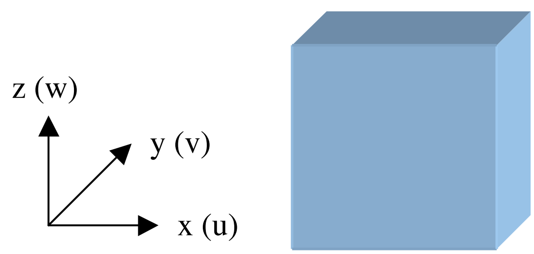
Equations of Motion Revisited

We have been discussing some of the important terms in the equations of motion for a fluid parcel. So far, we have discussed changes in acceleration, the Coriolis force, and the pressure gradient

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad \text{zonal (x) direction}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad \text{meridional (y) direction}$$

$$\frac{\partial w}{\partial t} + g = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \quad \text{vertical}$$



Note that the zonal equation contains the meridional velocity with the Coriolis parameter and vice-versa for the meridional equation. The acceleration terms are frequently small – setting acceleration to zero gives the geostrophic balance in the horizontal and the hydrostatic balance in the vertical.

Friction

Now that we understand how the pressure gradient and the Coriolis force can act to give the geostrophic balance, let's think about another force and that is *friction*. The simplest way to represent friction in the ocean is to assume that the faster a parcel is moving, the greater the friction. We can write this as

friction (x direction) =-Ju
friction (y direction) = -Jv
friction (z direction) = -Jw

where u, v, w are the velocities in the zonal (east-west), meridional (north-south) and vertical directions respectively. We will see how this form of friction can act to "spin down" the flow later.

Let's add friction to our equations above. For a particle moving in the x direction we have

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} - Ju$$

that is, acceleration is balanced by the Coriolis force, pressure gradient, and friction. Let's consider a couple of different balances

1) $-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

This is the *geostrophic balance* and occurs under steady conditions with no friction

2) $\frac{\partial u}{\partial t} - fv = 0$

This is the balance for *inertial oscillations* (we will explore this further later in the quarter).

3) $\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

In the absence of Coriolis force and friction, the *flow will go down the pressure gradient* (from high to low pressure)

4) $\frac{\partial u}{\partial t} = -Ju$

With no pressure gradient or Coriolis force, the flow will just slow down (note that the friction is larger with larger flow).

5) $0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} - Ju$

In this case, if there is an existing pressure gradient, the flow will once again go down the pressure gradient, but *be slowed by friction*.

6) $-fv = -Ju$

This is a balance between Coriolis force and friction and is analogous to the *Ekman balance* that we will review next week.

Part of the trick of being a physical oceanographer is figuring out what terms are important in the equations of motion. The rules of thumb are

1) If the motion is *larger than about 50km* or so, then the Coriolis force must be included. We will quantify this later.
2) If we are only interested in the mean motion, not the evolution of the fluid, then we can assume *steady state* (i.e. that the acceleration term is small).
3) *Friction* can be ignored in the interior of ocean basins, but near the boundaries (either side, top or bottom, it cannot be ignored.

Let's consider how friction would work in the real ocean.

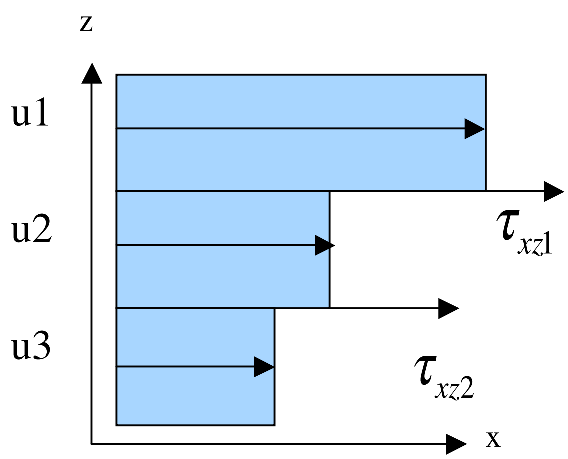
Assume we have flow in the interior of the ocean caused by a pressure gradient and in geostrophic balance (we neglect friction in the interior). At the bottom of the ocean, the fluid velocity slows to zero. The top of the ocean is forced by the wind stress. These statements are the "boundary conditions." The transition from the interior flow to these boundary conditions happens in a "boundary layer" where friction is important. The fluid velocity and stress must match smoothly between the boundary layers and the interior flow. Typically, we think of friction as being responsible for making this match between the interior (geostrophic) flow and the boundary conditions.

For the purposes of this discussion we assume that quantities vary only with *z*. The *x*-direction is taken to be the direction of the flow outside the boundary layer.

Stress

Let's examine how stress in a fluid works. Consider a laminar (non-turbulent) fluid flow, with three layers of fluid moving at different speeds.

When a molecule moves from one layer to another, it carries with it its momentum (here, density X velocity). For example, a molecule from layer 1 moving to layer 2 will have a larger momentum and will impart that momentum to other molecules in layer 2 during collisions. Conversely, a molecule from layer 3 that moves into layer 2 will have less momentum and collisions with layer 2 molecules will increase its momentum, The result of all this interaction on a molecular scale adds up to a force on the large scale that we observe. It is called the **shear stress**, a tangential force on the interface between the two layers moving at different speeds (we use the symbol τ for stress, which has the same units as *pressure*). The faster layer will slow down and the slower layer will speed up, with the stress being proportional to the difference between the speeds in the layers.



stress = force per unit area, tangential to that area

By analogy with the diffusion of properties like heat or salt, we have Newton's law of viscosity: "stress is proportional to the velocity gradient." The constant of proportionality is defined to be the molecular viscosity, μ.

