**Surface Gravity Waves**

For each of the waves that we will be talking about we need to know the governing equators for the waves. The linear equations of motion are used for many types of waves, ignoring the advective terms, as

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

The continuity equation still holds

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Knauss, Box 9.1 gives the mathematical details of the solution for *surface gravity waves*.

***I. Dynamical balance:*** pressure gradient term and local acceleration (+ gravity in the vertical equation.)

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

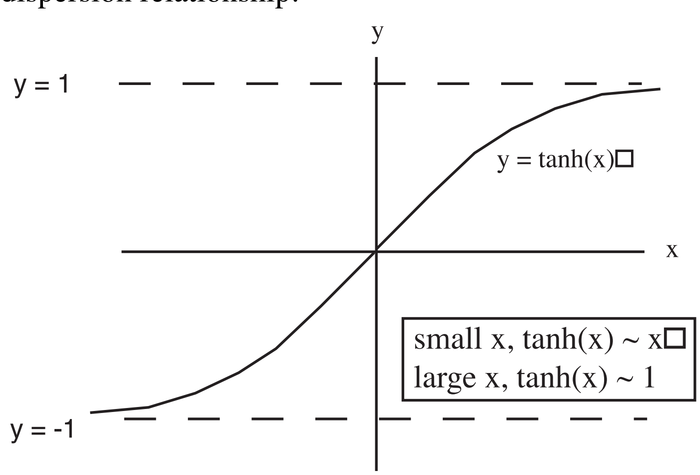
For these waves, we ignore rotation because of the short time and space scales (time scales less than the inertial period given by $2\pi / f$ ). We will consider later how these waves are modified by rotation.

***II. Dispersion relation***. Derived in book. *H* is the ocean depth.

$$\boxed{\omega^2 = gk \tanh kH}$$

We will consider two important limits of this dispersion relationship:

*A. Shallow water waves, or long waves***.**
For $k << 1/H$ ($\Lambda >>$ H) we are in the limit of small $kH$ so $\tanh(kH) \sim kH$. How long does the wavelength need to be for this approximation to be valid? If $\Lambda > 20$ H, the error in using this approximation is less than 3%.



Physically, this limit occurs when the depth of the water is shallow compared to the wavelength of the wave. *Wind waves approaching shore* behave like this limit (until they amplify sufficiently to become nonlinear and break.) *Tides and tsunamis*, very long waves, have a wavelength much greater than the open ocean depth, also are considered shallow water waves.

The dispersion relationship simplifies to

$$\omega = \sqrt{gH}\,k$$

Since the relationship is linear (for a given ocean depth), in this limit surface waves are nondispersive.

$$C = C_g = \sqrt{gH}$$

*B. Deep water waves*. For $k >> 1/H$ ($\Lambda << H$), the dispersion relation is in the large $kH$ limit, $\tanh(kH) \sim 1$. How deep does the water need to be for this approximation to be valid? If $\Lambda < 2H$, the error in using this approximation is less than 0.5%. This limit is applicable to wind-waves in the open ocean.

The dispersion relationship simplifies to

$$\omega^2 = gk$$

Since the relationship between $\omega$ and $k$ is not linear, deep water waves are dispersive and will have different phase and group velocities. Phase speed is given by

$$C = \frac{\omega}{\kappa} = \sqrt{\frac{g}{\kappa}}$$

Long waves (small $k$) have a faster phase speed than short waves. ("Long waves outrun short waves.") Group speed is given by

$$C_g = \frac{\partial\omega}{\partial\kappa} = \frac{g}{2\omega} = \frac{1}{2}\sqrt{\frac{g}{\kappa}} = \frac{C}{2}$$

*Example:* A storm offshore generates waves with a period of 20 s. If the storm is 1000km offshore, how long will it take the waves take to get to shore?

Although crests will travel at the phase speed, the energy from the waves generated by the storm will travel at the group speed

$$C_g = \frac{\partial \omega}{\partial \kappa} = \frac{g}{2\omega}$$

$$C_g = \frac{1}{2}\frac{g}{\omega} = \frac{1}{2}\frac{9.8m/s^2}{2\pi/20s} = 15m/s$$

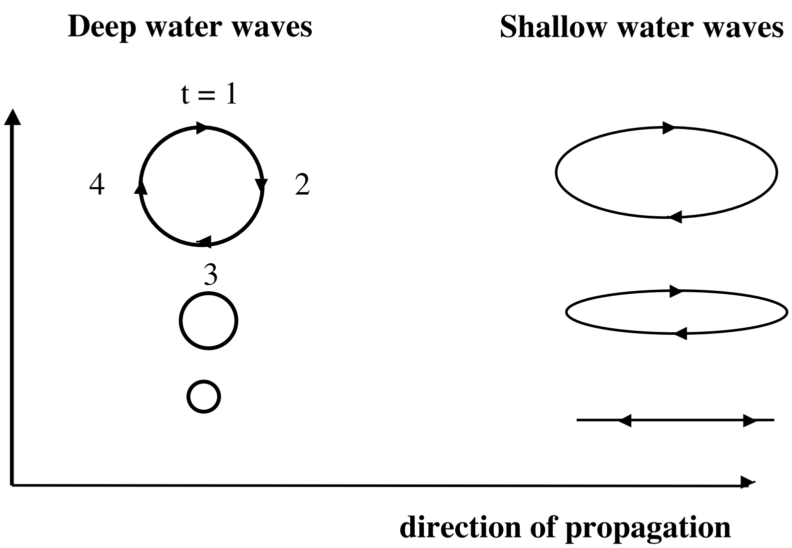
The time to get to shore will be given by the group velocity divided by the distance

$$\text{Time}=\frac{1000km}{15m/s} = 18hours$$

If the storm generates 10s waves, will it take more or less time to get to shore?

