Scaling in stationary states of one dimensional driven non-equilibrium processes

Marcel den Nijs, UW, Seattle

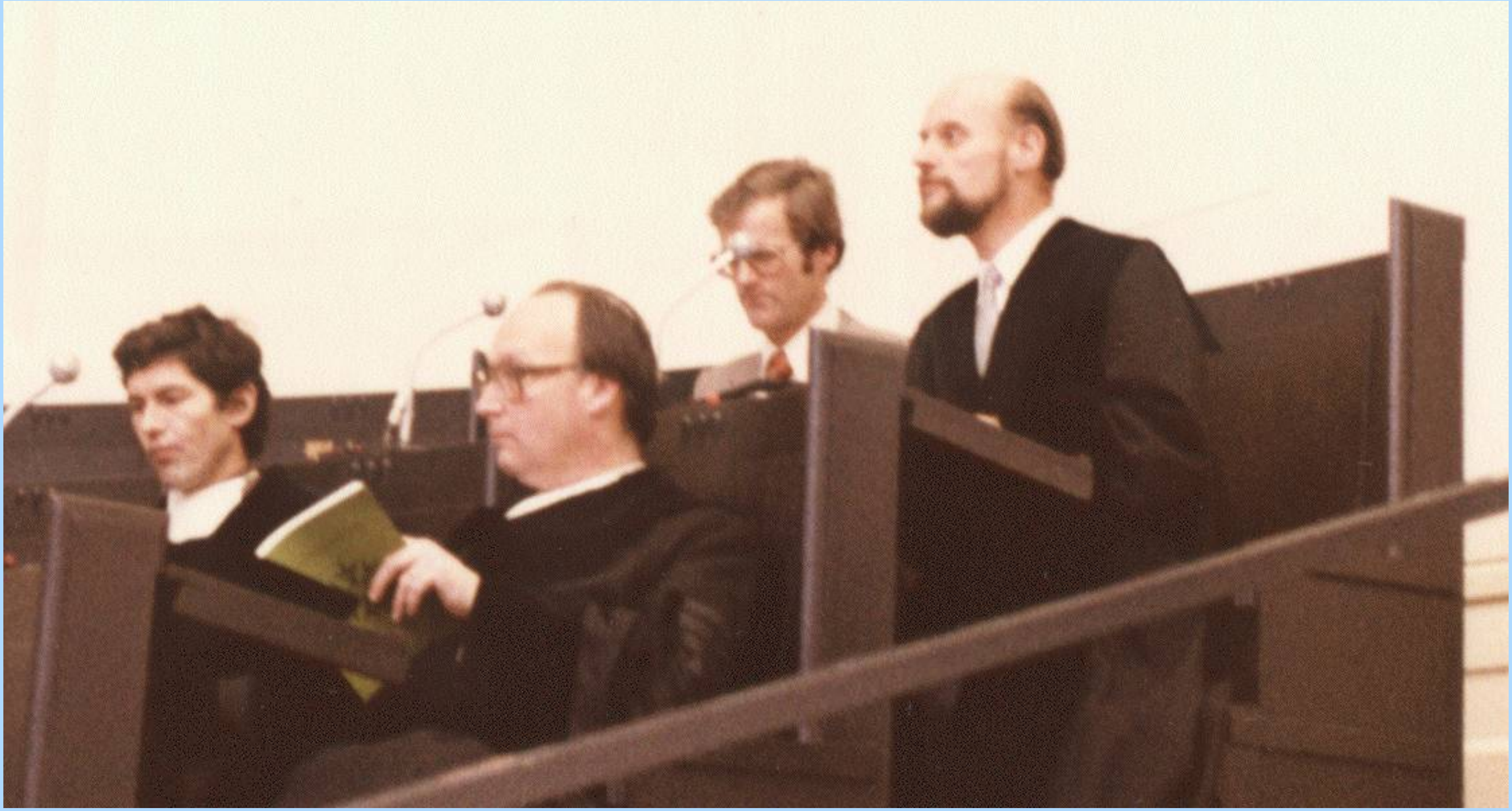
Hubert's Parting Party

October 07 2005

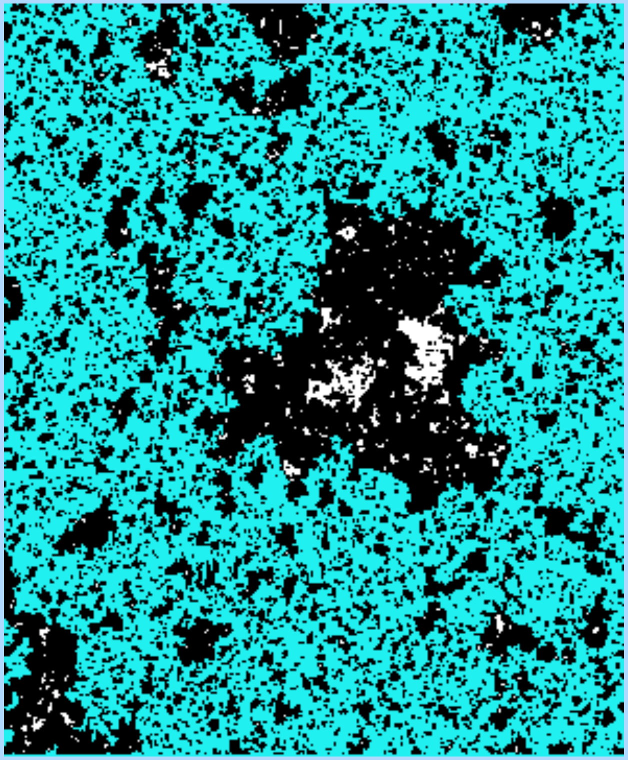
How does what we do now relate to what we studied then?

A. Two dimensional equilibrium critical phenomena in retrospect

B. Dynamic phase transitions in one dimensional driven non-equilibrium processes.



Dec 13, 1979



2D Ising configuration, Delft special purpose computer, mid 1980-ties

The distributions associated with an ensemble of critical configuration is invariant such that correlation functions and expectation values

$$G_{\gamma\delta}(r_1 - r_2) = \left\langle O_\gamma(r_1) O_\delta(r_2) \right\rangle - \left\langle O_\gamma(r_1) \right\rangle \left\langle O_\delta(r_2) \right\rangle$$

are scale invariant

$$M_\gamma = L^D \left\langle O_\gamma(r_1) \right\rangle \sim L^{D - x_\gamma} \sim L^{y_\gamma}$$

$$G_{\gamma\delta}(r, \{u_\alpha\}) = b^{-x_\gamma - x_\delta} G_{\gamma\delta}(b^{-1} r, \{b^{y_\alpha} u_\alpha\})$$

$$x_\alpha = D - y_\alpha$$

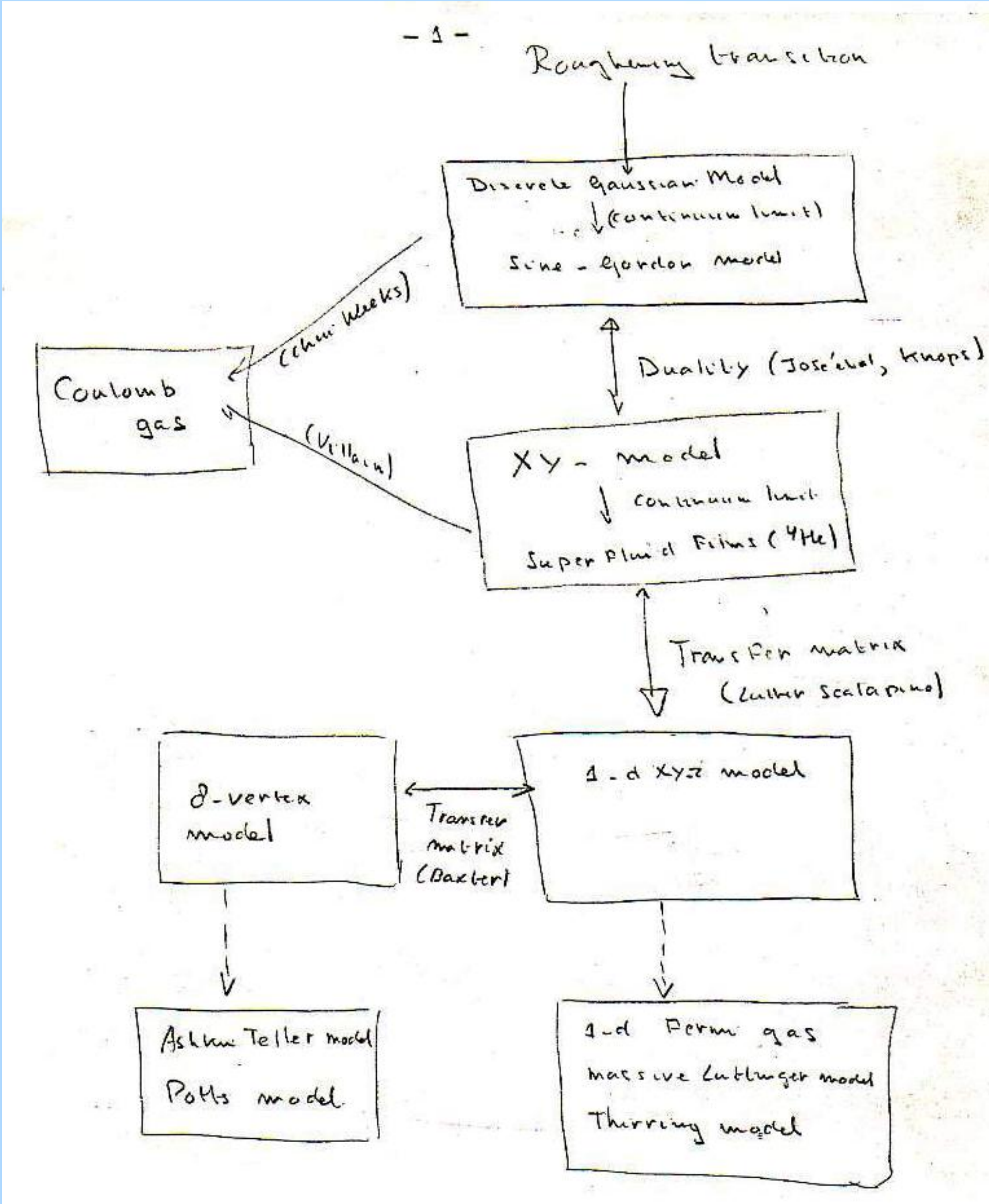
The set of critical exponents

$$\{y_T, y_H, \ldots.\}$$

represent the fractal dimensions of various geometric features in the critical configurations, like droplet

area, coast line, topological connectivity, etc

The various fractal dimensions of the object have independent values within scaling theory



Exact Coulomb gas mappings between 2D equilibrium critical phenomena universality classes

“All” 2D critical points map onto free massless scalar field theory (= line of fixed points = Baxter line).

