Clustering and conservation laws in one dimensional driven stochastic flow

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Predicting the structure of stationary states in driven stochastic processes, and the manner in which fluctuations relax back to it in time, remains an fundamental open issue. Stochastic driven flow in one dimensional channels, in its most basic form, belongs to the KPZ type universality class and has a trivial (disordered) stationary state. But the latter is generically unstable with respect to generalizations of the dynamic rules. The Arndt model, e.g., a two species stochastic driven flow process, represents interface growth where the number of up and down steps is conserved. We (Kyung Kim and MdN) show numerically that this extra conservation law, changes the growth dynamics to 2 factorized KPZ processes beyond a definite length scale in terms of quasi-particle type mixtures of up ad down steps. A second example is the introduction of non-local hopping events in driven stochastic flow (in collaboration with Meesoon Ha and Hyunggyu Park, KIAS). This creates strong local clustering and a dynamic first or second order phase transition to an empty road phase; which can be explained fully in terms of self consistent cluster methods. Supported by NSF grant DMR-0341341
Introduction

- Introduction
  part 1: interface growth with step conservation
  part 2: clustering in non-local hopping
General goals: non equilibrium driven stochastic processes display universal robust scaling behaviors

Examples: evolution of interface roughness; dynamic phase transitions such as queuing (traffic jams) behind obstacles (slow bonds) in driven stochastic flow, reaction-diffusion type population dynamics, such as directed percolation and directed Ising processes; self organized criticality, such as avalanche processes and scale free network dynamics.

Purpose: Identify and characterize those universal aspects.

Tools: experimental input (like flameless paper combustion), computer simulations of model processes; exact solutions of master equations; ultimately seek a generalization of conformal field theory for 1+1 dimensional processes (live at the edge of CFT. i.e., Luttinger liquid and 2D isotropic scaling equilibrium critical phenomena).

Timonen: flameless paper combustion

MdN+ Davidson: facet ridge endpoints in 2D equilibrium crystal shapes =KPZ

Wijngaarden: propagating front in high Tc super conductor
These processes reach a stationary state after
\[ t \sim N^{1/z} \]
with \( N \) the system size and \( z=3/2 \) the 1+1D KPZ dynamic exponent.

The stationary state is disordered with only short ranged correlations between steps in the interface (= particles in the driven flow representation).

But: this trivial stationary state is unstable towards all types of point defects (slow bonds), reservoirs, and various simple generalizations of the rule. This leads to clustering and/or non-trivial phase dynamic phase transitions.

- Can we predict dynamic stationary states on general principles, like we do in equilibrium type stationary states?

- Does clustering and/or additional conservation laws affect the dynamic exponent, i.e., the manner in which fluctuations relax back to the stationary state?
Asymmetric exclusion driven stochastic flow and BCSOS KPZ type interface growth

interpret the down steps as particles and the up-steps as vacant sites
stationary state, fluctuations, and group velocity:

The stationary state for ASEP processes with periodic boundary conditions is disordered random without any correlations; but fluctuations die out with KPZ dynamic exponent $z=3/2$ instead of the diffusion like $z=2$; and these fluctuations have non-zero group velocity

$$l \sim t^{1/z} \quad u_g = 1 - 2\rho$$
boundary induced phase transitions

Phase transitions take place in open road set-ups with reservoirs on both ends; (exact matrix formulation results of the stationary state by, e.g., Derrida et.al.)

In the maximum current (MC) phase the road controls the density, but in the low (high) density phase the input (output) reservoir (α or β) controls the bulk density.
queuing due to slow bonds

Studies of slow bonds in the ASEP goes back at least 10 years. The hopping rate through the slow bond is reduced to $p' = rp$.

Behind the slow bond a traffic jam develops. The issue us whether the queue is finite or infinite in length (does it scale with the system size in the thermodynamic limit, like in bose condensation); and also the detailed shape of the density profile.
critical point
\[ r_c = 0.80 \pm 0.02 \]
the order parameter vanishes as
\[ \Delta_b \sim |r - r_c|^{\beta} \]
\[ \Delta_b \sim N_s^{-\alpha} \]
\[ \beta = 1.5 \pm 0.01 \]
\[ x_\Delta = 0.370(5) \]

the density profile has always a power law tail near the slow bond:
\[ \rho(y) = \frac{1}{2} + \Delta_b + Ay^{-\nu} \]
with:
\[ \nu = \frac{1}{2} \] in the faceted phase (when \( \Delta_b \neq 0 \))
\[ \nu = \frac{1}{3} \] for weak slow bonds (\( r_c < r < 1 \))
\[ \nu = \frac{2}{3} \] for fast bonds (\( r > 1 \); never facets)
Part 1

interface growth with step conservation

with Kyung Kim (UW)

