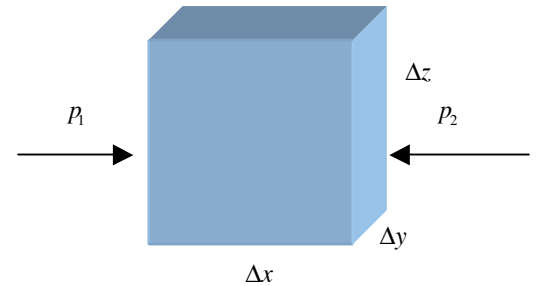


Equation of Motion

Consider a cube of fluid of density ρ and let the pressure increase from left to right. A pressure force is a pressure times the cross-sectional area with the force vector normal to the cross-section. The force on two sides of the cube would be as shown.



Then, according to Newton's first law

$$F = m a$$

where the mass of the fluid element is the density times the volume

$$m = \rho \Delta x \Delta y \Delta z$$

The acceleration and force on the fluid element will be given by

$$m \frac{\partial u}{\partial t} = F_1 - F_2$$

$$\rho \Delta x \Delta y \Delta z \frac{\partial u}{\partial t} = F_1 - F_2 = (p_1 - p_2) \Delta y \Delta z$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\Delta p}{\Delta x} \rightarrow -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad \text{as } \Delta x \rightarrow 0$$

This is the simplest form of the momentum equation for one dimension.

An analogous equation can be obtained for the other horizontal dimension. Conventionally this is written as

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

Finally, we can derive a force balance equation in the vertical. In this case however, in addition to the pressure gradient force, the force of gravity will act on the cube. This force will be mass times gravitational acceleration $mg = \rho \Delta x \Delta y \Delta z g$, and is downward. Since conventionally we measure things to be positive upward, this force is negative.

Adding this force to the force balance equation gives us that, for vertical velocity w ,

$$\rho \Delta x \Delta y \Delta z \frac{\partial w}{\partial t} = (p_{bottom} - p_{top}) \Delta x \Delta y - \rho \Delta x \Delta y \Delta z g$$

and, in the limit of the volume going to zero,

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$$

Note that, in the absence of flow, we have a balance of vertical forces between pressure gradient and gravity. This balance is called the *hydrostatic balance*, and can be expressed mathematically as

$$\frac{\partial p}{\partial z} = -g\rho,$$

an expression called the *hydrostatic relation*.

Pressure in a fluid column (*hydrostatic relation*)

Newton's law applies to a column of fluid, as

$$F = M a$$

where the acceleration here is gravity, g , and it is directed toward the center of the earth ("downward"). This holds for each element of fluid, dM , in the water column

$$dM = \rho A dz$$

So the force contributed by each fluid parcel, dF , is

$$dF = \rho A dz g$$

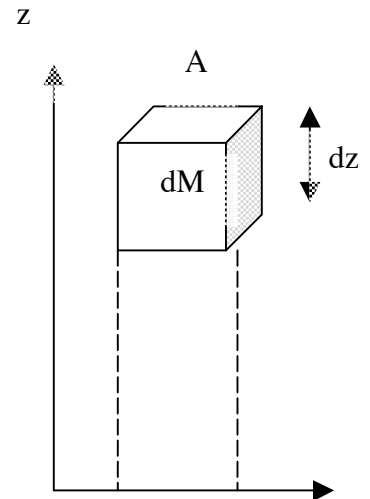
and the force exerted by the entire column is the sum of the forces from each parcel

$$F = g A \sum_i \rho_i dz_i$$

or, for a continuously stratified fluid

$$F(z) = g A \int \rho(z) dz$$

In the ocean we generally consider the force per unit area, or the pressure p , in place of force. Dividing through by the area A of the column gives



$$p(z) = g \int_z^0 \rho(z) dz$$

or, the differential form of the *hydrostatic equation*

$$\frac{\partial p}{\partial z} = -g \rho$$

where we remember that pressure is force per unit area.