“All” temperature, magnetic field, etc, fluctuations can be represented as topological excitations (vortices and/or spin waves)

exact values and relations between critical exponent

←Hubert Knops lecture notes mid 1970-ties

I Potts Model — exact results; extended scaling relations

II Coulomb gas formulation; why?

III Potts Model = Whitney Polynomial
= generalized 6-vertex model
= generalized Body-Centered Solid-On-Solid Model
= Coulomb gas with modulo 4 charges

IV Self dual line = exact soluble 6-vertex model
= Coulomb gas with $Q = \pm 4$ charges

$q<4$ only bound pairs
$q=4$ Kosterlitz-Thouless transition
$q>4$ free $Q=\pm 4$ charges

$x = 2 - y$

$\cos\left(\frac{\pi y}{2}\right) = \frac{1}{2}\sqrt{q}$

V Energy-Energy correlation function ; $Q = +2$ charges

(+2) — ≠ — (+2)
(−4) screening
$Q=-4$ mod. 4 point charge

$$X_T = \frac{3}{x} - 1$$

den Nijs (1979)
Emery and Black (1981)

VI Spin-Spin correlation function ; $Q = 1-y, -1-y$ charges.

(1−y) — ≠ — (−1−y)
(+2y) metal box

$$X_{H_1} = \frac{1-y^2}{4x}$$
$$X_{H_2} = \frac{9-y^2}{4x}$$
⋮

Nienhuis et al (1980)
Pearson (1980)
den Nijs (1982)

Coulomb gas reformulations of the Potts model and O(n) model (MdN, Nienhuis,…)

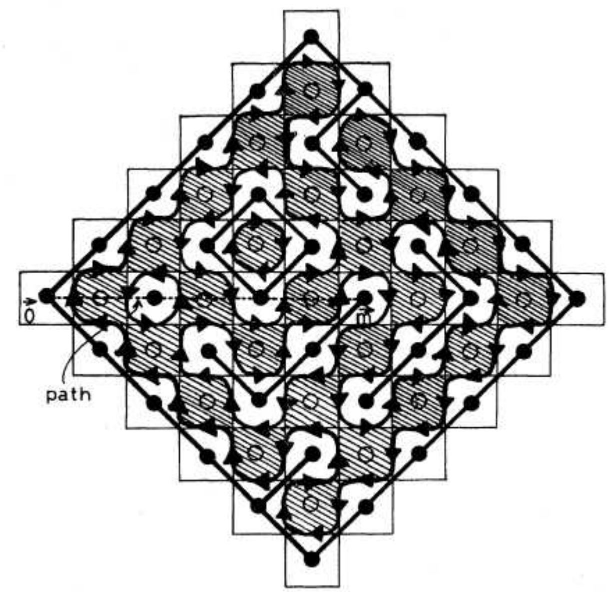
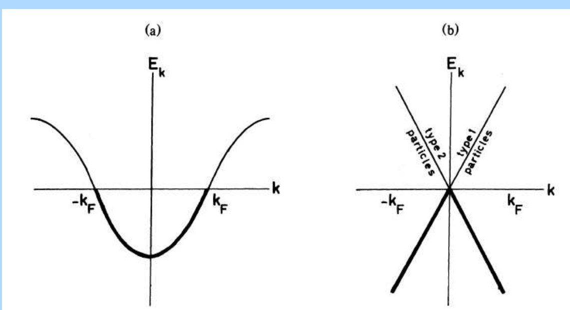
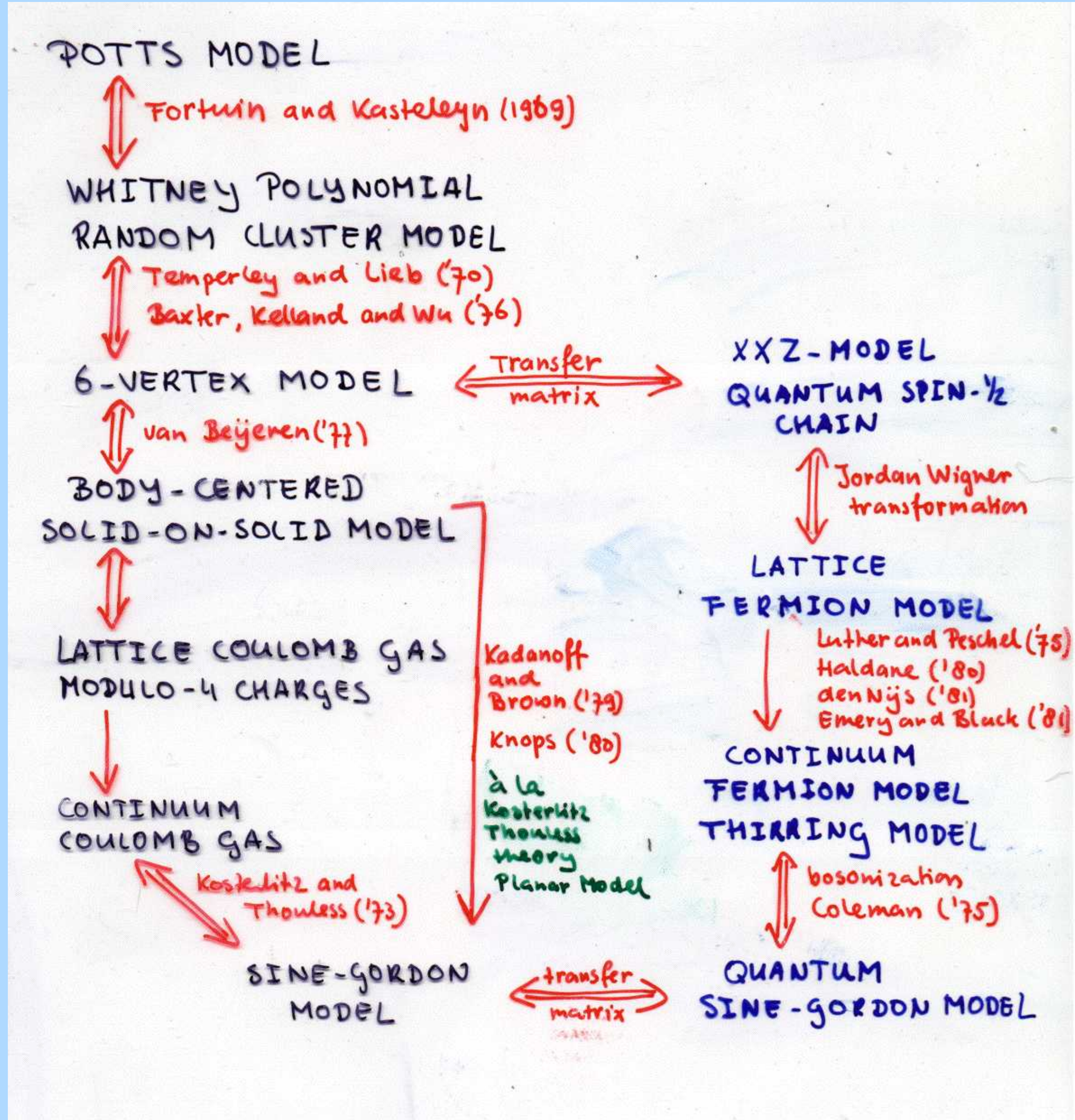


FIG. 1. Polygons in graph formulation of Potts model. Closed circles are sites of lattice $\mathscr{L}$, open circles are sites of dual lattice $\mathscr{D}$; polygons follow bonds of surrounding lattices $\mathscr{S}$; dotted line is path of summation in Eq. (9).

Equivalence between D+1 dimensional equilibrium statistical mechanics and ground state properties of D dimensional QFT

One dimensional conductors, e.g. nano tubes, Tomonaga-Luttinger liquids and bozonization





2 D Gaussian model

$$H = s^2 \sum_{\vec r} \frac{K_x}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{K_y}{2}\left(\frac{\partial\phi}{\partial y}\right)^2 \qquad \phi(\vec r)$$

$$+ \frac{(\beta-1)}{s^2}\cos[2\phi(\vec r)]$$

$$+ \frac{u_4}{s^2}\cos[4\phi(\vec r)]$$

(free) scalar field theory (log transfer matrix Gaussian model)

$$H = s\sum_x \frac{1}{2K_y}p^2 + \frac{1}{2}K_x\left(\frac{\partial\phi}{\partial x}\right)^2 \qquad [p(x),\phi(x)] = -i\,\delta(x)$$

$$+ \frac{(\beta-1)}{s}\cos[2\phi(x)]$$

$$+ \frac{u_4}{s}\cos[4\phi(x)]$$

(massless) Thirring model

$$H = s\sum_x i\,\frac{1}{2}\left(\pi K_x + \frac{1}{\pi K_y}\right)\left[\psi_1^+\frac{\partial}{\partial x}\psi_1 - \psi_2^+\frac{\partial}{\partial x}\psi_2\right]$$

$$+ \frac{1}{2}\left(\pi K_x - \frac{1}{\pi K_y}\right)\left[\psi_1^+(x)\psi_1(x)\,\psi_2^+(x)\psi_2(x)\right]$$

$$+ (\beta-1)\left[\psi_1^+(x)\psi_2(x) + \psi_2^+(x)\psi_1(x)\right] \qquad \{\psi_i^+(x),\psi_j(y)\} = \delta(i-j)\,\delta(x-y)$$

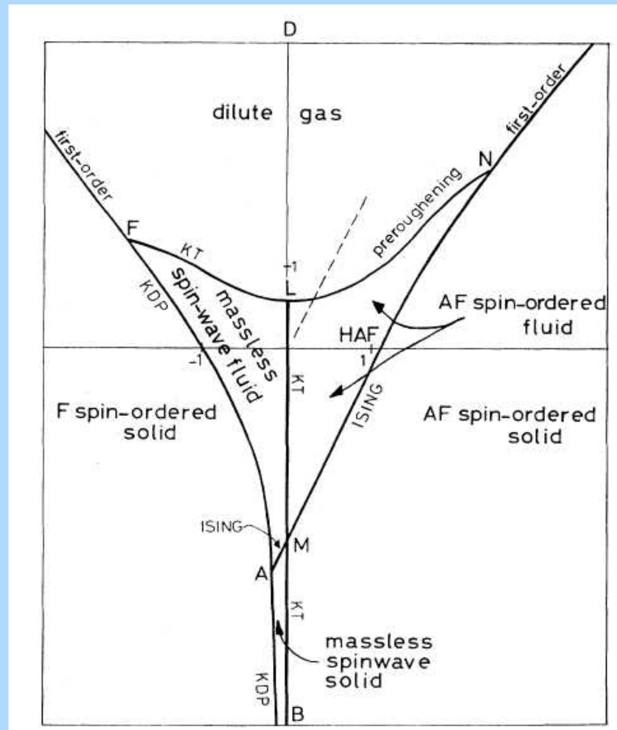
$$+ s\,u_4\left[\psi_1^+(x)\psi_2(x)\,\psi_1^+(x+s)\psi_2(x+s) + c.c.\right]$$