***III. Water parcel motions:***

For deep water waves, water parcels make circular orbits. The radii of these circular orbits decreases with depth exponentially, with the vertical decay scale proportional to wavelength. For shallow water waves, water parcels make elliptical motions. The horizontal excursion is independent of depth, while the vertical excursion decays linearly with depth and collapses to purely horizontal motion near the bottom.




*Why this difference?*

The vertical component of the equation of motion is:

$$\frac{\partial w}{\partial t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

For *deep water* waves, the force due to the vertical wave pressure ( $p' = p - p_{hydrostatic}$ ) gradient is significant compared to the hydrostatic pressure gradient. The perturbation pressure gradient (horizontal and vertical) decays away from the sea surface and the accelerations that lead to water parcel motions also decay away from the sea surface.

Horizontal and vertical motions are comparable in size. The wave pressure decays approximately exponentially: $p' = e^{kz}$.

For *shallow water* waves, the wave pressure gradient (p') can be neglected and pressure is hydrostatic, i.e, the horizontal pressure gradient just below the sea surface is felt throughout the depth of the fluid (which is why the horizontal water parcel motions do not change with depth). Vertical velocity is such as to conserve mass given convergences and divergences in the horizontal flow field.

Let's look at the shallow water limit quantitatively (this is the full solution to a later in-class exercise).

We get horizontal velocity from the *horizontal momentum equation* (acceleration = pressure gradient force)

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

The sea surface displacement is hydrostatically related to the horizontal part of the pressure gradient:

$$p(x,t) = \rho_0 g\eta = \rho_0 g a\cos(kx - \omega t)$$

So the pressure gradient force is

$$F_p = \frac{-1}{\rho_0}\frac{\partial p}{\partial x} = gak\sin(kx - \omega t)$$

We then *integrate* in time to get the horizontal velocity

$$u = \frac{gak}{\omega}\cos(kx - \omega t)$$

The *vertical momentum equation* is simply the hydrostatic balance at all times. We will get the vertical velocity from the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

First we *differentiate* $u$ with respect to $x$, then integrate with respect to $z$.

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} = \frac{gak^2}{\omega}\sin(kx - \omega t)$$

$$w = \frac{gak^2}{\omega}\sin(kx - \omega t)z + const.$$

To determine the constant, we apply the boundary condition that $w = 0$ at the bottom, $z = -H$ to find

$$w = \frac{gak^2}{\omega}\sin(kx - \omega t)(z + H)$$

Horizontal velocity does not depend on $z$, so $w$ is just a linear function of z. So the amplitude of vertical velocity gets smaller towards the bottom, and the ellipses become flatter and flatter approaching the bottom. Vertical velocity is zero under crests and troughs and rising or falling in between. Put them all together and you get the ellipses sketched above.

We can estimate the excursion that any parcel will take. At a fixed location, say x=0,

$$u = \frac{gak}{\omega}\cos(\omega t)$$

The excursion in the x direction will then be given by

$$x = \frac{gak}{\omega^2}\sin\omega t$$

The maximum distance that a fluid parcel will get from the origin will be

$$x = \frac{gak}{\omega^2} = gak / gHk^2 = a / Hk\,.$$

Notice that long wavelength waves (and therefore lower frequency cause a larger excursion of fluid parcels.

Notice that the size of the vertical velocity is much smaller than the horizontal velocity, that is

$\frac{|w|}{|u|} \approx kH << 1$ for shallow water waves. Thus, the motion is mostly horizontal as the fluid moves to conserve mass as the sea surface height changes. Note that the velocity with which particles move is *not* the phase or the group velocity, but rather, their speed depends on the phase of the wave and the amplitude of the wave. For linear waves, the particles do not have any net motion in the direction of the wave.

Example:
Assume that a tidal wave of period 12.42 hours exists in an estuary 50m deep. What would the wavelength of the wave be (assuming it is a shallow water wave)?

The dispersion relation tells us that

$$\omega = \frac{2\pi}{T} = \sqrt{gH}\,k = \sqrt{gH}\,\frac{2\pi}{\Lambda}$$

Thus, solving for wavelength we get

$$\Lambda = \sqrt{gH}\,T = \sqrt{9.8 * 50}12.42 hours * 60 \min / hr * 60 s / \min = 990 km$$

What is the maximum velocity of the tidal waves?

$$u = \frac{gak}{\omega} = ga\frac{T}{\Lambda} = 9.8 m / s^2 * 1 * 12.42 hr * 60 \min / hr * 60 s / \min / 980000 m = .45 m / s$$

What is the maximum tidal excursion?

$$x = a / Hk = 1 / 50 / 2 / \pi * 980000 = 3.1 km$$