In a homogeneous (constant density) ocean, the pressure at depth H is

$$p(-H) = g \rho_o \int_{-H}^0 dz = g \rho_o H$$

For a layered ocean where each layer has different density and thickness h.

$$p(-H) = g \sum_i \rho(i) h(i) = g(\rho(1)h(1) + \rho(2)h(2) + \dots)$$

The hydrostatic relation gives that the vertical gradient of pressure is proportional to the density of water. So, this can be integrated from the sea surface (where pressure is atmospheric pressure) down to get the pressure at depth, if we know the density distribution over the intervening depths. Namely, at depth $-H$,

$$p(-H) = p_{atmos.} - g \int_{\eta}^{-H} \rho(z) dz = p_{atmos.} + g \int_{-H}^{\eta} \rho(z) dz$$

where η is sea surface elevation. This expression states that, *at any depth, the pressure will simply be given by the atmospheric pressure plus the weight of the water above the point where we are measuring it.*

Let's examine the pressure gradient in some simple examples

Case I. constant density fluid with density ρ_0

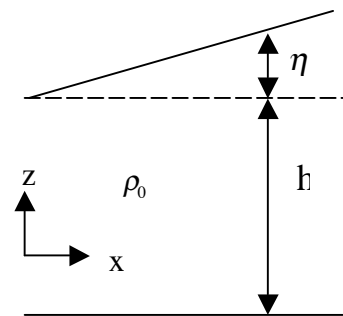
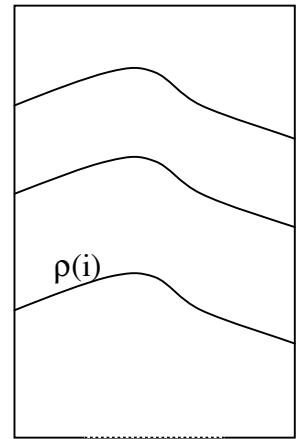
The hydrostatic pressure will be given by $p = -g\rho_0 z$. Remember that z is positive upward so that the pressure in this case is simply proportional to the depth.

Where the sea surface height is sloped, then there will be an additional contribution to the hydrostatic pressure given by

$$p = -g \rho_o h + g \rho_o \eta$$

where η is the sea surface height departure from some nominal or mean height h. If we designate the slope of the sea surface as

$$i_x = \frac{\partial \eta}{\partial x}$$



then the pressure gradient will be given by

$$\frac{1}{\rho_a} \frac{\partial p}{\partial x} = g i_x$$

Case II. two layer fluid with densities ρ_1 and ρ_2

In this case, we have a sea surface height slope, as well as a sloping interface between the two layers. The interface deviation is given by η_1 .

In this case, the hydrostatic pressure in layer 2 at depth z will be given by

$$\begin{aligned} p &= g\rho_1\eta + g\rho_1(h_1 - \eta_1) + g\rho_2\eta_1 - g\rho_2(z + h_1) \\ &= g\rho_1\eta + g\eta_1(\rho_2 - \rho_1) + g\rho_1h_1 - g\rho_2(z + h_1) \end{aligned}$$

Note that it is the horizontal gradient in pressure that we are usually interested in, given by

$$\frac{1}{\rho_1} \frac{\partial p}{\partial x} = g \frac{\partial \eta}{\partial x} + g' \frac{\partial \eta_1}{\partial x}$$

where the *reduced gravity* term is defined as

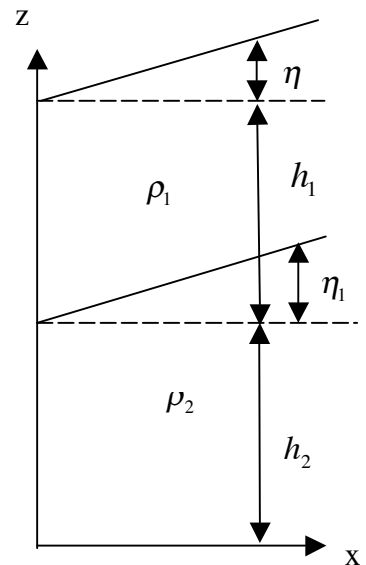
$$g' = g \frac{\Delta\rho}{\rho}$$

Archimedes's Principle and Buoyancy

Consider the following imaginary situation. We have a motionless sea where pressure is in a hydrostatic balance. We take a small cube of water, like we had at the beginning of this note, and somehow (don't ask me how!) replace it with water of different density, say ρ_{new} . How would this affect the balance of force?