Introduction

- part 1: interface growth with step conservation
- part 2: clustering in non-local hopping
Arndt et. al. model: each site is vacant or occupied by only one + or one - particle. + (-) particles only move to the right (left). Choose update sites at random; 
p = hopping probability to empty site; 
r = passing probability (+-) -> (-+);

This is an interface growth process with singe particle deposition step edge growth (p), no deposition on flat surface segments and only dimer deposition (r) otherwise.

We exclude the backwards passing (-+ ) -> (+-), brick evaporation (it leads to stronger clustering and condensation “like” effects)

Without the dimer constraint and when allowing flat segment growth (if only demanding, dh=1,0,-1 between nearest neighbors) this would be conventional Kim-Kosterlitz type KPZ growth

The dimer constraint implies conservation of number of up- and down-steps (except at surface edge).

Does this conservation law change the KPZ dynamics?

3 special lines:
• \( \rho_+ = \rho_- = 0.5 \): pure BCSOS type KPZ growth
• \( r=2p \): disordered stationary state
• \( r=p \): two (entwined) pure KPZ processes: + particles when blind to the difference between - particles and vacancies; the same for - particles when bind to + and 0
The conventional method for determining the (KPZ) scaling dynamics does not work (very well) due to oscillations (except at \( r=2p \)):

\[ W \sim t^\beta \quad \text{for intermediate times} \]
\[ W \sim L^\alpha \quad \text{in the (finite size, L) stationary state;} \]
\[ t \sim L^z \quad \text{finite size time scale to reach the stationary state} \]

with for KPZ: \( z=1.5 \) and \( \alpha+z=2 \).

Instead we evaluate numerically the time evolution of the two point correlators:

\[
G_{++}(x) = \langle n_+ (x_0) n_+ (x + x_0) \rangle - \langle n_+ \rangle \langle n_+ \rangle
\]
\[
G_{--}(x) = \langle n_- (x_0) n_- (x + x_0) \rangle - \langle n_- \rangle \langle n_- \rangle
\]
\[
G_{+-}(x) = \langle n_+ (x_0) n_- (x + x_0) \rangle - \langle n_+ \rangle \langle n_- \rangle
\]

interface time evolutions starting from random rough initial state

interface time evolutions from flat initial state
At r=p the process reduces to a pure ASEP type KPZ process in 2 different subspaces:

The + particles, when blind to the difference between 0 and -, obey a pure ASEP process. The same is true for the -particles when blind towards + and 0.

In the stationary state $G_{++}(x) = G_{--}(x) \sim \delta(x)$, but

$$G_{+-}(x) = \langle n_+(x_0)n_-(x + x_0) \rangle - \langle n_+ \rangle \langle n_- \rangle$$

has an exponential tail (on the x>0 side only) with range $\xi = 2\sim 4$ lattice units at $\rho_+ = \rho_- = 0.25$.

The area exactly compensates $G_{+-}(0)$,

$$A_{+-} = \int_0^\infty G_{+-}(x) dx = -G_{+-}(0)$$

This implies perfect screening.

$A_{+-}$ represents the probability to find a -particle near a tagged +particle. The tag at x=0 excludes an amount $\rho_-$ of -particles from x=0. All of this remains within length scale $\xi$ in front of the tag.
In the stationary state each + (and each -) tagged particle carries a cloud of opposing charge with it.

The tagged charge is fully screened.

The content of the cloud changes constantly by the flow.

Pair condensation would create a globally flat surface, beyond length scale $\xi$.

Screening makes it more likely to find +− near each other than −+, implying stationary state skewness of the surface, sharper valley’s than hill tops. But that is not very special.

What does full screening mean?
Perfect screening implies that the two KPZ processes (the one for the + particles with blindness to - particles versus vacant sites; and the one for the -particles with +0 blindness) totally decouple beyond length scale $\xi$.

$\xi$ diverges in the $\rho_+ = \rho_- = 0$. 5 limit, the pure BCSOS type single KPZ process.

