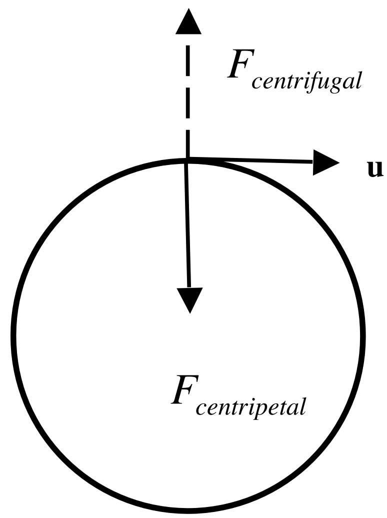
Inertial Oscillations

Imagine shooting a hockey puck across an ice-covered (i.e., frictionless) surface. As the hockey moves at speed *u* in a given direction for a time long enough that the Earth rotates a significant amount, the Coriolis force (magnitude = *fu*) will begin to deflect it to the right (N Hem). Since the Coriolis force acts at right angles to the motion, it does not change the *speed* of the hockey puck, but does change its *direction*. In the new direction of travel, the Coriolis force continues to deflect it to the right. Eventually, the hockey puck makes a circle.

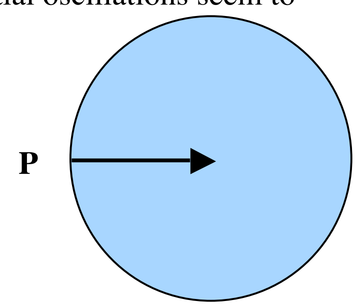


This is similar to the circular motion of a ball on a string. In the case of the ball, your arm exerts the inward force (the *centripetal* force) that causes the ball to continuously change direction and move in a circle. For our hockey puck, the Coriolis force is providing this inward normal "force".

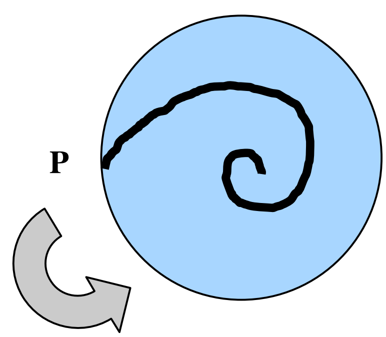
From the point of view of its own rotating reference frame, the ball feels a *centrifugal force* (a fictional force equal and opposite to the centripetal force) outwards that is balanced by the string tension pulling inwards.

Thought experiment:

We talk about the Coriolis force as a "fictitious force", yet inertial oscillations seem to exist because of it. In this experiment, we will try to reconcile these two notions.

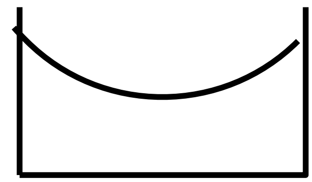


1) Consider a bucket of water. If the bucket is at rest, then the surface will be flat. Now freeze the water in the bucket. Push a hockey puck from the edge of the bucket **P** towards the middle.

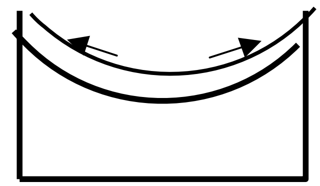


2) If you look at the same motion from a rotating reference frame, what would the motion look like?

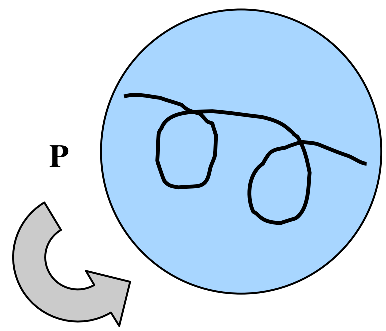
3) Now consider the same bucket of water, but this time give it a uniform rotation. What will the surface look like? The "centrifugal force", actually the tendency for the fluid to move in a straight line (*inertia*) will create a curved surface, higher near the outside edge.



4) Now freeze the water in the rotating bucket, and stop the rotation. Release a hockey puck from the edge of the bucket. What will the motion be? Oscillations along the curved surface.



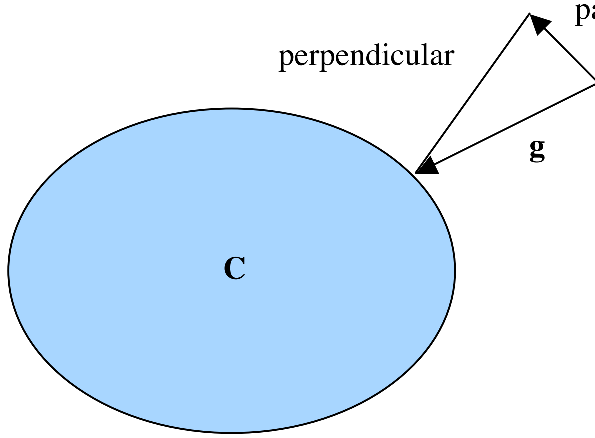
5) Finally, look at the motion from a rotating reference frame (from the top). We see oscillations moving up and down the curved surface, at the same time being deflected into circular motions.



