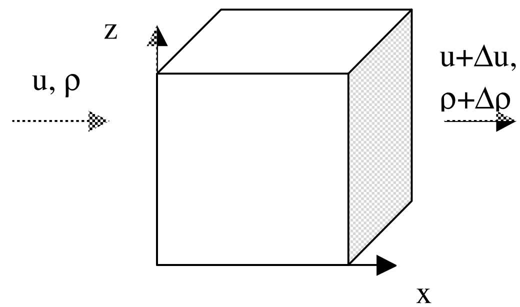
Conservation

To understand how the ocean moves, we will examine some conservation properties, starting with the conservation of mass. In Elementary Physics, quantities like mass, velocity and temperature were associated with a discrete body; however, in a fluid they are specified at each location and vary continuously from one place to the next. We will derive mathematical equations for conservation appropriate for the continuum as follows.

Consider a box at a fixed location in the fluid with the dimensions Δx, Δy, and Δz. What is the time rate of change of mass M in the box?



Rate of change of mass
= mass flux into the box
minus mass flux out of the box

$$\frac{\Delta M}{\Delta t} = \rho\, u\, \Delta y\, \Delta z - (\rho + \Delta\rho)\,(u + \Delta u)\, \Delta y\, \Delta z$$

Dividing through by the volume V of the box gives

$$\frac{1}{V}\frac{\Delta M}{\Delta t} = \frac{\Delta\rho}{\Delta t} = \frac{-1}{\Delta x}\left[(\rho + \Delta\rho)\,(u + \Delta u) - \rho\, u\right]$$

Taking the limit as Δx→0 and Δt→0, gives

$$\boxed{\frac{\partial\rho}{\partial t} = \frac{-\partial(u\rho)}{\partial x}} \quad (2.1)$$

where the derivatives are *partial derivatives*. Recall that a partial derivative of a function is just the derivative with respect to a single variable, holding the others constant. For example, if

f(x,y,z,t) = ax + by + czt, then $\frac{\partial f}{\partial x} = a, \quad \frac{\partial f}{\partial t} = cz.$

The above equation is for the conservation of mass (actually density) in one dimension (x). Expanding and rearranging terms gives another version

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} = -\rho\frac{\partial u}{\partial x}$$

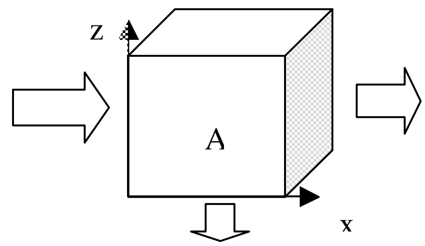
where the LHS (left hand side) is the change in density following a water parcel around. If we allow density and velocity to vary in three dimensions we obtain the full mass conservation equation

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \quad (2.2)$$

Now the LHS is the variation in density in three dimensions following the water parcel and the RHS is the *divergence* of the velocity field. Divergence is a measure of the expansion of the fluid.

Simplifications and special cases:

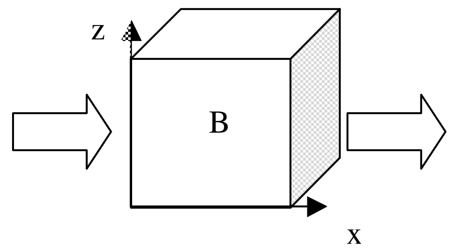
A) An incompressible fluid. The amount of water in the cube does not change. This is also a statement of mass conservation in this case and is the one that we will use most often in this class.



$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0 \quad (2.3)$$

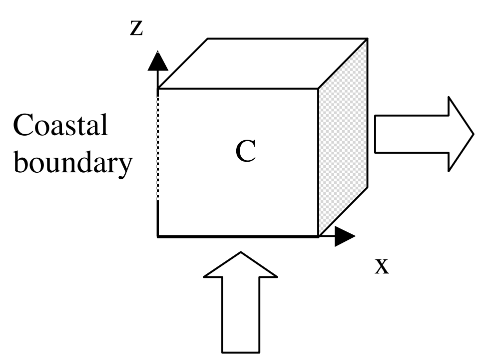
The flow field is *non-divergent*. In example A, the field is non-divergent.

B) Horizonally non-divergent with no vertical motion



$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0=-\frac{\partial w}{\partial z}$$

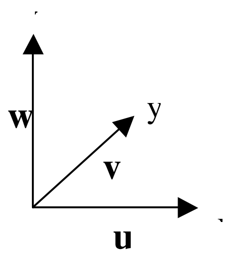
C) Coastal upwelling. Near-surface offshore flow at the coast is compensated by vertical flow from below the surface layer (and onshore flow farther down in the water column).