$\xi$ is the crossover length scale from single KPZ type growth (inside) to (KPZ)$^2$ (outside)
The probability wave packet broadens following conventional KPZ scaling; numerically as:

$$W \sim t^{1/z} \text{ with } z = 1.53(3).$$

Starting from a disordered initial state, the correct local pairings set themselves up quickly.

In $G_+$, conservation of probability implies the emission of a -particle wave packet traveling to the left with group velocity $u_g = 1 - 2\rho_+$. 
$W \sim t^{1/z}$

$z = 1.52(2)$. 

$\text{width of wavepacket } p=1.0 \ r=1.0 \ L=3200 \ z=1.534[5]$ 

$\text{height of wavepacket } p=1.0 \ r=1.0 \ L=3200 \ z=1.533[4]$ 

$p=1.0 \ r=1.0 \ \rho=0.25$
At $r < p$ the stationary state $G_{+-}$ correlator remains localized, but a - cloud on the $x < 0$ side appears and the screening seems (at first) imperfect.

$G_{++} = G_{--}$ develops a structure as well. Two tagged + particles (and tagged - pairs) attract each other.

These attractions are dynamically generated. The cloud of - particles in front of a tagged + particle increases, because at $r < p$ the crossing probability is smaller than the free hopping rate, and creates bottleneck.

The + particles can not ignore the difference between - particles and vacancies anymore. The - clouds in front of tagged + particles become mutually visible.

The $x < 0$ cloud of the tagged + particle in $G_{+-}$ reflects this visibility of the - clouds of + particles behind it.

Clouds overlap since $\xi > 1/\rho_+$. Tagged + particles attract because: “when the forward - cloud of a tagged + engulfs a tagged + in front of it, that + particle slows down”.
The enhanced $x>0$ cloud and the small $x<0$ cloud represent the enhanced probability for finding $+$ particles near the tagged $+$-particle.

This is expressed formally by the quasi-particle mapping

$$\tilde{n}_+ = \beta n_+ + \alpha n_- \quad \tilde{n}_- = \beta n_+ + \alpha n_-$$

Empirically we find that it transforms the 2-point correlators onto those at $r=p$.

It removes all the tails in $G_{++} = G_{--}$ as well as the $x < 0$ tail in $G_{+-}$.

It Leaves $\xi$ (almost) invariant. $\alpha + \beta$ is tuned such that the amplitude matches the $A_+$ at the corresponding point along the $r=p$ “fixed line”.

Higher order correlations, must factorize, be very small for this to work so well.

Conclusion: at $r<p$ the process decouples into 2 independent KPZ processes as well, but now in terms of quasi particles.
At $r>p$ the same quasi particle transformation applies.

All correlations becomes smaller.

The - cloud in front of the tagged + particle is decreased, because + - particles cross each other more readily than their free hopping rate.

Tagged alike particles dynamically repel each other.

The minute surplus -cloud at $x<0$ of the tagged + particle reflects that - particles can not flow away fast enough after passing the tagged +.
Example: the $\rho_+ = \rho_- = 0.25$ line.

the quasi particle transformation “rotates away” all interaction effects.

$$\tilde{n}^- = \beta n_+ + \alpha n_-$$

$$\tilde{n}^+ = \alpha n_+ + \beta n_-$$

It removes all structure in $G_{++} = G_{--}$ and the $x<0$ tail in $G_{+-}$.

$\xi$ is almost invariant;

$\alpha + \beta$ is tuned such that the transformed amplitude matches the $A_+$ at the corresponding point with that $\xi$ along the $r=p$ “fixed line”.
lines of constant area $A_{+-}$ and width $\xi$ of the - cloud in front of a tagged + particle

quasi particle mapping
contour lines in the phase diagram

(lines numerically, and thus noisy)

lines of constant quasi-particle mixing $\alpha/\beta$
The dynamic exponent remains KPZ like with $z = 1.5$ in the entire phase diagram.

For example, in our simulations at $r/p = 0.5$, the wave packets emitted from the local area near the tagged particle spread numerically with

$W \sim t^{1/z}$, $z = 1.54$ (2)
At $r>p$ the amplitudes of the correlations become smaller, increasing the noise in the wave packet determination.

E.g., at $r/p = 10/7$, $z = 1.51(3)$
At $r=2p$ the stationary state is completely disordered; the amplitudes of wave packets, etc, in the correlation functions vanish. $\xi$ becomes very short and likely is zero. In that case the process fully decouples into $(KPZ)^2$.

