Physics Qual Exam Problems

spring 1996 – autumn 2011

University of Washington

September 13, 2012

Preface

This is a compendium of problems from past qualifying exams for physics graduate students at the University of Washington. This compendium covers the period preceding Autumn 2011 when the Department changed the format from a classic stand alone Qualifying Exam, (held late Summer and early Spring) into the current course integrated Masters Review Exam (MRE) format. The problems from the post Autumn 2011 period can be found in the separate MRE problems compendium.

UW physics graduate students are strongly encouraged to study *all* the problems in these two compendia. Students should not be surprised to see a mix of new and old problems on future exams.

The level of difficulty of the problems on the old Qualifying Exams and the new Masters Review Exams is the same. All problems from the Qualifying Exams that cover material beyond the first and second quarters of the quantum mechanics and electromagnetism courses have been removed from this compendium.

Problems are grouped into four chapters:

1. Classical Mechanics
2. Electromagnetism
3. Quantum Mechanics
4. Thermodynamics and Statistical Mechanics

The actual exams for each section contained typically 2 problems. Their relative weight can be judged from the point assignments on the problems. The exam for each section had a maximum possible score of 100 points. Not all problems from all exams are listed in this compendium, because some are used several times or are very similar, while others do not apply to the current material anymore.

Many faculty have contributed to the preparation of these problems, and many problems have received improvements from multiple people. Consequently, it is impossible to give individual attributions for problems.

If you notice any typographical errors (no doubt there are still some), please send a note to the chair of the Exam Committee (currently Marcel den Nijs) so improvements can be made.

Finally, here are bits of advice given to the students studying for the stand alone old version of qualifying exam, most of this still applies to the current MRE format:

- Try to view your time spent studying for the Qual as an opportunity to integrate all the physics you have learned (and not just as a painful externally imposed burden).
- Read problems in their entirety first, and try to predict qualitatively how things will work out before doing any calculations in detail. Use this as a means to improve your physical intuition and understanding.
- Some problems are easy. Some are harder. Try to identify the *easiest* way to do a problem, and don't work harder than you have to. Make yourself do the easy problems *fast*, so that you will have more time to devote to harder problems. Make sure you *recognize* when a problem is easy.
- Always include enough explanation so that a reader can understand your reasoning.
- At the end of every problem, or part of a problem, look at your result and ask yourself if there is any way to show quickly that it is *wrong*. Dimensional analysis, and consideration of simplifying limits with known behavior, are both enormously useful techniques for identifying errors. Make the use of these techniques an ingrained habit.
- Recognize that good techniques for studying Qual problems, such as those just mentioned, are also good techniques for real research. That's the point of the Qual!
- Good luck!

Notation

Boldface symbols like $\boldsymbol{r}$ or $\boldsymbol{k}$ denote three-dimensional spatial vectors. Unit vectors pointing along coordinate axes are denoted as $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, *etc.* Carets are sometimes (but not always) placed over quantum operators to distinguish them from c-numbers. Dots are sometimes used as shorthand for time derivatives, so $\dot{f} \equiv df/dt$. Implied summation conventions are occasionally employed.

Physical constants appearing in various problems include:

c	vacuum speed of light
e	electron charge
m_e	electron mass
m_p	proton mass
ϵ_0	vacuum permittivity
μ_0	vacuum permeability
$Z_0 \equiv \mu_0 c$	vacuum impedance
$h \equiv 2\pi\hbar$	Planck's constant
g	Earth's gravitational acceleration
G_{N}	Newton gravitational constant
k_{B}	Boltzmann's constant

Trying to memorize SI values of all these constants is *not* recommended. It is much more helpful to remember useful combinations such as:

$\alpha \equiv e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$	fine structure constant
$(300\,\mathrm{K})\,k_{\mathrm{B}} \approx \frac{1}{40}\,\mathrm{eV}$	room temperature
$m_e\,c^2 \approx 0.5\,\mathrm{MeV}$	electron rest energy
$m_p\,c^2 \approx 1\,\mathrm{GeV}$	proton rest energy
$a_{\mathrm{B}} \equiv \hbar/(\alpha m_e c) \approx 0.5\,\mathrm{\AA}$	Bohr radius
$\frac{1}{2}\alpha^2 m_e c^2 \approx 13.6\,\mathrm{eV}$	Rydberg energy
$\hbar\,c \approx 200\,\mathrm{MeV\ fm}$	conversion constant
$1/\epsilon_0 = \mu_0\,c^2 \approx 10^{11}\,\mathrm{N\,m^2/C^2}$	conversion constant
$m_{\mathrm{Pl}} \equiv \sqrt{\hbar c/G_{\mathrm{N}}} \approx 10^{19}\,\mathrm{GeV}/c^2 \approx 0.2\,\mu\mathrm{g}$	Planck mass

Contents

Chapter 1

Classical Mechanics

1.1 Anisotropic Oscillators

A. [**10** points] Recall that the Poisson bracket $\{a,b\}_{\text{PB}} = \sum_k \left(\frac{\partial a}{\partial q_k} \frac{\partial b}{\partial p_k} - \frac{\partial a}{\partial p_k} \frac{\partial b}{\partial q_k} \right)$. Let $f = f(\vec{q}, \vec{p}, t)$ be an arbitrary function on phase space. Prove that

$$\frac{df}{dt} = \{f, H\}_{\text{PB}} + \frac{\partial f}{\partial t} \,.$$

B. A two-dimensional oscillator has kinetic and potential energies

$$\begin{aligned} T(x,y) &= \tfrac{1}{2} m \,(\dot{x}^2 + \dot{y}^2)\,, \\ V(x,y) &= \tfrac{1}{2} K (x^2 + y^2) + C\,xy\,, \end{aligned}$$

where K and C are constants and m is the mass of the particle.

i. [**10** points] Show by a coordinate transformation that this oscillator is equivalent to an anisotropic harmonic oscillator with Lagrangian

$$L = \tfrac{1}{2} m \,(\dot{\eta}^2 + \dot{\xi}^2) - \tfrac{1}{2} A\,\eta^2 - \tfrac{1}{2} B\,\xi^2\,,$$

where η and ξ are the transformed coordinates, and A and B are functions of K and C. Express the constants A and B in terms of K and C.