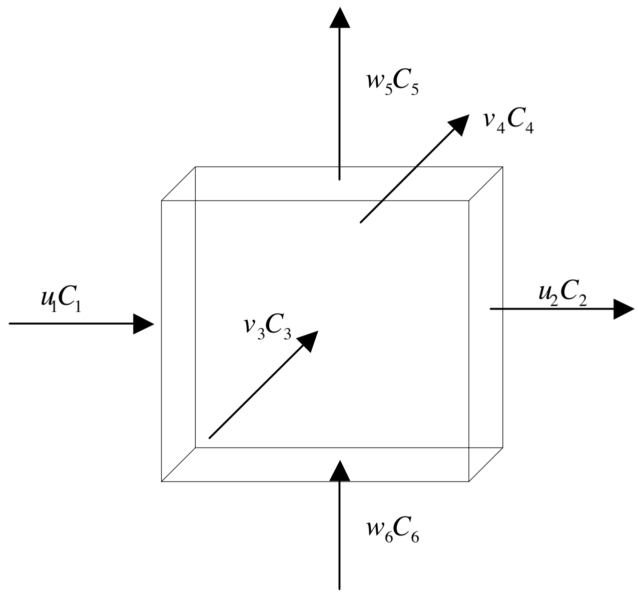


$$\frac{\partial u}{\partial x}=-\frac{\partial w}{\partial z}$$

Advection

We can apply the same balance argument to derive an equation that describes how the concentration of a constituent of set water, such as salt and nutrients, changes. If we consider a cube of fluid as shown below, the change in the concentration of a tracer (say, salt) within the cube will be given by the sum of the fluxes across each face of the cube.





The total amount of the tracer in the volume will be given by the concentration times the volume of the cube C dx dy dz. The change in the amount of tracer will be given by the flux of tracer across the faces of the cube

$$dxdydz\frac{\partial C}{\partial t} = (w_6C_6 - w_5C_5)dxdy + (u_1C_1 - u_2C_2)dzdy + (v_3C_3 - v_4C_4)dxdz$$

Now we divide by the volume of the fluid to get the conservation equation

$$\frac{\partial C}{\partial t} = (w_6C_6 - w_5C_5)/dz + (u_1C_1 - u_2C_2)/dx + (v_3C_3 - v_4C_4)dy$$

Now as the volume gets very small, we can turn this into

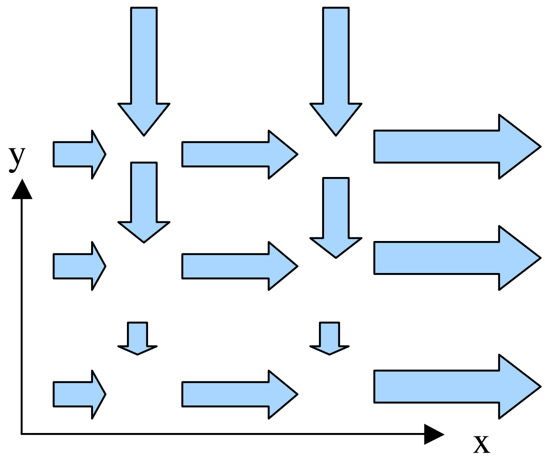
$$\begin{gathered}\frac{\partial C}{\partial t} = -\frac{\partial wC}{\partial z} - \frac{\partial uC}{\partial x} - \frac{\partial vC}{\partial y} \\ \frac{\partial C}{\partial t} + \frac{\partial uC}{\partial x} + \frac{\partial vC}{\partial y} + \frac{\partial wC}{\partial z} = 0\end{gathered} \quad (2.4)$$

This conservation statement will hold for temperature and density as well.

For the dynamics that we are interested in, we can assume that the ocean is *incompressible*. We know that this is not strictly true, that is, sea water is compressible and is denser at depth. By assuming incompressibility, we are in effect ignoring sound propagation in the ocean, which is fine for the purposes of the dynamics that we will be discussing.

When we make the incompressible assumption, we what are saying is that the amount of fluid within the cube remains the same. This gives us the statement that the fluid is *non-divergent*

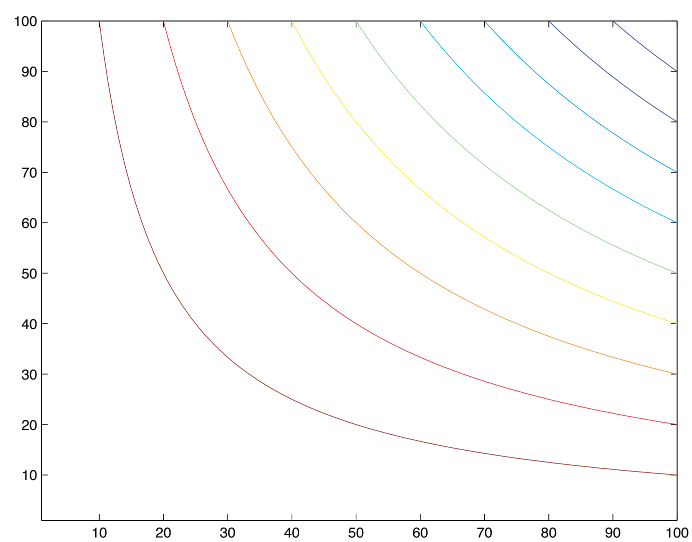
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$