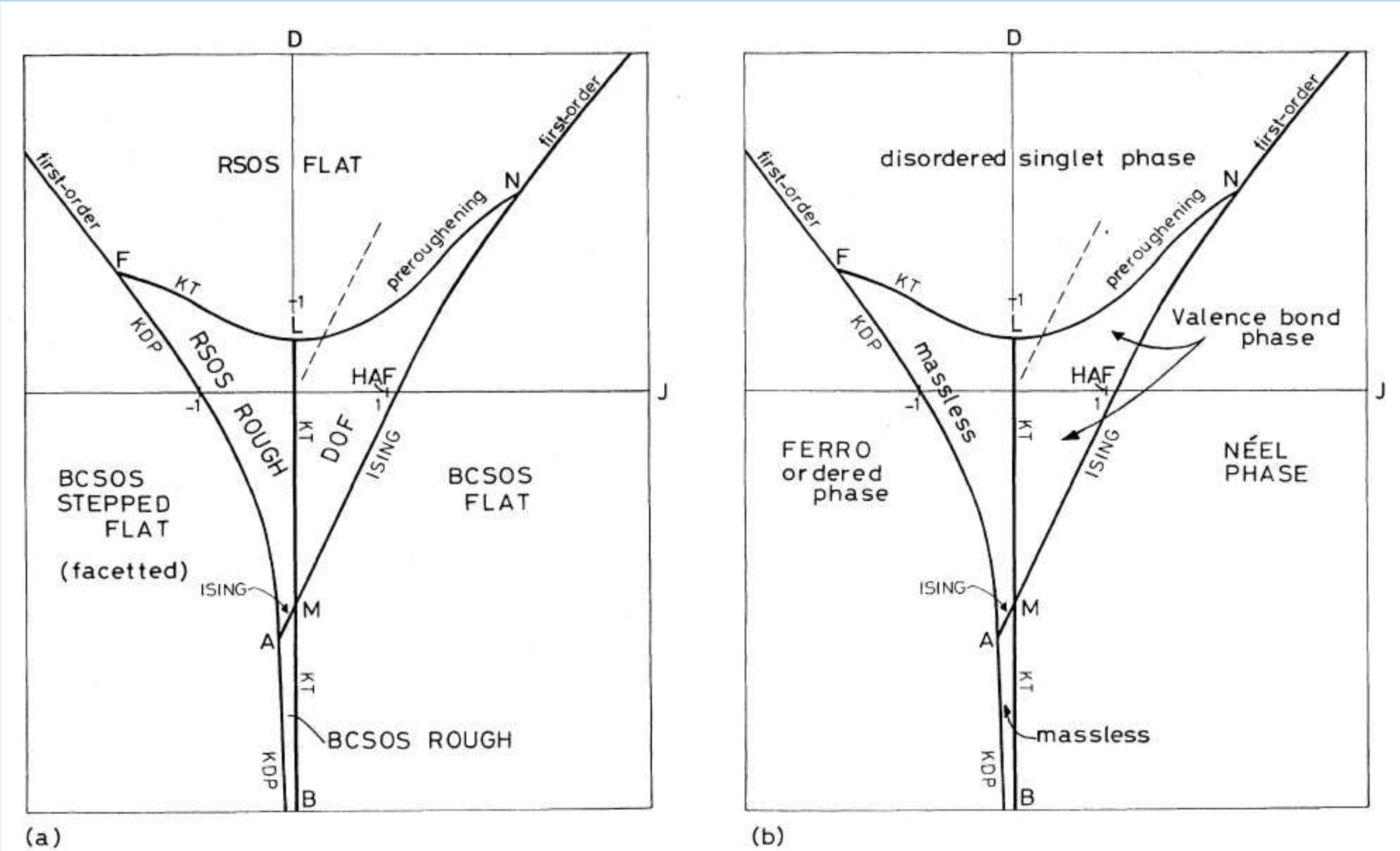
Scaling limit equivalents of the Ashkin-Teller model

at every site : 2 Ising spins (S_i, T_i)

$$H = \sum_{\langle ij\rangle} K\,S_iS_jT_iT_j + (J_1+K)\,S_iS_j + (J_2+K)\,T_iT_j$$

Preroughening transitions too!





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RSOS FLAT
first-order
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RSOS ROUGH
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BCSOS STEPPED FLAT
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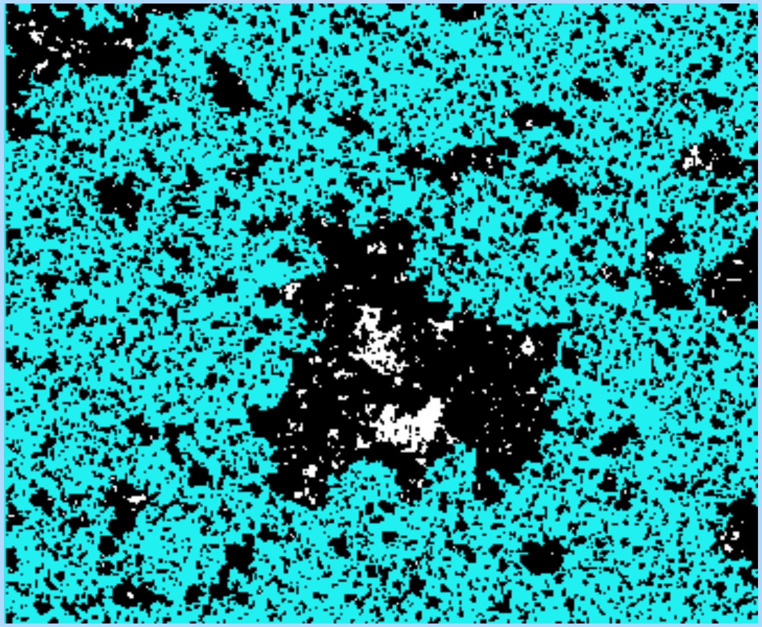
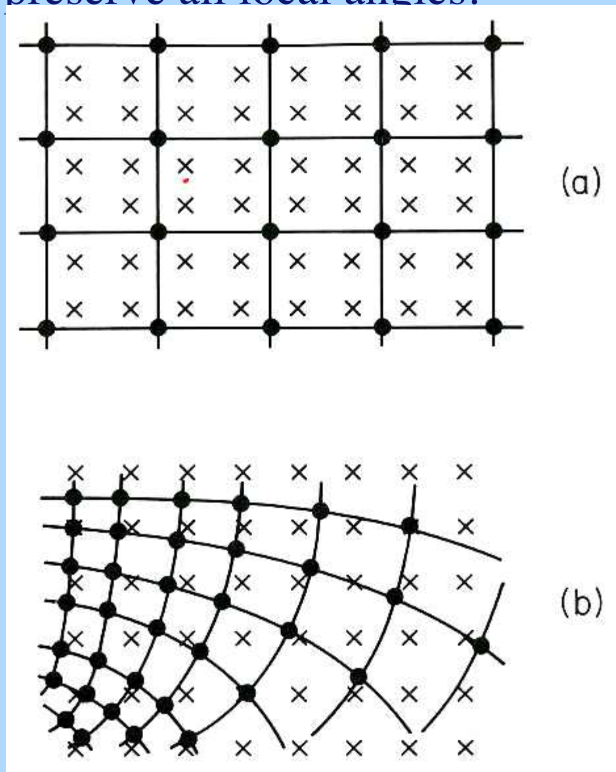


Conformal invariance (>1982):

(Polyakov, Zamalochikov, Shenker, Friedan, Cardy, …)

The ensemble of critical configurations {S(z)} with z=x+iy is invariant under all analytic complex functions w(z); i.e., deformations of space that preserve all local angles.



The local geometrical variables mix and their critical dimensions become linked to each other.

Confirms critical exponent relations found earlier by the Coulomb gas method

Can handle higher order correlation functions

2D critical phenomena and 1+1D strongly interacting quantum Hamiltonians share now the very elite status, in that they are analytically well understood.

(Conformal invariant, Lorentz invariant, boson-fermion equivalences, loop gas mappings, elliptic functions class exact solutions of specific discretized space/time models, etc)

Can we generalize this?
Can we escape for other classes of scale invariant phenomena slavery to numerical simulations interpreted with scaling analysis) as well?

- Upper critical dimensions, mean field theory, and field theoretical ε-expansion type RT classify universality classes and are helpful to prove the existence of "fixed points" but do not provide the exact scaling dimensions values.
- The conformal group in D>2 is too small.
- Dynamic phase transitions in non=equilibrium driven systems have typically anisotropic scaling in space-time (non-conformal invariant even in 1+1D)

Schramm-Loewner evolution

We return to the halfplane version of the Loewner equation, which will be our setting for the remainder of the thesis. For $\kappa \geq 0$, set $\lambda(t) = \sqrt{\kappa} B_t$, where B_t is standard Brownian motion. Then chordal SLE_κ is the random family of conformal maps generated by λ, that is, the family of maps solving the following stochastic differential equation:

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t},$$
$$g_0(z) = z.$$

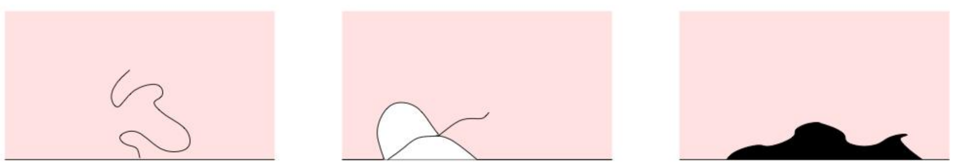


Figure 1.2: The SLE_κ trace is a simple curve, a non-simple curve, or a space-filling curve depending on the value of κ. The colored region is the domain G_t.