This is the simplest type of time dependent motion caused by the Earth's rotation and is called an **inertial oscillation**.

In our thought experiment we saw that surface curvature plus rotation give us inertial oscillations in our frozen bucket, but why do they exist on earth? To understand this, we must recognize that the earth is not a sphere. Instead, because it is rotating, it bulges at the equator and is *ellipsoidal*.

The gravity vector **g** points toward the center of the Earth **C**. Because of the ellipticity there is a component of gravity *parallel* to the Earth's surface, as well as a component perpendicular to the surface. The component parallel to the surface is the restoring force for the inertial oscillations.



The equations of motion for the inertial oscillation are given by:

$$\frac{\partial u}{\partial t} - fv = 0$$

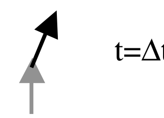
$$\frac{\partial v}{\partial t} + fu = 0$$

This is a set of coupled equations, which means that a change in the x-direction velocity, u, will cause a change in the y-direction velocity v, and vice-versa. To see how this works let's try an initial condition that $\mathbf{u} = (u,v) = (0,U)$ at $t = 0$ (give the hockey puck a push towards the north).

To see how these coupled equations interact, let's look also at time t=Δt

$$u(\Delta t) = u(0) + f\,v(0)\Delta t = f\,U\Delta t$$

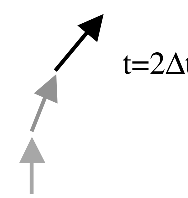
$$v(\Delta t) = v(0) - f\,u(0)\Delta t = U$$



and the next time step, t=2Δt

$$u(2\Delta t) = u(\Delta t) + f\,v(\Delta t)\Delta t = 2f\,U\Delta t$$

$$v(2\Delta t) = v(\Delta t) - f\,u(\Delta t)\Delta t = U\left(1 - f^2\Delta t^2\right)$$

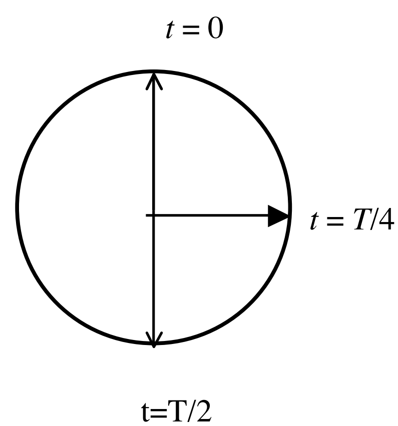


and you can see how the circular motion proceeds.

The solution of these equations for u and v (check and see!) is a circular oscillation with period $T = 2\pi/f$. $\omega = f$ is called the **inertial frequency**.

$$u = U\sin(ft)$$

$$v = U\cos(ft)$$



The velocity vector rotates to the *right* with time in the northern hemisphere and to the *left* in the southern hemisphere.

The intertial period is given by

$$T = \frac{2\pi}{f}$$

We can calculate the period, which is different at different latitudes. At 10N, it is 69 hours, at 30N it is 24 hours, and at 45N it is 16.9 hours.

How large is the radius of the circle for these oscillations? From the balance between centrifugal and Coriolis forces,

$$U^2/r = fU$$
$$r = U/f.$$

You can show that for a velocity of 5 cm/s, and $f = 1\times10^{-4} s^{-1}$, the radius of an intertial circle would be 500 m.
Inertial oscillations do not exist in a pure form like this. However, for both surface and internal gravity waves, in the limit as the frequency approaches *f,* are motions that have the character of inertial oscillations. Gravity waves cannot exist at frequencies lower than the inertial frequency. A spectrum anywhere in the ocean usually yields a strong peak at the inertial frequency.

Rotation effects on surface gravity waves.

Rotation effects will only be important for long waves with low frequencies – so we can assume outright that these are *shallow water* waves, even in the open ocean. Rotating surface gravity waves are called **Poincare waves** or **inertia-gravity** waves. Away from land boundaries, the tide propagates as an inertia-gravity wave.

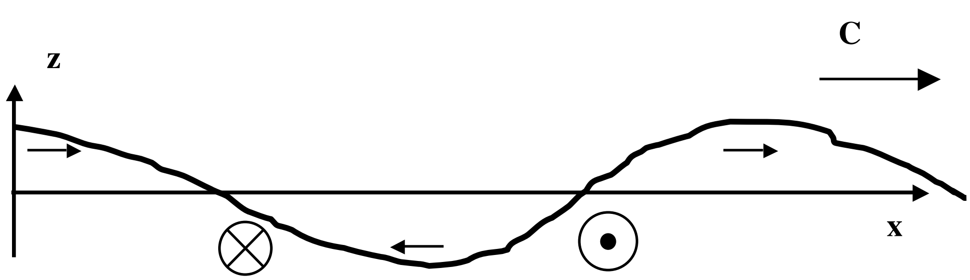
Dynamical balance
These waves have the same dynamics as shallow-water surface gravity waves, with the addition of the Coriolis force:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