To see what this means, consider a flow in the x-direction that is changing in the x-direction. If we want to conserve mass, then there must be either a vertical velocity or a velocity in the y direction to feed the changes in flow (just as in the coastal upwelling circulation example).

In two-dimensions, given this velocity field, we can define a stream function (this is very much like the dynamic height fields or sea level that we looked at earlier in the quarter). The *stream function* is defined by

$$u=-\frac{\partial \psi}{\partial y} \qquad v=\frac{\partial \psi}{\partial x}$$



With w = 0, the stream function exactly satisfies

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

For the flow field that we looked at above, we have a stream function that looks like the figure to the right.

If we release a particle in this flow field, it will move along one of the streamlines. Its speed will depend on the gradient of the stream function, that is, where the lines are closer together, the flow is stronger.

Now back to the conservation equation for salinity or equivalent *conservative* tracer. The conservation equation (2.4) can be expanded to give

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}+C(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z})=$$
$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=0$$

assuming the fluid is *non-divergent*. Now, how do we understand these equations? What this last equation says is that as we follow a fluid parcel in the flow field, the tracer is conserved and this is often written as

$$\boxed{\frac{DC}{Dt}=\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=0} \quad (2.5)$$

which is known as the Lagrangian or *material* or *advective* derivative. This says that in the absence of diffusion, as you follow a fluid parcel in a moving fluid, the concentration will remain constant and it is conserved. In the Eulerian framework, we look at the changes in tracer at a fixed location, denoted by the term $\frac{\partial C}{\partial t}$.

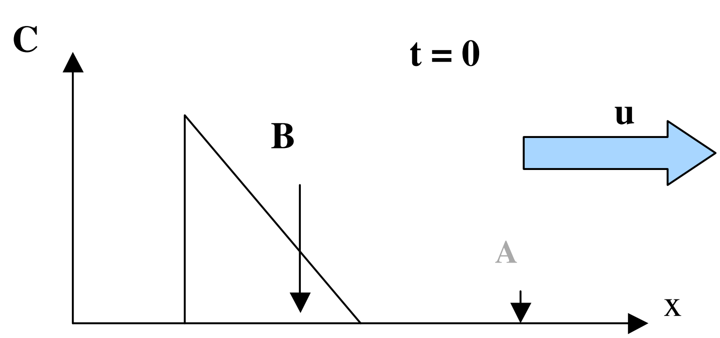
You may also be familiar with biologically important tracers such as oxygen. Then, we expect that below the surface layer, the oxygen will be respired and decrease over time. This can be written as

$$\frac{DO}{Dt}=-\hat{O}$$

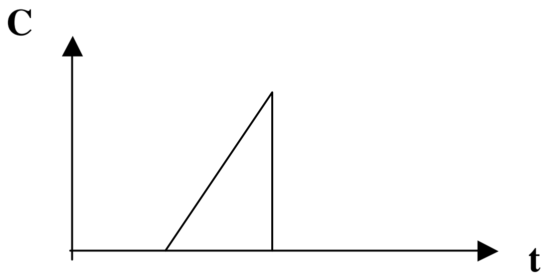
where the right hand side represents the respiration (or depletion of oxygen) following a fluid parcel. This is a *non-conservative* tracer.

As an example, consider a uniform zonal flow field. In that case, we have

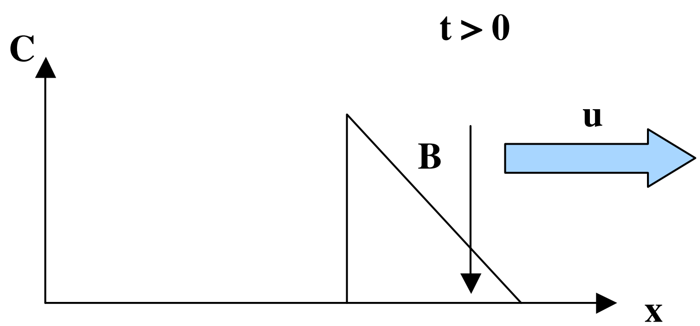
$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=0$$



