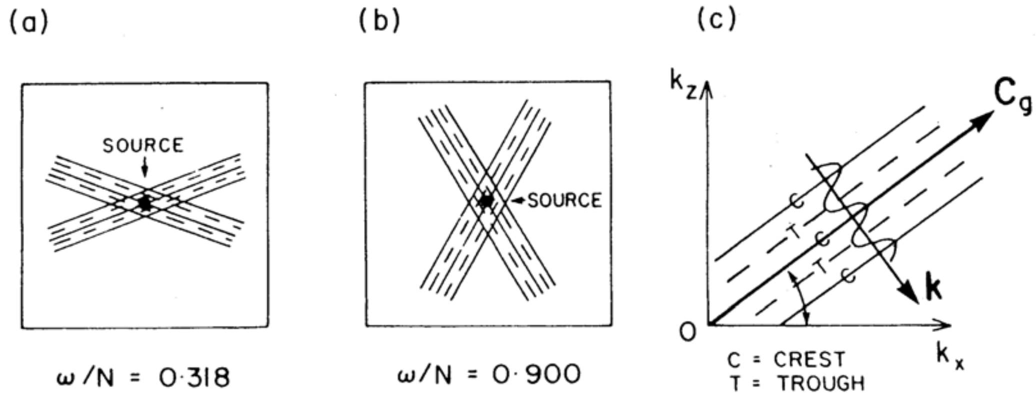


FIG. 12.16 *Phase configurations of internal gravity waves. Source is at centre and rays spread out in X-formation with (a) $\omega/N = 0.318$, (b) $\omega/N = 0.900$. Full lines represent troughs and dashed lines represent crests with propagation across the rays; (c) shows the relations between group velocity* ($\mathbf{C}_g$) *and phase velocity direction* ($\mathbf{k}$).