From: PhD thesis, Joan Lind, UW (Math Dept, Rohde), Spring 2005

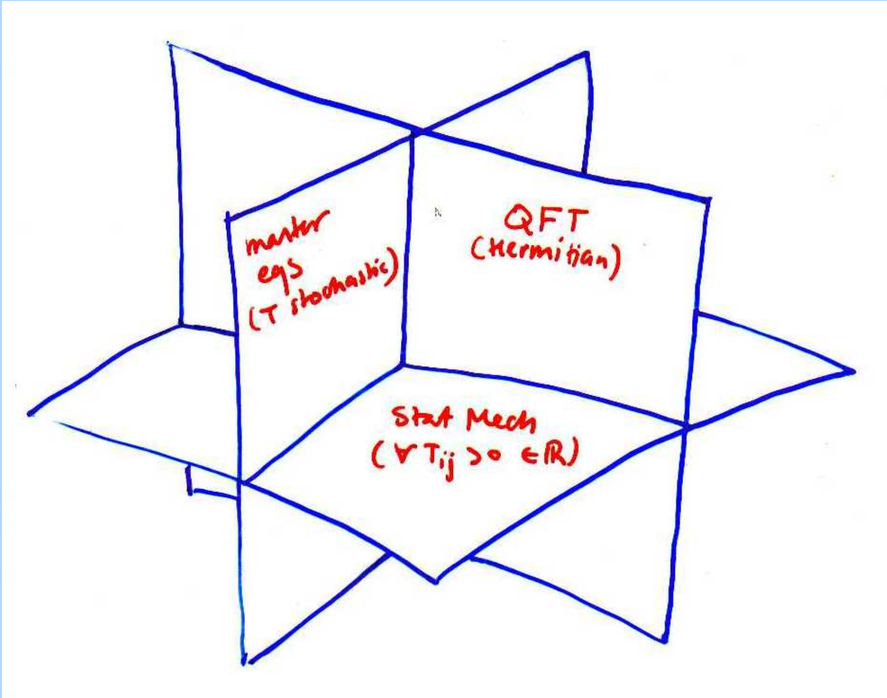
Seeking generalizations from within:

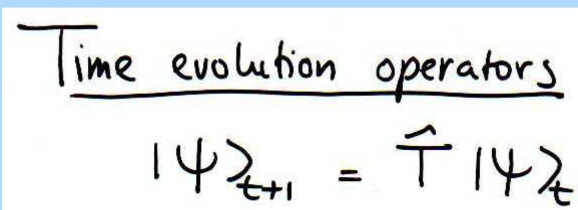
SLE

The Coulomb gas and conformal field theory critical dimension relations are rederived (more rigorously) in terms of fractal object generated by an iterative dynamic process.

Mostly mathematicians are working on this + Cardy+ Nienhuis + …..

New results yet ?
New physics yet ?





Seeking generalizations by connecting to different physical processes:

1+1D driven stochastic non-equilibrium processes

-transfer matrices of 2D equilibrium stat mech
-time evolution operators 1+1D QM (path integrals)
-master equations 1+1D dynamic processes

Master Equations require stochastic transfer matrices (left eigen vector of ground state is the disordered state)

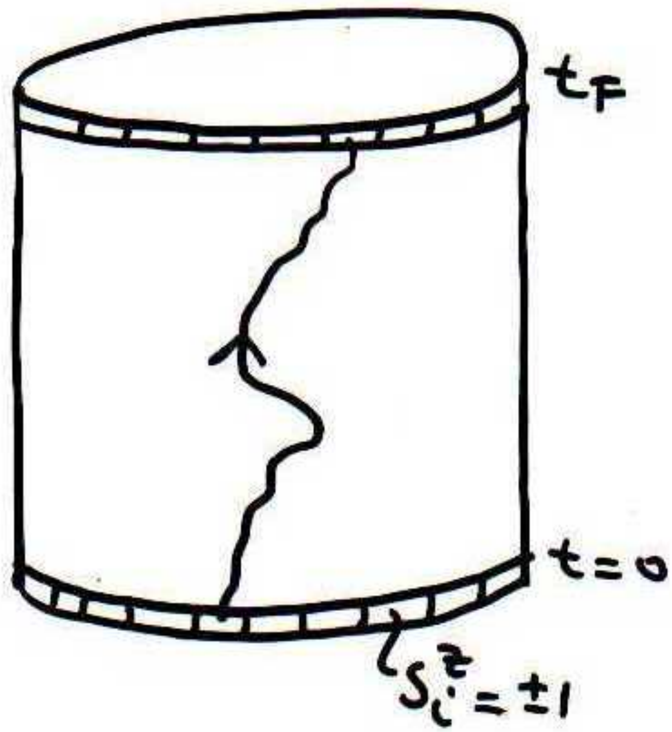
D dimensional Quantum Mechanics:

$$\langle \psi | \psi \rangle = 1$$

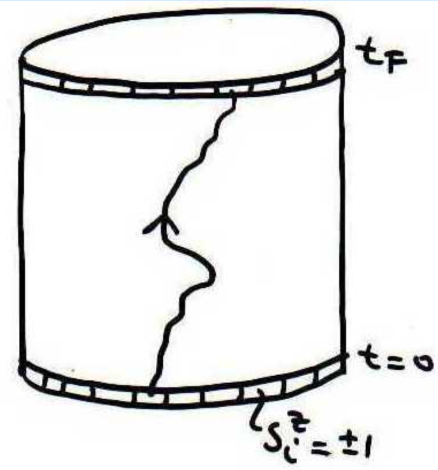
$$\hat{T} = e^{-i\hat{H}} \qquad \hat{T}^{+}\hat{T} = 1$$

$$\langle \{S_i^z\}_{t_F} | \hat{T}^{t_F} | \{S_i^z\}_0 \rangle$$

= transition probability



D+1 dimensional Equilibrium Statistical Mechanics:



$$e^{-F/k_BT} = Z(\{S_i^z\}_{t_F} \mid \{S_i^z\}_0)$$

$$= \langle \{S_i^z\}_{t_F} \mid \hat{T}^y \mid \{S_i^z\}_0 \rangle$$

partition function for specific bound. cond.

$y_F = i\, t_F$ Euclidian time

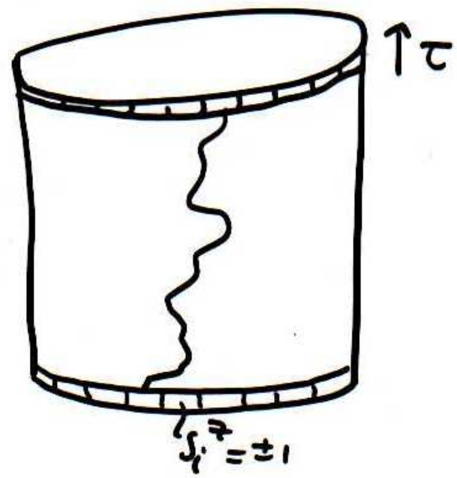
$$|\Psi\rangle_t = \sum_{\{S_i^z\}} Z(\{S_i^z\}_t \mid \{S_i^z\}_0)\; |\{S_i^z\}\rangle$$

$$|\Psi\rangle_{t+1} = \hat{T}\,|\Psi\rangle$$ builts an extra y-slice onto the D+1 spatial lattice

↑ matrix elements are Boltzmann factors for interactions concerning slice $it < y < i(t+1)$ only

Master equations:

$$|\psi\rangle_t = \sum Z_t(\{S_i^z\})\,|\{S_i^z\}\rangle$$



$Z_t(\{S_i^z\})$ = prob. to be in microstate $\{S_i^z\}$ at time t

$$\sum_{\{S_i^z\}} Z_t(\{S_i^z\}) = 1$$

define: $\langle D| = \sum_{\{S_i^z\}} \langle \{S_i^z\}|$ "disordered state"

$$\langle D|\psi\rangle_t = \sum_{\{S_i^z\}} Z_t(\{S_i^z\}) = 1$$

$\langle D|\hat{T}|\psi\rangle_t = \langle D|\psi\rangle_{t+1} = 1 \Rightarrow \langle D|\hat{T} = \langle D|$ left eigenvector with $\lambda_0 = 1$

How do 1+1D stochastic dynamic processes (Master equations) fit in the 2D equilibrium and QFT universe?

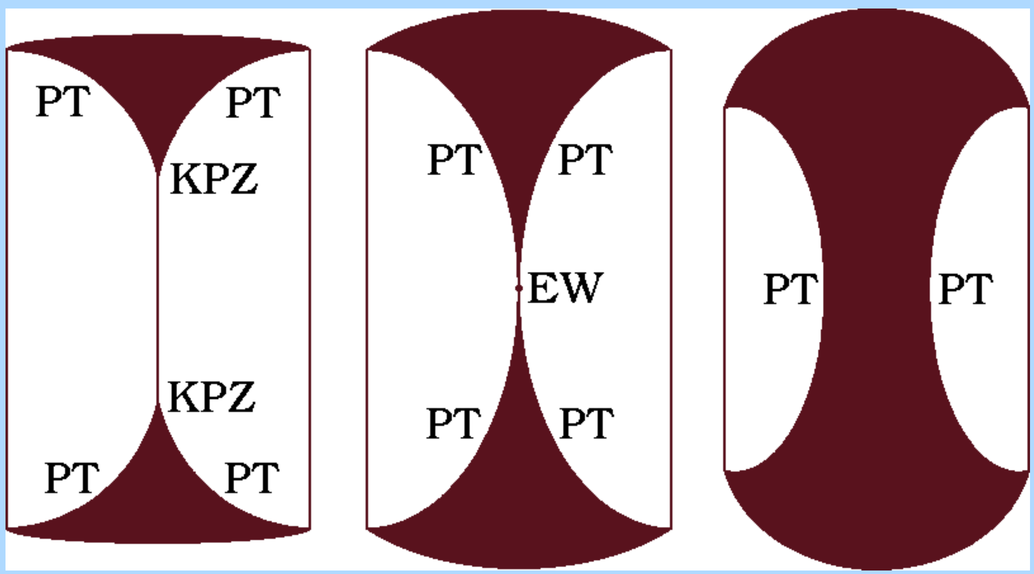
- Anisotropic scaling, non-conformal! (Lifshitz point like)
- Stochastic dynamic processes are located at the edge conformal invariant scaling.
- 1+1D dynamic phase transitions, like directed percolation, require stationary states with long range effective "interactions".
- Those with no or short range interactions are unstable to edge effects (boundary and point defect induced phase transitions.
- Some exact solutions but no non-conformal Coulomb gas method generalizations yet !…??…no novel "free field theories".
- KPZ growth and Asymmetric Exclusion processes (discretized Burgers equation) serve as bench marks and illustrations for this.