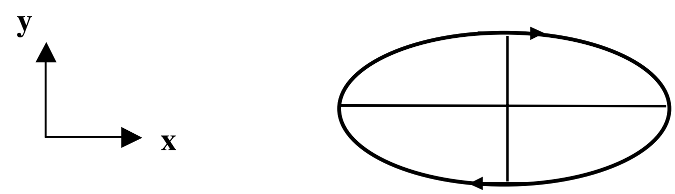
Solution form

We will take *x* to be in the direction of propagation of the wave (which is not changed by rotation) and *y* in the horizontal direction perpendicular to the direction of propagation. The sea surface elevation has the same cosine function as for non-rotating waves, but water parcel motions are modified by rotation.

Again, we are working on very large horizontal scales (in order for rotation to be important), so vertical velocity is negligible. Water parcel motions are nearly horizontal.



In the *xy*-plane, water parcels make *anticyclonic* elliptical motions. The ratio of the length of major axis (x) of the ellipse to the minor axis (y) is ω/f. For high frequency motions, the ellipse elongates along the direction of propagation and we recover the non-rotating solution. For $\omega = f$, the orbits become circular, like inertial oscillations.



Dispersion relation

The dispersion relation is similar to that for shallow water waves, except with the addition of f

$$\omega = \sqrt{f^2 + gHk^2}$$

The minimum frequency for these waves ($k \to 0$, large wavelength) is f, the inertial frequency.

Group velocity is given by

$$C_g = \frac{\partial \omega}{\partial k} = \frac{gHk}{\sqrt{f^2 + gHk^2}} \quad (18.1)$$

Unlike non-rotating shallow-water waves, these waves are *dispersive*. Unlike deep water waves, short(er) waves outrun long(er) waves.

Examining two limits shows how these rotationally modified gravity waves relate to waves we have already studied:

(i) for the shortest waves (large k, large ω/f) $C_g = \sqrt{gH}$ and we approach non-rotating shallow water surface gravity waves. This is the maximum group speed for inertia-gravity waves. This is also the limit for $f = 0$.

(ii) for the longest waves (small k), $C_g = 0$. These motions are *non-propagating* inertial oscillations.

We can *estimate how long the wave needs to be for rotation to be important*. The first term under the square root in (18.1) is as large as the second term when

$$f^2 \approx gHk^2$$

Take the length scale of the wave to be $L \sim 1/k$ (it's a scaling argument so we don't worry about the 2π.) Then solving for L gives

$$L_R = \frac{\sqrt{gH}}{f}$$

This length scale is given the name **Rossby radius of deformation** (for barotropic or external motions, in this case). This is an important length scale for ocean dynamics of all kinds (not just these waves), as it is a measure of the importance of rotation for a given phenomenon. For length scales much smaller than the Rossby radius of deformation, rotation is unimportant. How big is this scale? In the open ocean at midlatitudes $L_R = \dfrac{\sqrt{(9.8)(4000)}}{5\times 10^{-5}} \approx 4000km$.

For internal motions we can also define the **internal or baroclinic Rossby radius**

$$L_R = \frac{\sqrt{g'H_1}}{f} = NH/f$$

In this case, H is the vertical scale of motion, not the ocean depth. Since g' << g and the scale of the motion is less than the full ocean depth, the baroclinic Rossby radius is much smaller, of order 10km - 100 km. For $g' = 0.02m/s^2$, $H_1 = 500m$ and the same value of f as above, $L_R = 32km$. and Rotation becomes important for much shorter internal gravity waves than for surface gravity waves.

Rotation effects on internal gravity waves

Dispersion relation

The dispersion relation for internal waves is modified to become

$$\omega^2 = f^2 \sin^2\varphi + N^2\cos^2\varphi$$

[Note: Knauss gives the equivalent dispersion relation

$$\tan^2\theta = \frac{\omega^2 - f^2}{N^2 - \omega^2}$$

where now the angle θ is the angle that the group velocity makes with the horizontal.]

This more complete dispersion relation makes it clear that the minimum frequency for internal waves is the inertial frequency, *f*. For this limit, the angle that the wavenumber vector makes with the horizontal is 90°. Energy propagation and water parcel oscillations are nearly horizontal. This "wave" (which is not really a wave since phase speed goes to zero in this limit) looks like a set of vertically stacked inertial oscillations.

Water parcel motion

In a similar way to surface gravity waves, the effect of rotation on internal waves is to cause *anticyclonic* elliptical motions of water parcels, with motion perpendicular to the plane of wave propagation at crests and troughs.

The fluid parcels thus have elliptical motions in the plane perpendicular to the phase propagation and parallel to wave crests. These waves thus have motions that are similar to internal waves and inertial oscillations. In the high frequency limit, the particle motions are up and down, like buoyancy oscillations. In the low frequency limit, the particle motions are horizontal and the particle motions are nearly circles in the horizontal plane.