We assume at first the pressure distribution is unaffected, so that there will be no horizontal density gradient and no horizontal flow. In the vertical, again the pressure forces does not change but the gravitational force does, since the density and thus the mass of water in the cube is different now. Newton's equation for the cube in the vertical becomes

$$\rho \Delta x \Delta y \Delta z \frac{\partial w}{\partial t} = (p_{bottom} - p_{top}) \Delta x \Delta y - \rho_{new} \Delta x \Delta y \Delta z g.$$



But, since the water column was originally in a hydrostatic balance,

$(p_{bottom} - p_{top})\Delta x\Delta y = \rho_{original}\Delta x\Delta y\Delta z g$. So the force balance can be written as

$$\rho \Delta x \Delta y \Delta z \frac{\partial w}{\partial t} = \rho_{original} \Delta x \Delta y \Delta z g - \rho_{new} \Delta x \Delta y \Delta z g = -g(\rho_{new} - \rho_{original})\Delta x\Delta y\Delta z$$

The cube experiences net force, which is equal to the weight of the water ($g\rho_{new}\Delta x\Delta y\Delta z$) it displaced ($g\rho_{original}\Delta x\Delta y\Delta z$). This is the celebrated *Archimedes's Principle*, which states that the weight of an object placed in a fluid is reduced by the weight of the fluid it displaces. This reduction in weight is called *buoyancy*. We see that this net reduction of weight is due to pressure force applied from the surrounding fluid, which is in hydrostatic balance.

Moreover, the sign of this net force depends on the difference between the original and the new densities. If the former is greater than the latter, the resultant force is upward, and results in an upward flow – lighter fluids tend to rise. If the former is less than the latter, the force is downward – heavier fluids tend to sink.

Stable and Unstable Stratification

As we discussed at the beginning, the ocean is *stratified* – there is a variation of sea water density in the vertical, which results from variation of temperature and salinity in the vertical. This stratification is almost always bottom-heavy, that is, density increases with depth. This is intuitively understandable, since we expect lighter water to rise and heavier water to sink. But we can make this understanding rigorous by considering the force balance, as we just did.

Consider a typical, bottom-heavy ocean and think of moving a parcel of water a little upwards. Since density increases with depth, this parcel of water is heavier than the parcel it replaced. So the force on it is downward and the resultant acceleration tends to send the water back down. Similarly, displacing a water parcel downward will result in its being lighter than the surrounding water, and the parcel will accelerate upward. In this way, buoyancy will tend to “push back” water parcels to the depths from which they came from, and consequently we don't expect large vertical motion that would rearrange densities in the water column. In this way, a bottom-heavy stratification is said to be *stable*.

In contrast, if the water column was top-heavy, then a parcel displaced upward would be lighter than the surrounding, and would experience upward acceleration, away from the original depth. Further, as it travels upward it would become ever lighter than the surrounding, so the acceleration would continue indefinitely. Similarly, downward displacement would result in indefinite downward acceleration. The net result will be generation of vigorous vertical motion that transports light water upwards and heavy water downwards. We expect that a slightest provocation would result in large vertical motions in a top-heavy water column, and such stratification is said to be *unstable*. When

we see top-heavy density gradient in the ocean, it is always associated with vigorous vertical motion, such as convection and breaking internal waves.

There is one important subtlety to consider. When a parcel of water is displaced, not only is it of different density than the surrounding water; it is also under different pressure than it was at the original location. Since the sea water expands and contracts with decreasing and increasing pressure, we must take this effect into account when making a buoyancy argument. The water column is stably stratified only if displacement of a water parcel would not result in acceleration away from the origin with the density change of the parcel taken into account.