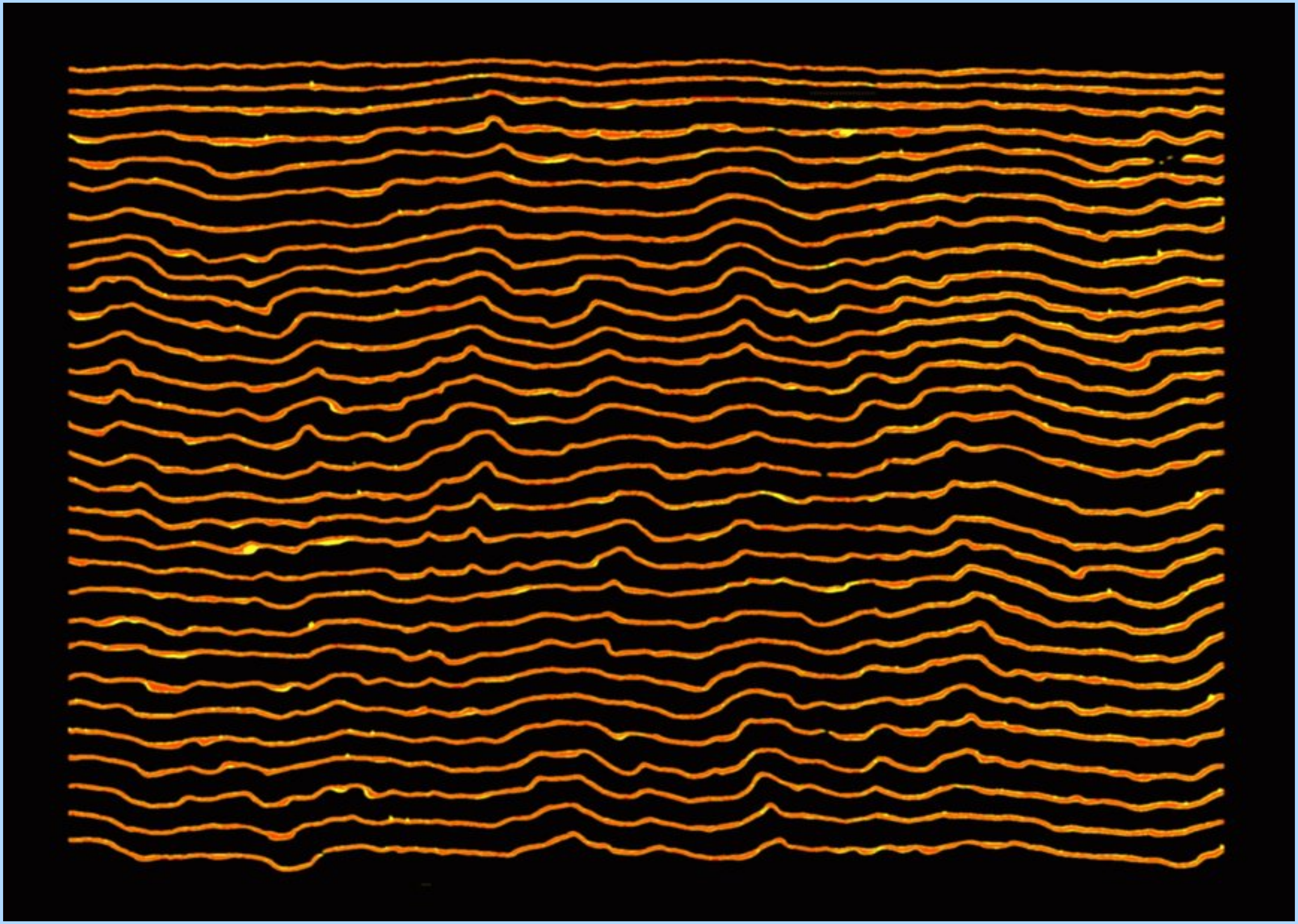
driven stochastic flow through1D channels

1+1D KPZ surface growth

2D polymer localization

facet ridge end points in 3D equilibrium crystals

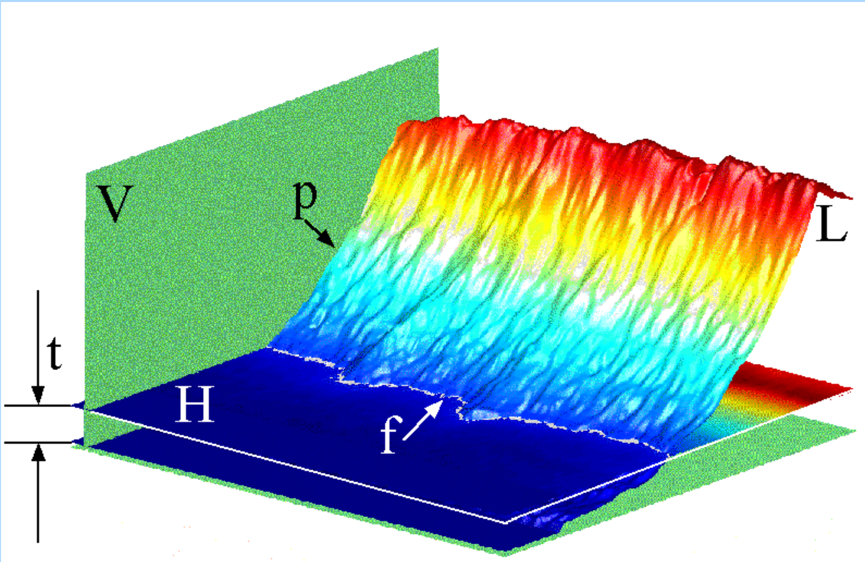




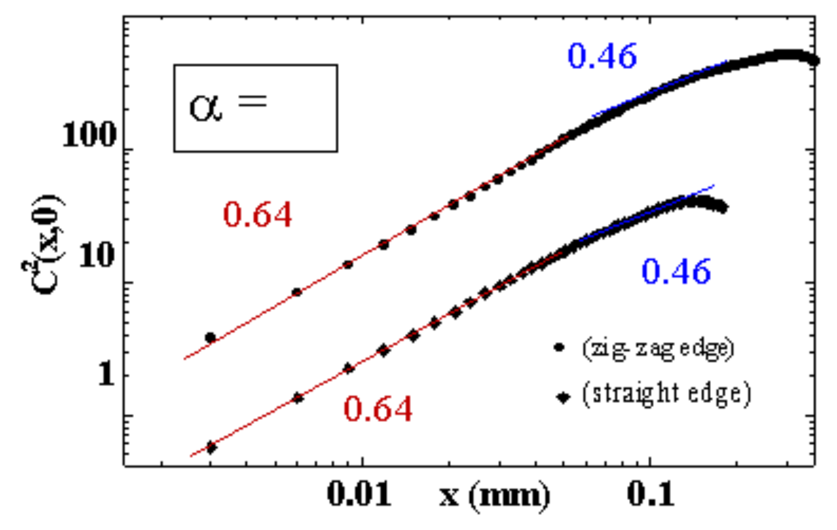
Another example of KPZ growth:

Flux front propagation in High Tc super conductors

Wijngaarden's group at the VU in Amsterdam PRL 83, 2064 ('99)



Spatial distribution of vortex density (plotted along the vertical axis) in a $YBa_2Cu_3O_7$ thin film in a field of 11 mT.



Surface roughness

The moments of the height distribution

$$W_n(L,t) = L^{-1} \sum_r \langle (h_r - h_{av})^n \rangle$$

scale as

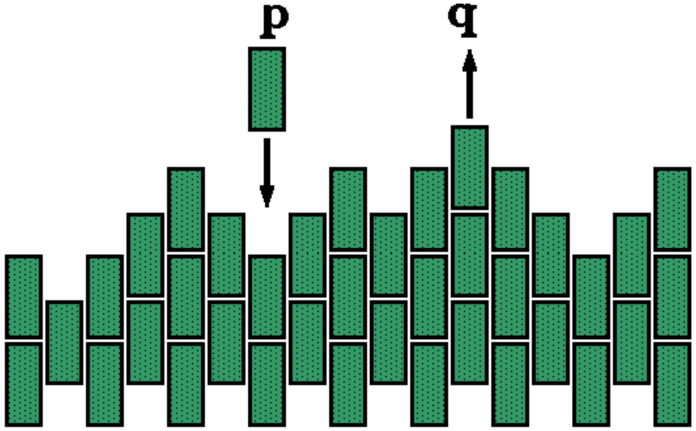
$$W_n(t,L) = b^{n\alpha} W_n(b^{-z}t, b^{-1}L)$$

$$\alpha = \tfrac{1}{2} \qquad z = 2 - \alpha = \tfrac{3}{2}$$

1+1 dimensional KPZ growth exact results

- The 1D KPZ stationary state is still trivial; the Gaussian distribution. In the BCSOS brick growth model the up/down steps are placed at random
$$\rightarrow \alpha = \frac{1}{2} \rightarrow z = 2 - \alpha = \frac{3}{2}$$
- The exact Bethe Ansatz solution of the BCSOS model confirms $z = 2 - \alpha = \frac{3}{2}$.
- The linear theory fixed point is unstable with relevant crossover exponent $y_\lambda = \frac{1}{2}$. Crossover scaling yields $z = z_{EW} - y_\lambda = 2 - \frac{1}{2} = \frac{3}{2}$, with as only assumption that λ is a redundant scaling field for KPZ growth.

BCSOS (brick laying) growth



- Rectangular building blocks (brick wall on its side).

– Nearest neighbour heights differ by only $\Delta h = S_n^z = \pm 1$.

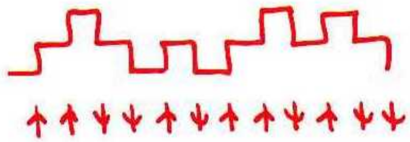
Growth rule: Select at random one of the columns. If this column is the bottom (top) of a local valley (hill top), a particle adsorbs (disorbs) with probability p (q) Local slopes are inactive ($\rightarrow \lambda < 0$).

– Early numerical studies: Meakin, Family, $\cdots$.

– In 1D, surface fully characterized by spin-$\frac{1}{2}$ type step variables $\rightarrow$ Master equation: XXZ quantum spin chain.

– Bethe Ansatz exact solution in 1D: Dhar, Gwa/Spohn.

$$|\psi\rangle_{t+1} = \hat{T}\,|\psi\rangle_t$$



$$\hat{T} = \frac{1}{N}\sum_i \left[1 + p\,(S_i^+ S_{i+1}^- - 1)\,\delta(\downarrow\uparrow) + q\,(S_i^- S_{i+1}^+ - 1)\,\delta(\uparrow\downarrow)\right]$$

$$= 1 - \frac{p+q}{N}\sum_i \left[\frac{1}{4} - \frac{1}{4} S_i^z S_{i+1}^z - \frac{p}{p+q} S_i^+ S_{i+1}^- - \frac{q}{p+q} S_i^- S_{i+1}^+\right]$$

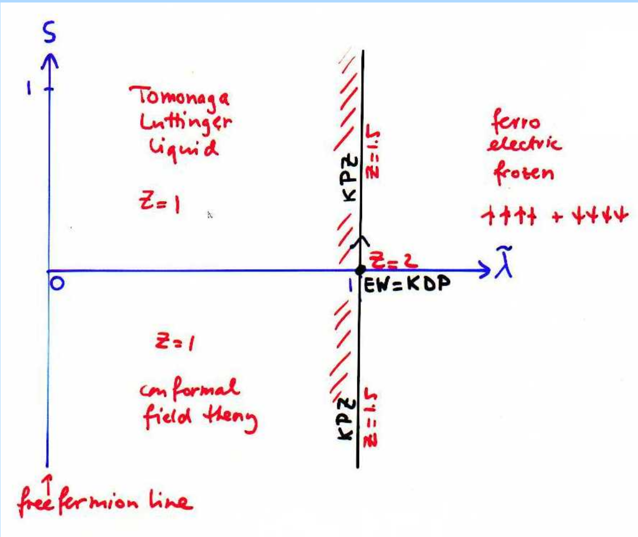
$$= 1 - \frac{p+q}{N}\hat{\mathcal{H}} \simeq e^{-\frac{p+q}{N}\hat{\mathcal{H}}}$$

↑ $p+q$ rescales time intervals

$$s = \frac{p-q}{p+q}$$

$$\hat{\mathcal{H}} = \sum_i \left[\frac{1}{4} - \frac{1}{4}\vec{S}_i \cdot \vec{S}_{i+1} - \frac{s}{2}\left(S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+\right)\right]$$

↑ no growth case ≡ ferro electric Heisenberg point!