Here we can determine the dynamic exponent the conventional way: $z = 1.51 (2)$. 
Conclusions of PART 1:

1. Interface growth with conserved number of up and down steps adds a conservation law to the KPZ type growth process.
2. It remains in the KPZ universality class; but actually as factorized into two independent KPZ processes beyond a finite correlation length $\xi$.
3. At $r=p$ this factorization is in terms of + steps being blind to the difference between down steps and no steps; and the same for - steps being blind to $dh=0,1$.
4. At $r<p$ and $r>p$ the factorization is in terms of quasi particle mixtures
   \[ \tilde{n}_+ = \alpha n_+ + \beta n_- \quad \tilde{n}_- = \beta n_+ + \alpha n_- \]
5. $\xi$ represents the crossover length scale between single KPZ behavior and the $(KPZ)^2$; it also represents as clustering length scale

Next: Extend to brick evaporation case, i.e., allowing backwards hopping of - + pairs where cluster condensation transition (does not) takes place. Relate this to zero range process descriptions, etc?
Part 2
clustering in non-local hopping

with Meesoon Ha and Hyunggyu Park

Introduction
part 1: interface growth with step conservation
- part 2: clustering in non-local hopping
non local hopping, stick & slip driven stochastic flow

rule: choose a site at random; if occupied:
• the particle jumps with probability 1-p forward by only one site
• with probability p it makes a non-local hop to the site immediately behind the particle in front of it.
• the most forward particle, if chosen, jumps with certainty into the exit reservoir
• a particle jumps onto site x=1, if chosen and empty, from the entry side reservoir with probability $\alpha$.

Phase diagram contains 3 phases:
• C reservoir controlled phase
• MC bulk control phase
• ER empty road phase, with zero bulk density (only a finite number of particles on the road)

C-ER transition is second order
MC-ER transition is first order

At the transition lines $J=p$
bulk density as function of non-local hopping rate $p$ at increasing values of reservoir exit rate $\alpha$

**Numerical results:** at the second order transition the bulk density scales as

$$\rho(\varepsilon, L) = \mathbf{b}^{-x_\rho} \rho(b^{y_\varepsilon} \varepsilon, b^{-1} L)$$

$$\rho \sim \varepsilon^\beta \quad \rho \sim L^{x_\rho}$$

$$\beta = \frac{x_\rho}{y_\varepsilon} = 1 \quad y_\varepsilon = x_\rho = \frac{1}{2}$$

scaling function collapse at second order transition for $\alpha=0.2$
More numerical results:

- At the C-ER and MC-ER transitions $J=p$.

- No singularity in $J$ at the 2-nd order transition! The critical fluctuations decouple from the current fluctuations.

- The first-order transition takes place at $p=0.3$ and does not vary with $\alpha$.

- The second order line lies beneath $\alpha=p$.

- The critical end point lies at $\alpha=0.6$, $p=0.3$.
The stationary state becomes more and more clustered with increasing $p$.

Out of mean field theory would carry a higher current.

Clustering rules the phase transitions.

The first order transition at $p=0.3$ is driven by a turn about of the drift velocity of free clusters, which empties out the channel by clusters falling back into the entry reservoir.

The nature of the second order transition is set by "mother cluster" behavior (the cluster attached to the entry reservoir at $x=0$). It shields the bulk from current fluctuations; and thus transforming $J(\alpha)$ into an independently fluctuating variable; a control like variable.
Free clusters deep inside the bulk

Assume at least one particle exist to the left of the free cluster. Assume this free cluster is a mesoscopic stable stationary moving object, and that its internal density $\rho_c=1-v_c$ is large enough to maintain coherence (say $\rho>1/4$).

Treat such free cluster self-consistently, as if they are meso/macroscopic objects.

Free cluster equations:

\[ J_c = (1 - rv_c)v_c = p + \rho_c u_D \]
\[ v_{x+1} = \frac{J_c}{1 - rv_x} \]
\[ u_D = u_F = u_R \quad , \quad u_F = r \frac{p}{\rho_F} \quad , \quad u_R = 1 - \rho R - p \rightarrow \rho_F \cdot \rho_R = p \]

\[ \alpha = 1.0 \quad  \quad  t = 512 \quad  N_s = 256 \]

\[ P = 0 \quad \quad  0.1 \quad \quad  0.2 \quad \quad  0.3 \quad \quad  0.4 \quad \quad  0.5 \]
F-type free cluster state (front limited)
\[
\rho_c = \rho_F = \frac{1}{2r} \left( \sqrt{4p - 3p^2} - p \right)
\]
\[
J_c = r\rho_c = \frac{1}{2} \left( \sqrt{4p - 3p^2} - p \right)
\]
\[
u_D^{F} = r - \frac{p}{\rho_c} \rightarrow u_D^{F} = 0 \text{ at } p = \frac{1}{3}
\]
\[
p_s = \frac{1}{2} \left[ 3 - \sqrt{5} \right] = 0.191
\]