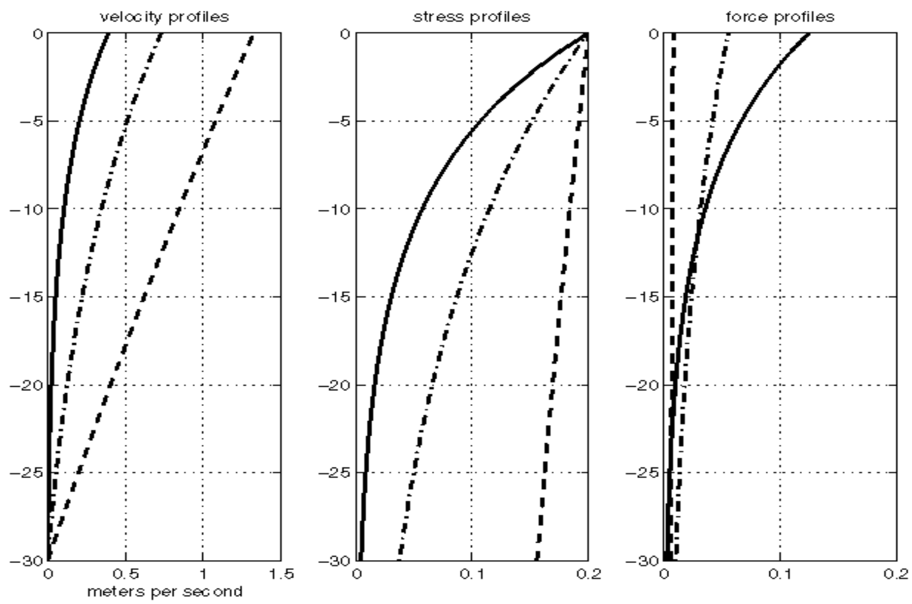
$$\tau = \mu \frac{\partial u}{\partial z}$$

The force on a parcel of fluid is given by the *vertical gradient* in the stress

$$F_{friction} = \frac{\partial \tau_{xz}}{\partial z}$$

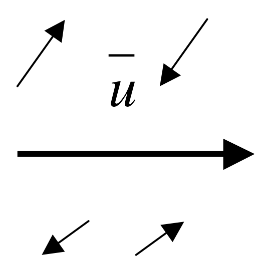
analogous to the *pressure gradient force*.

To understand the relationship between velocity shear, stress, and the force on a parcel of fluid, look at the figure below. Assume that initially there is no velocity in the water column, and then we impose a constant stress (from the wind) at the surface (z = 0).



After a short time, the surface stress accelerates the flow (solid line) and the shear begins to penetrate into the water column. The force is largest at the surface where the vertical derivative of stress is largest. Some time later, water deeper in the column has been accelerated (dash-dot line), stress has penetrated deeper, and the force and acceleration of the water column have decreased. Later, the water column has been accelerated so that the vertical shear is nearly constant (dashed line), stress is nearly constant, and the force on the parcel has been decreased to nearly zero. The water column stops accelerating and the flow is in a steady state with the wind stress. Note that there is still a (constant) *stress*, the velocity profile increases from near-zero to a large surface value, and that momentum may continue to be transferred between layers, but the column is no longer accelerating.

Turbulent flows.



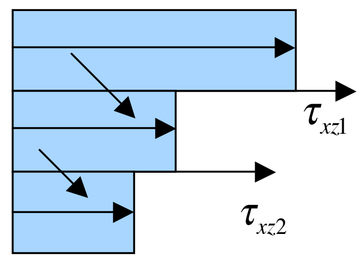
The ocean is a turbulent fluid, in contrast to the previous example. This means that we cannot specify or measure the velocity field (or other fluid properties) precisely – there is always a random component. We distinguish more organized aspects of the fields and the random or turbulent components as, for example,

$$u = \bar{u} + u'$$

where the overbar represents a time average, and u' is the random or *eddy* component. We try to quantify the statistics of the random components and relate those statistics to the mean field. In the diagram to the right, the mean flow is positive and the eddy components of u have some organization: when w'> 0, then u'> 0 and when w'< 0, then u'< 0. So there is a correlation between u' and w', that is, $\overline{u'w'} > 0$.

In physical oceanography, we do not usually work on the molecular level. However, we use an analogy between the motion of fluid parcels and molecular motion to define eddy processes. For a turbulent flow, we parameterize the stress, which is actually defined by the correlation, in terms of the shear of the mean flow, as

$$\tau = -\rho \overline{u'w'} \approx \rho A_v \frac{d\overline{u}}{dz} \quad (6.1)$$



where τ is the turbulent shear stress and A_V is the vertical **eddy viscosity**. (We will drop the overbar for the rest of these notes, i.e., *u* will represent the time-mean flow.) This statement describes mathematically the tendency of the eddy motions to reduce the velocity shear; we call this a *down-gradient* diffusion of momentum and it is the usual case in the ocean.

To appreciate the difference between molecular and eddy diffusion, let's compare some of the numbers. The value for dynamic molecular viscosity is $\mu = 1 \times 10^{-3} kg / ms$. We can also divide this by the density of water to get the kinematic viscosity $\nu = 1 \times 10^{-6} m^2 / s$

In contrast, the eddy viscosities are much larger with the horizontal value being much larger than the vertical value

$A_h = 10^2 - 10^4 m^2 / s$

$A_v = 10^{-4} - 10^{-2} m^2 / s$