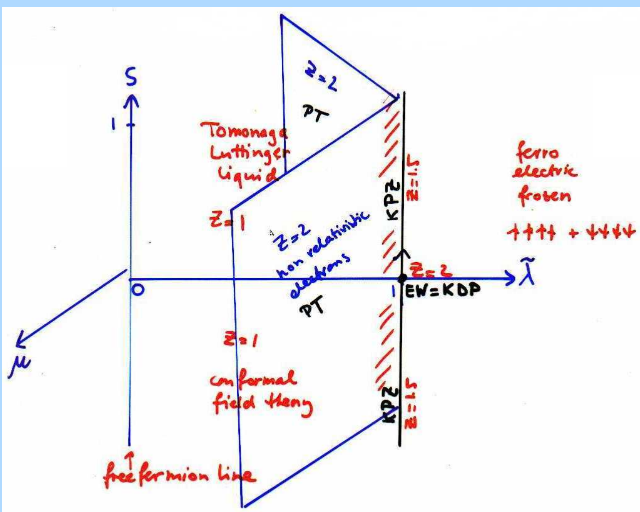


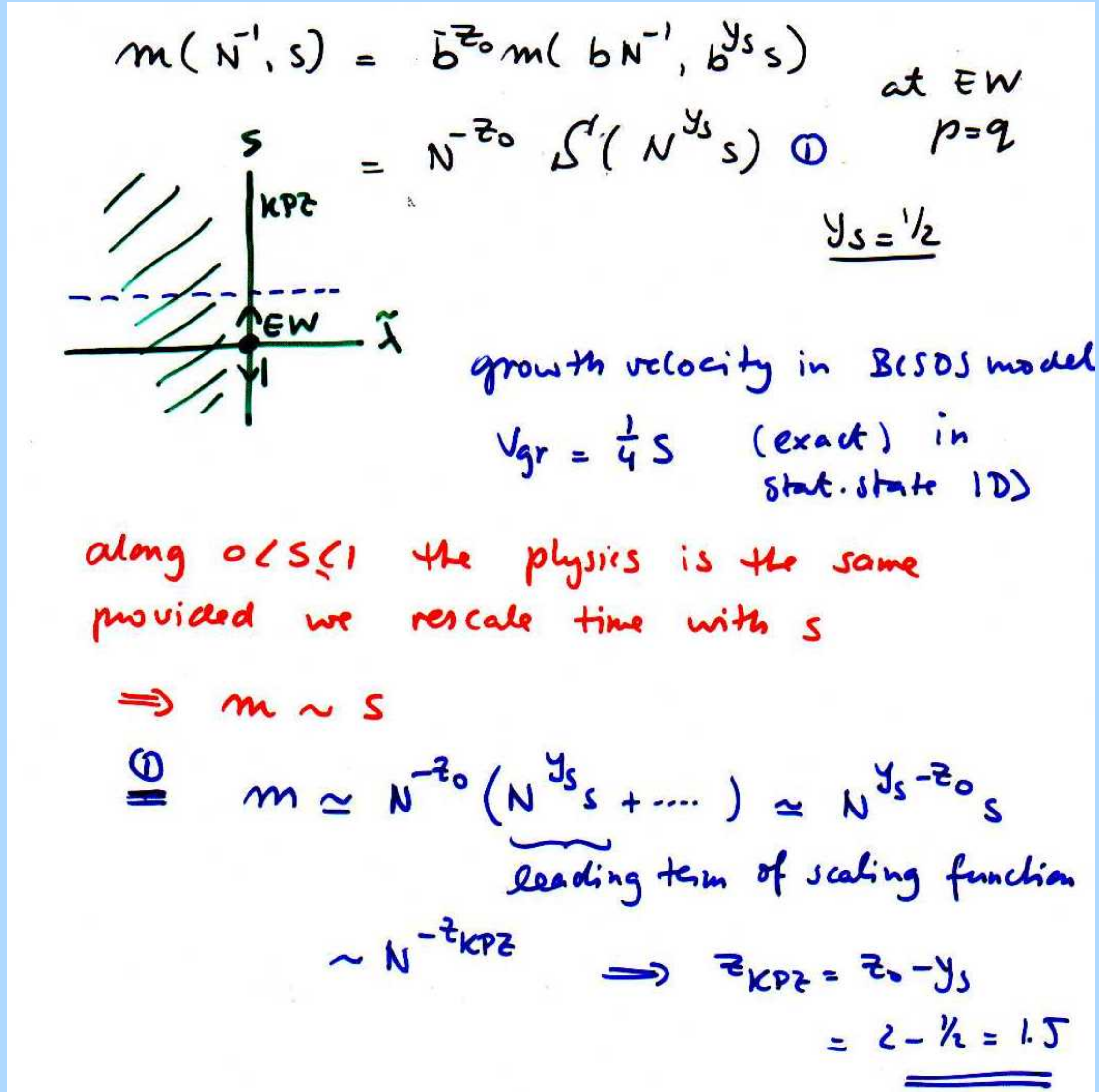
Only 2 known "free field theories":

- Relativistic electrons (CFT) with z=1

- Non-relativistic electrons (PT, EW) with z=2

"z-theorem":
unstable dynamics has always larger dynamic exponent (for z>1)





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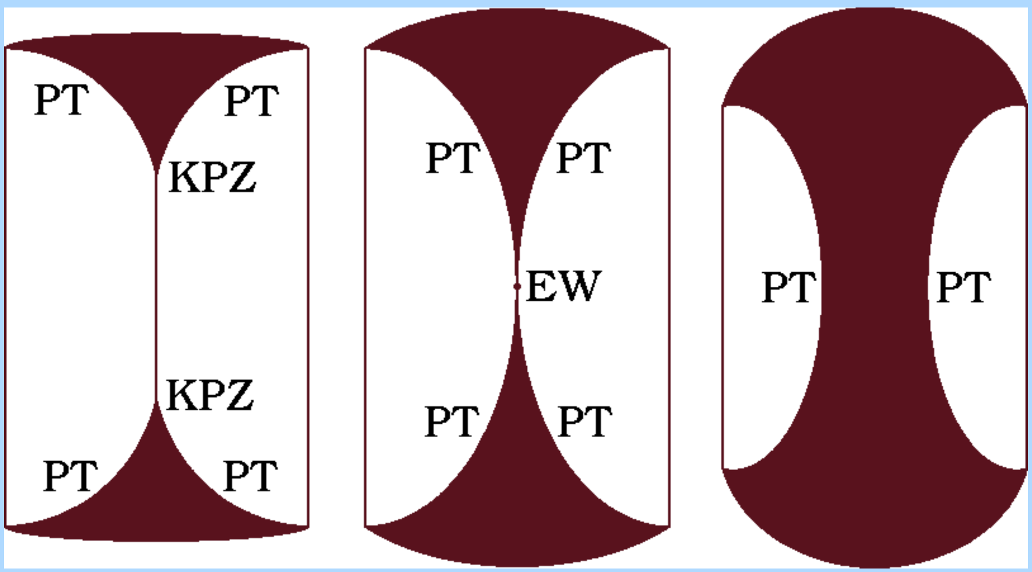
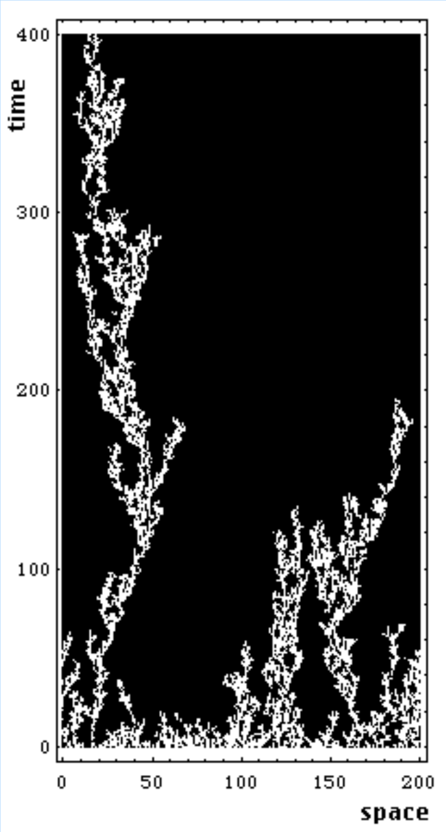
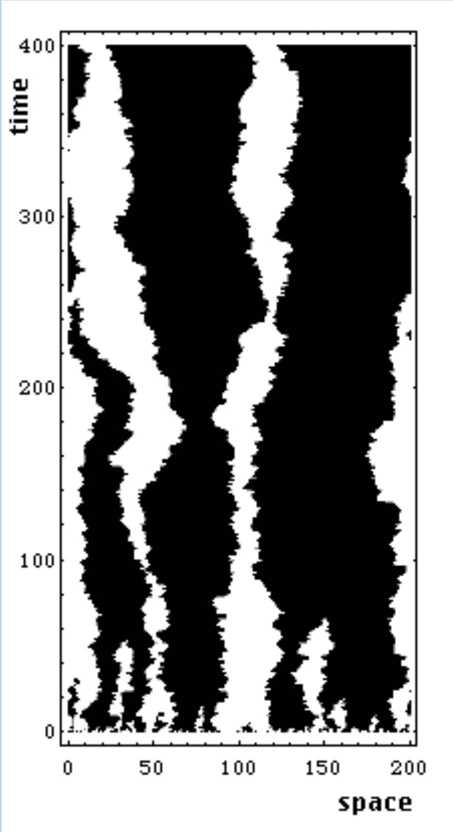
Conclusion:

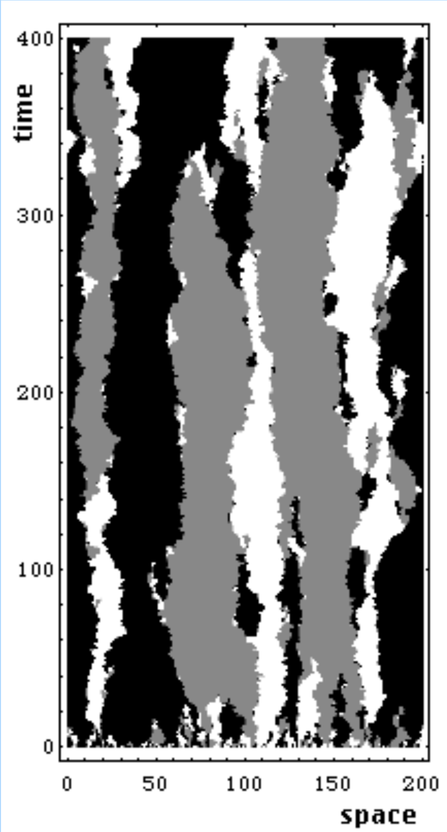
Master equations of 1+1D stochastic processes with z>1 live on the edge of the z=1 conformal invariant world



Can we construct some type of anisotropic scaling deformations of CFT?