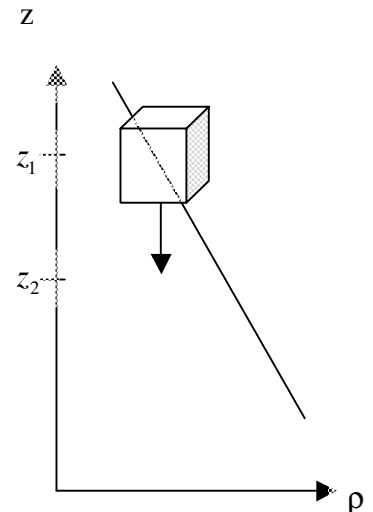
Sea water is compressible, that is, when pressure is exerted on it, it can compress, and have a smaller volume. As a thought experiment, consider a fluid parcel at the surface of say 1 cm³ volume. We move this parcel with temperature 0.58 degrees. If we then move this parcel down to 5000 m, without letting it exchange heat with the surrounding water in the process, the parcel becomes compressed and its temperature rises. What happens is that the fluid parcel increases its temperature to 1 degree C. The in situ temperature is now 1 degree C. The potential temperature of this water parcel is defined as 0.58 degree. The potential temperature θ is defined as the temperature that a fluid parcel would have if it is brought adiabatically (without exchange of heat) to the surface or the reference level. We can also define potential density in a similar way and is given by σ_θ .

The effect of compressibility is taken into account by considering potential temperature and potential density; however, determining water column stability over a large depth range from potential density based on a single reference pressure can give erroneous results, since compressibility of water is sufficiently variable over large pressure ranges. For example, when sea surface pressure is used as a reference (sigma theta), the Antarctic Bottom Water in South Atlantic appears to come out lighter than the overlying North Atlantic Deep Water. This inversion disappears when a deep reference pressure (such as 4000 decibars, sigma-4) is used.

The ocean is stably stratified, that is, lighter fluid overlies heavier (denser) fluid. Generally, if a fluid parcel becomes denser than the parcel below it, the denser fluid will sink and mix with the fluid below until the column is again stably stratified (*convection*). Examples of processes that increase density are cooling at the ocean's surface, or evaporation, which increases salinity.

But what happens initially if a parcel of fluid is displaced a small amount vertically, so that its density no longer matches that of the surrounding fluid?

Thought experiment:



Imagine moving a water parcel from depth z_1 to depth z_2 in a water column with density $\rho(z)$. The parcel initially had density $\rho(z_1)$, but the surrounding fluid has density $\rho(z_2)$. Does the parcel try to return to its original position?

It depends on the *compressibility* of the fluid. Recall that the density depends on the pressure, which according to the hydrostatic relationship, depends on the depth, z .

Case I. Incompressible fluid (no change in density with depth, $k \rightarrow 0$). Parcel retains its original density, so is less dense than the ambient fluid \Rightarrow upward buoyant force.

Case II. Very compressible fluid (large change in density with depth, $k \rightarrow \infty$). Parcel's density increases with depth faster than the ambient density $\rho(z)$, and parcel continues to sink.

Case III. Slightly compressible fluid. Parcel's density increases with depth more slowly than the ambient density $\rho(z)$, so parcel is less dense than the surrounding fluid \Rightarrow upward buoyant force.

The ocean usually belongs to case III (although there are some interesting case II exceptions). In attempting to return to their original depth z_1 parcels will usually overshoot, so that they are then lighter than the ambient fluid. In this case they experience a downward buoyant force and continue to oscillate for some time. The frequency of the vertical oscillations N is given by

$$N^2 = -\frac{g}{\rho} \frac{\partial \rho_{potential}}{\partial z}$$
 and N is known as the Brunt-Vaisala buoyancy frequency. Note that

we use potential density to take into account the compressibility of sea water.

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