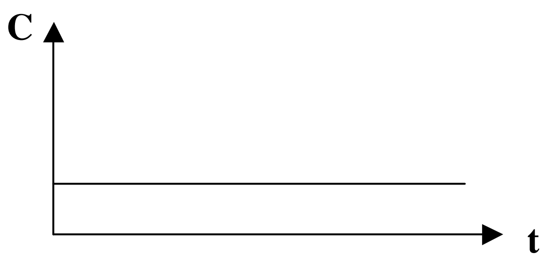
If we place a lump of fluid with high concentration C at a certain location, what will happen to the concentration over time at a fixed location, say A? It will look like the following graph: as the tracer arrives, the concentration measured at A will gradually increase, until the rear end of the lump arrives, at which point the concentration drops abruptly to zero.



Consider what the concentration will be at point A. This is the Eulerian frame of reference.



What, however, will the concentration be at B if we are moving with the velocity U (in the Lagrangian frame of reference)?

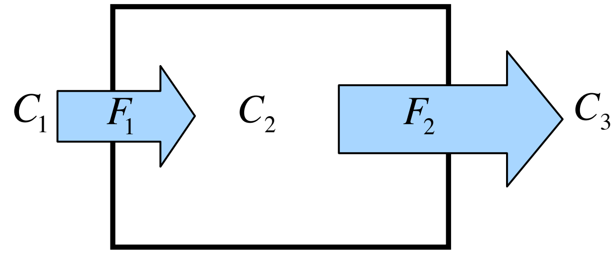


Since B is moving with the flow, it will always see the same concentration. Thus, over time, C at B is constant.

We will look at other examples (2-dimensional etc) in lab.

Diffusion

Another process that can change the concentration of a property of the fluid is diffusion. First consider a thought problem in which we have a barrel with salty water on the bottom and a fresh cap of water on the top. We know that, although there is no motion in the barrel, the salty water will diffuse upward into the fresh water and the fresh water will diffuse downward into the salty water until the salt is uniform. This occurs through the molecular process of diffusion.



The next thing to consider is how diffusion acts to change the concentration of a tracer in the ocean. We once again look at our cube of fluid and ask how diffusive fluxes can change the concentration within the cube. For simplicity consider this in one dimension only.

Now the molecular fluxes are going to be proportional to the gradients in the concentration. Now if $C_3 < C_2$

$$F_2 = -\kappa \frac{C_3 - C_2}{\Delta x}$$

That is, the flux will be from the higher values of tracer to the lower values. We can write this in differential form as

$$F = -\kappa \frac{\partial C}{\partial x}$$

Now consider the conservation statement for the box. We have

$$\frac{\partial C}{\partial t} = F_1 - F_2 = -\kappa \frac{C_1 - C_2}{\Delta x} - \kappa \frac{C_3 - C_2}{\Delta x}$$

$$= \kappa \frac{\partial^2 C}{\partial x^2}$$

so that the concentration will change when there is a divergence (or change) in the flux across the box. Thus, if the concentration changes linearly with x,

$C = ax + b, \quad then \quad \frac{\partial^2 C}{\partial x^2} = 0$ and there will be no diffusion of the tracer.

For salinity and temperature we have for molecular diffusion

$$\kappa_s = 1.5 \times 10^{-9} m^2 / s$$

$$\kappa_T = 1.5 \times 10^{-7} m^2 / s$$

The molecular diffusion of heat and salt is very small, much smaller than the observed diffusion in the ocean.

One way to quicken the molecular transfer is to *stir* the water. If we stir the water in the barrel, we will cause the salinity to become uniform more rapidly. How does this happen? Stirring increases by many times the surface areas of the interface between the salty and fresh water. Mixing still takes place on the molecular level, but the interfaces sharpen the concentration gradients and causes molecular diffusion to happen more quickly. In the ocean turbulence does this stirring. We describe this turbulent process using *eddy diffusion* just as we did for momentum earlier. As with momentum, the coefficients for eddy diffusion are much larger than for molecular diffusion and the coefficients are different for vertical and horizontal processes.

$$A_h = 10^2 - 10^4 m^2 / s$$

$$A_v = 10^{-4} - 10^{-2} m^2 / s$$

Generally, this rate of stirring is not the same in all directions. For reasons we will come to shortly, much more work is required to move water parcels in the vertical (strictly speaking, across gradients in density) than in the horizontal, and vertical turbulence is supressed relative to horizontal turbulence. In three dimensions then we have

$$\frac{\partial C}{\partial t} = A_h\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) + A_v\frac{\partial^2 C}{\partial z^2} \quad (2.6)$$

Both *advection and diffusion* can take place at the same time. In this case, the equation will be modified as

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = A_h\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) + A_v\frac{\partial^2 C}{\partial z^2} \quad (2.7)$$