MC-type free cluster state
(bulk controlled)
\[
\rho_R = \frac{1}{2}, \rho_c = 1 - \frac{1}{2r}, \rho_F = 2p
\]
\[
J_c^{MC} = \frac{1}{4r}, \; u_D^{MC} = \frac{1}{2} - p
\]

The free cluster drift velocity in the forward limited F-state changes sign at p=1/3
Beyond the critical endpoint, $\alpha<0.6$, the transition is 2-nd order and induced by starvation of particle influx.

Empirically: $J(\alpha)$ is analytic at critical line, act as control variable. The injection current picture (decoupled current fluctuations) explains the linear vanishing of the order parameter.

\[
J(\alpha) = p \frac{N - N_c}{N} + J_c \frac{N_c}{N}, \quad \rho_c = \frac{N_p}{N_c}, \quad \rho = \frac{N_p}{N}
\]

\[
J(\alpha) = p(1 - \frac{\rho}{\rho_c}) + (p + \rho_c u_D) \frac{\rho}{\rho_c} = p + u_D \rho \quad \rightarrow \quad \rho = \frac{J(\alpha) - p}{u_D} \rightarrow \beta = 1
\]

How is this decoupling of the current fluctuations achieved?
Try a self consistent “mother cluster” approach:

A stable forward growing mesoscopic object, screening all current fluctuations, and emitting free clusters by breaking spontaneously at regular but stochastic intervals and places.

\[
J = \alpha v_1 = (1 - rv_c)v_c = p + \rho_c u_F, \quad u_F = r - \frac{p}{\rho_F}
\]

\[
v_{x+1} = \frac{J}{1 - rv_x - pP_x}, \quad P_{x+1} = v_x P_x
\]

\(P_x = \) probability for entire road between \(x = 0\) and \(x\) to be empty

- Clusters behave as independent objects.
- The mother cluster carries the current fluctuations.
- The free clusters carry the bulk density fluctuations.
GOOD:
- The F and MC mother cluster states similar to the free clusters states. This explains why J becomes independent of α in the “MC phase” and the C-MC phase boundary is reproduced reasonably well.
- The locations of the critical line and critical endpoint are reproduced accurately.

Mother cluster self consistency equations

\[ J = \alpha v_1 = (1 - rv_c) v_c = p + \rho_c u_F \quad , \quad u_F = r - \frac{p}{\rho_F} \]

\[ v_{x+1} = \frac{J}{1 - rv_x - pP_x} \quad , \quad P_{x+1} = v_x P_x \]

\( P_x \) = probability for entire road between \( x = 0 \) and \( x \) to be empty

BAD:
- In the R state, the front boundary condition can not be satisfied. The solution is only stable and “stationary” in its rear. Moreover, in the bulk mother cluster density is too low (iteration to the unstable low density fixed point).
- The R state is identical to the global mean field solution with uniform density (clustering unstable). Acts like a finite open ended system.

The “mother cluster “does not truly exist beyond the length scale where \( P_x \rightarrow 0 \); that distance defines the onset of a strongly fluctuating non-stationary region where new free clusters assemble.
At $p_c$ the bulk density obeys uncorrelated random noise type finite size scaling

$$\rho \sim L^{x_\rho} \quad \text{with} \quad x_\rho = \frac{1}{2}$$

This represents the uncorrelated random current fluctuations inside the “mother-cluster”. $p_c$ is not special for those fluctuations, but the absence of free clusters integrates them over $t \sim L$ (the time of flight).
Conclusions  PART 2:

1. The stick-slip non-local hopping ASEP generalization has a first or second order transition into an empty road phase.

2. The stationary state has strong local clustering.

3. The first-order transition is induced by the drift velocity of free clusters turning negative.

4. Clusters behave as independent objects.

5. The second-order transition has simple critical exponents associated with the fact that the current behaves as an independent fluctuation control variable; all current fluctuations are generated and limited to the “mother clustering” near the x=1 entry point.