Directed percolation, directed Ising, directed q=3 Potts, etc, processes with dynamic phase transitions in their stationary states also live on the edge of CFT.

All have stationary states with long range "effective interactions" in Boltzmann like formulation, $P(\Gamma)=\exp[-E(\Gamma)]$ (needed to evade van Hove's theorem)

The exact values of their scaling indices are yet unknown even in 1+1D

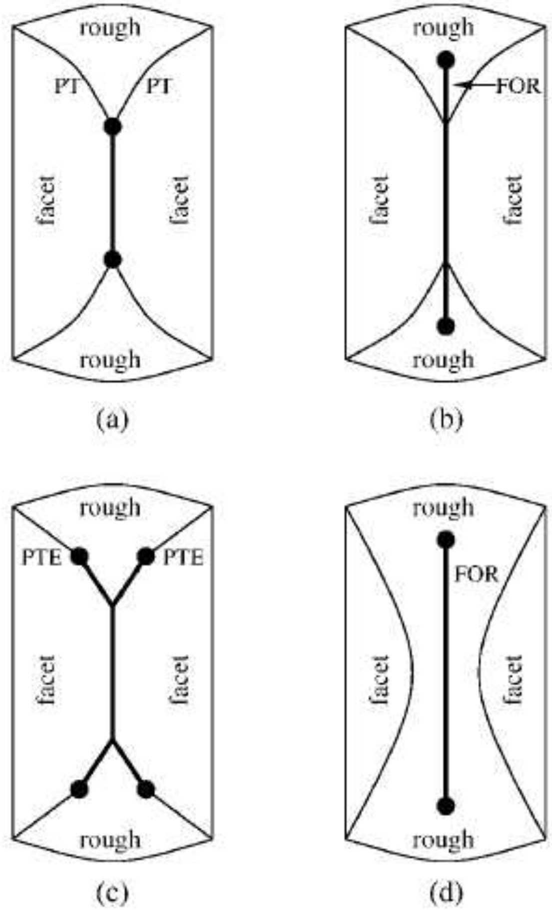


FIG. 1. Equilibrium crystal shapes in the BCSOS model with enhanced interaction range: (a) ECS in the exactly soluble square lattice BCSOS model with stochastic FRE point. (b) ECS with a first-order line extending into the rough area. (c) ECS with first-order facet-to-round boundaries and PTE points. (d) ECS with a spontaneous tilted rough phase, i.e., with a first-order ridge inside the rough phase.

Davidson and MdN 1999

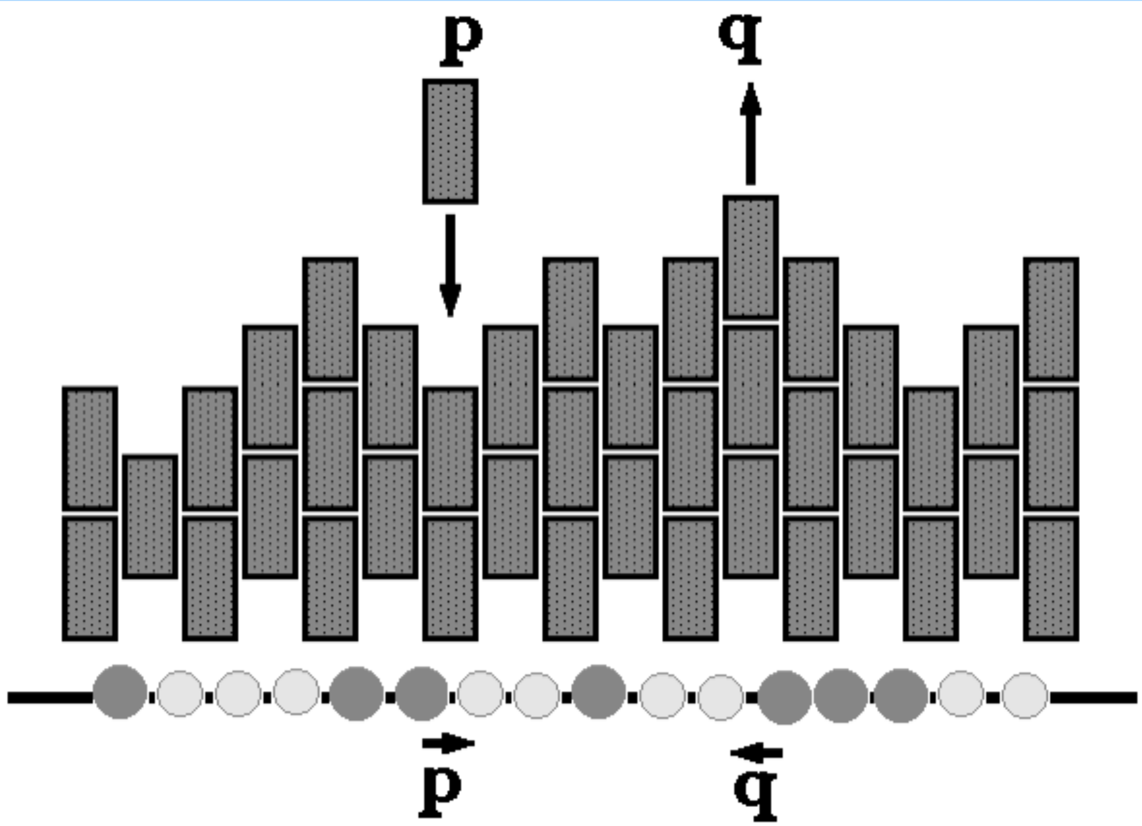
Stochastic matrix critical behavior is typically unstable, consistent with the living on the edge of the CFT world

Adding interactions to the exactly soluble 6-vertex model KPZ line (conical points), in the non-stochastic direction leads immediately to:
- first-order rough-to-flat edges,
-sharp ridges inside the rough (rounded) phase,
-and more.

Similar sensitivity within the stochastic subspace as well asymmetric exclusion process (ASEP)

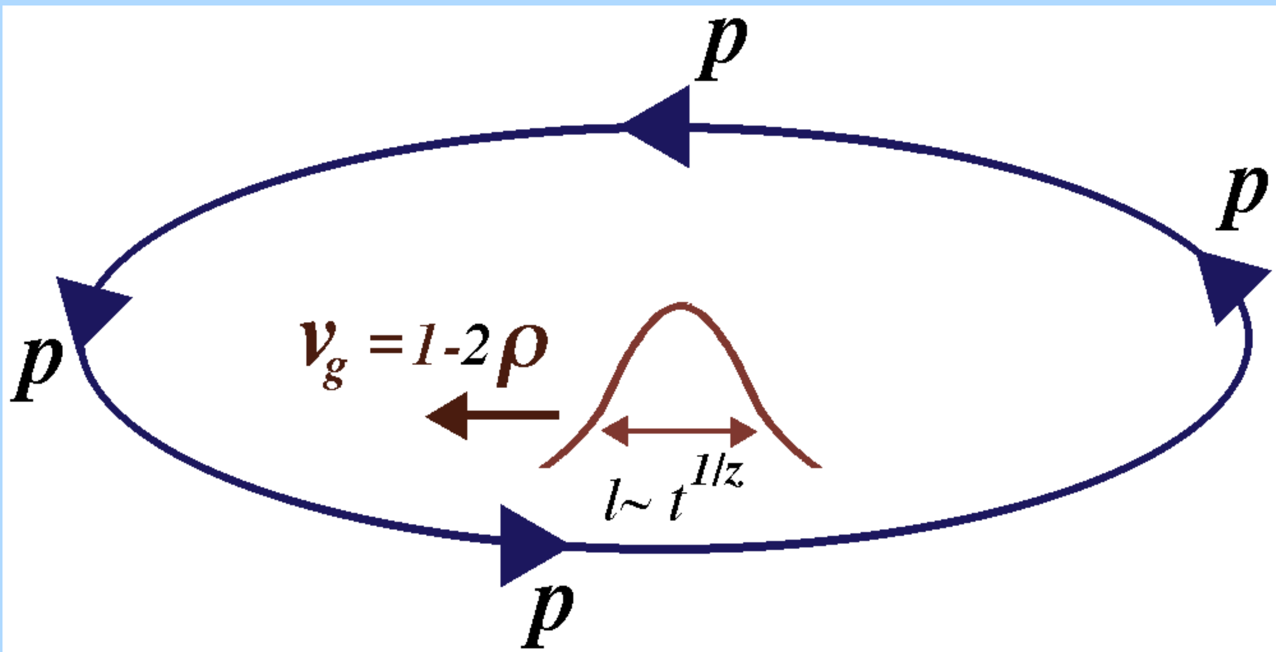
The BCSOS interface model (KPZ growth) is equivalent to a driven flow of particles with hard core repulsive interactions:

Interpret the $S_n^z = -1$ down-steps as particles and the $S_n^z = +1$ up-steps as empty sites



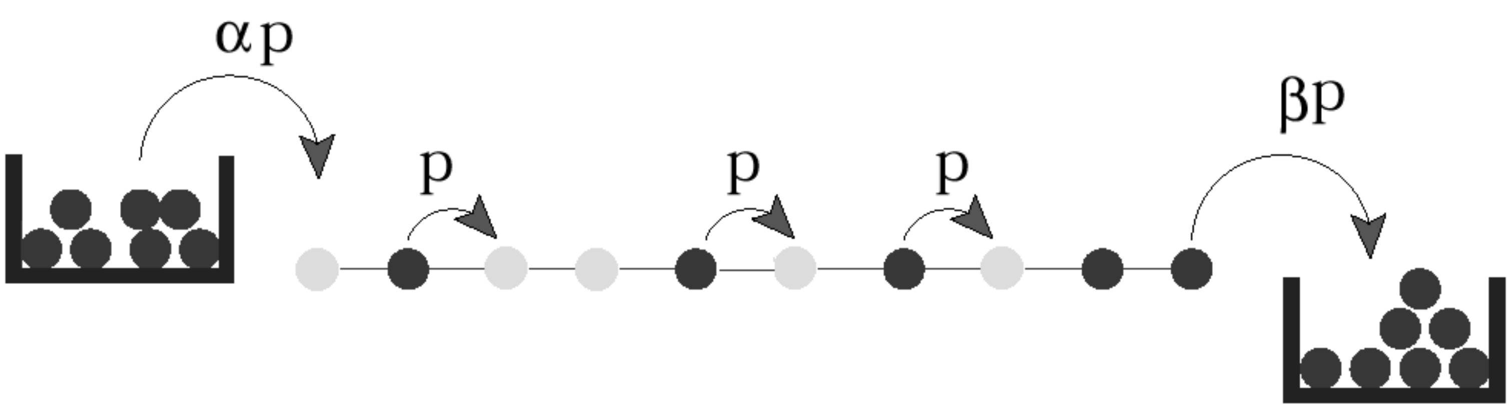
stationary state, fluctuations, and group velocity

The stationary ASEP state for periodic boundary conditions is disordered, random, without any correlations, but fluctuations scale in time as $l \sim t^{1/z}$ with the KPZ dynamic exponent $z = \frac{3}{2}$, and move with group velocity $v_g = 1 - 2\rho$ (tilt of KPZ surface).



boundary induced phase transitions

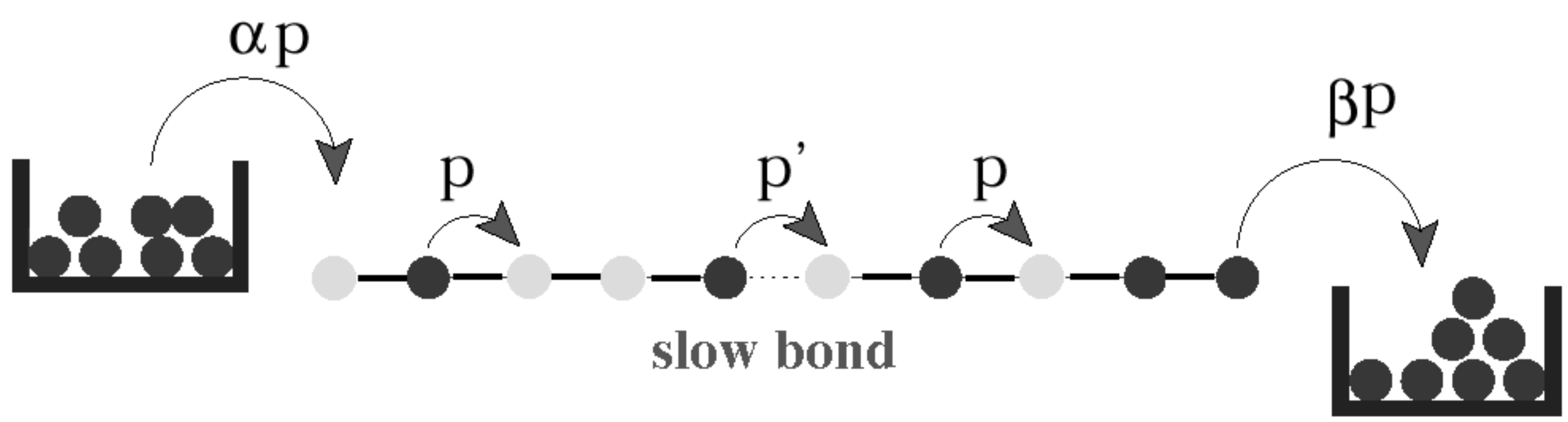
Phase transitions take place in open road set-ups with reservoirs on both ends; (exact matrix formulation results of the stationary state by, e.g., Derrida *et.al.*)



In the maximum current (MC) phase the road controls the density, but in the low (high) density phase the input (output) reservoir (α or β) controls the bulk density.

queuing due to slow bonds

Behind the slow bond a traffic jam develops. The issue us whether the queue is finite or infinite in length (does it scale with the system size in the thermodynamic limit, like in bose condensation); and also the detailed shape of the density profile.



critical point

the order parameter vanishes as

$\Delta_c \sim |r - r_c|^{\beta}$

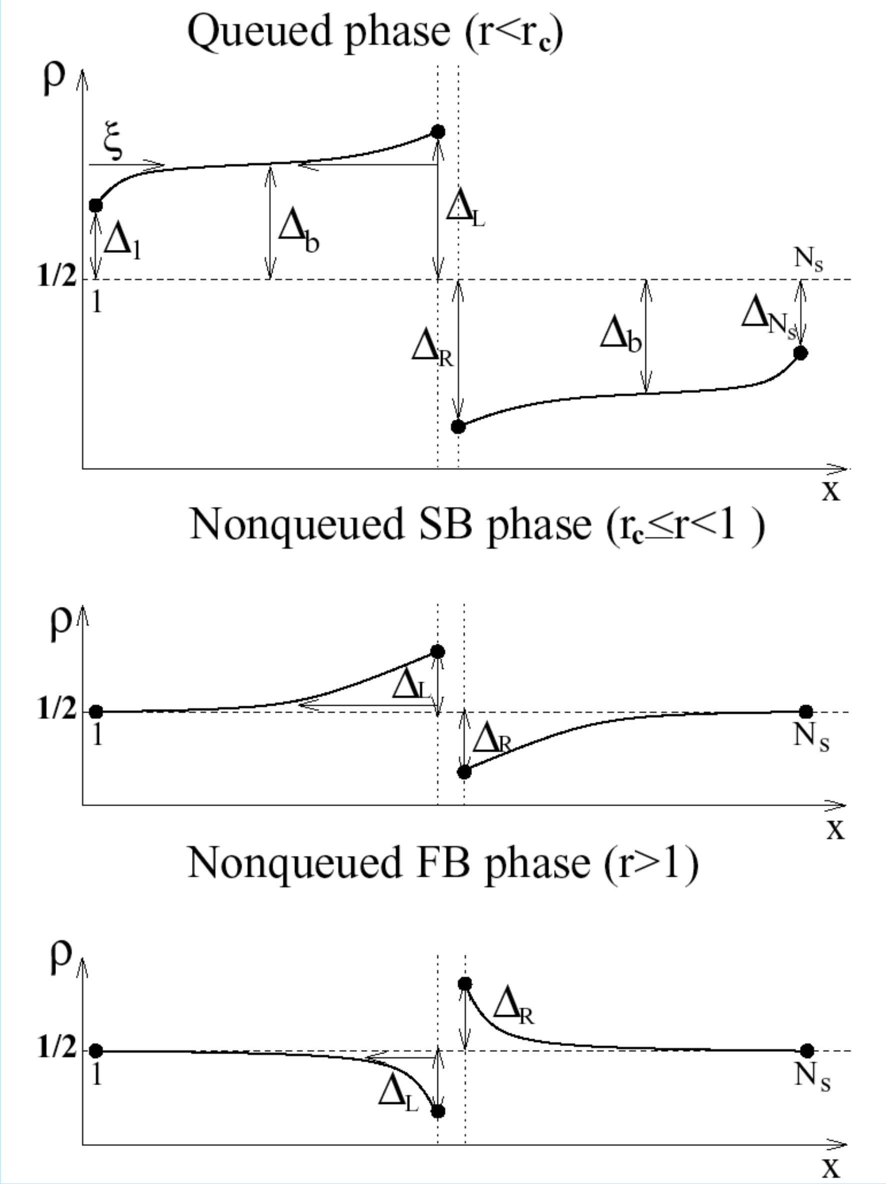
$r_c = 0.80 \pm 0.02$
$\beta = 1.5 \pm 0.01$

Finite size scaling of $\Delta_b \sim N_s^{-x_\Delta}$ at $r = 0.80$.
$x_\Delta = 0.370(5)$

Data collapse of the FSS scaling function

$\Delta_b(N_s, \epsilon) = N^{-x_\Delta} \mathcal{S}(N_s^y \epsilon)$

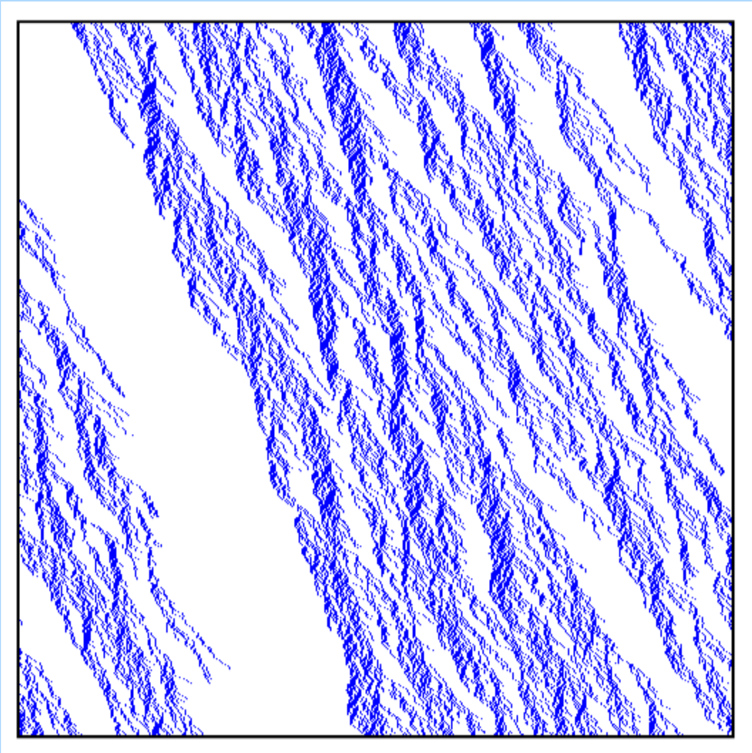
with $x_\Delta = \beta y$



Non local hopping ASEP: (MdN, Meesoon Ha, Huynggyu Park, 2005)

Jump forward with probability p to site immediately behind next particle, with probability 1-p only to next site

The stationary state immediately clusters.
Dynamic phase transition into an empty road state.
Crossover from a first-order to a critical line



Conclusions and Questions:

- Is the z=1 conformal invariant Coulomb gas 2D critical phenomena world unique? Hopefully not…..
- stochastic 1+1D processes live at the edge of the z=1 world. (hinting to possible anisotropic deformations of CFT??)
- z-theorem: unstable “multi-critical” z>2 anisotropic scaling criticality crosses over to smaller z criticality.
- stationary states in dynamic processes must have long range effective interactions to sustain dynamic phase transitions (basically van Hove’s theorem)
- Those with short range interactions are unstable with respect to boundary effect and bulk defects and also subject to phase transitions (traffic jams and clustering).

“Still a lot of fun before I want to retire!”