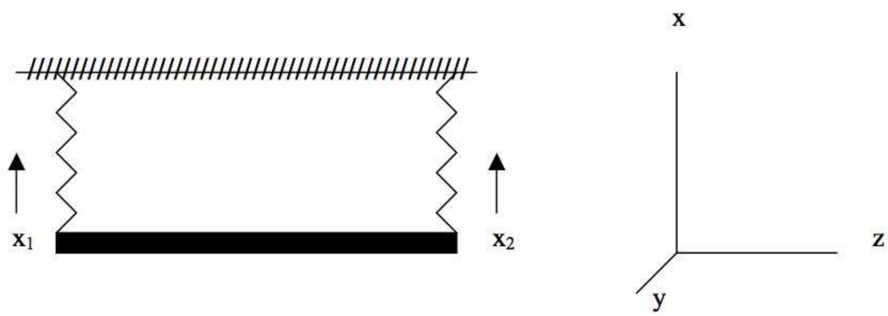
ii. [**5** points] Use a Legendre transform to derive the Hamiltonian for the transformed problem.

iii. [**10** points] Find two independent constants of motion for the problem, and verify this fact using the result of part A.

iv. [**10** points] If $C = 0$, find a third independent constant of motion. Again verify this fact using the result of part A.

1.2 Bar on Springs

A rigid uniform bar of mass M and length L is supported in equilibrium in a horizontal position by two massless springs attached at each end.



The identical springs have the force constant k. The motion of the center of mass is constrained to move parallel to the vertical x-axis. Furthermore the motion of the bar is constrained to lie in the xz-plane.

A. [**5** points] Show that the moment of inertia for a bar about the y axis through its center of mass is $ML^2/12$.

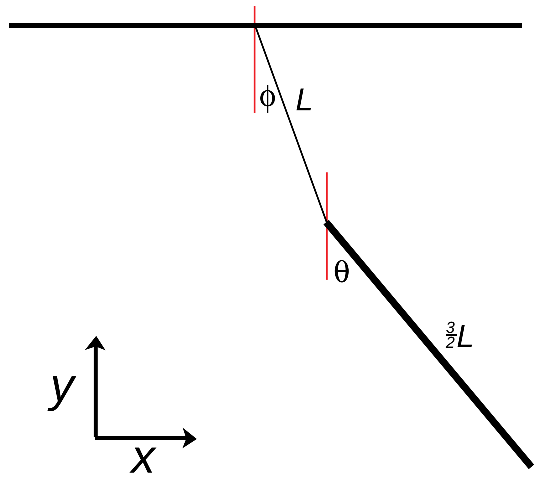
B. [**15** points] Construct the Lagrangian for this bar-spring arrangement assuming only small deviations from equilibrium.

C. [**15** points] Calculate the vibration frequencies of the normal modes for small amplitude oscillations.

D. [**5** points] Describe the normal modes of oscillation.

1.3 Bar on String

A thin uniform bar of mass M and length $\frac{3}{2}L$ is suspended by a string of length L and negligible mass, as shown in the figure. [Note: The moment of inertia of a thin uniform bar of length l and mass m about its center of mass, perpendicular to its length is $\frac{1}{12}ml^2$.]



A. [**8** points] In terms of the variables θ and ϕ shown in the figure, what is the position and velocity of the center of mass of the bar in the xy-plane?

B. [**8** points] Write the Lagrangian for arbitrary angles θ and ϕ, and write the Lagrangian appropriate for small oscillations.

C. [**7** points] Find the Euler-Lagrange equations and show that the equations of motion for the angles θ and ϕ are

$$L\ddot{\theta} + L\ddot{\phi} + g\theta = 0, \qquad L\ddot{\phi} + \tfrac{3}{4}L\ddot{\theta} + g\phi = 0\,.$$

D. [**8** points] Write down the form of the normal modes of the system and solve for the frequencies of the normal modes.

E. [**10** points] Describe, both quantitatively and qualitatively, the motion of each normal mode.

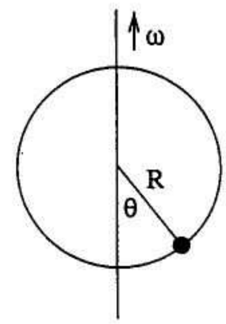
Consider the situation where initially the system is at rest with $\theta = \phi = 0$. Starting at time $t = 0$, a constant force of magnitude F is applied horizontally to the bottom of the rod.

F. [**7** points] How are the equations of motion that you found in part C modified by the force.

G. [**6** points] After a very short time Δt, how are θ and ϕ related?

1.4 Bead on Rotating Hoop

A circular wire hoop is rotating about a vertical axis (along a diameter) with constant angular velocity ω. A bead of mass m, is free to slide without friction on the hoop. A convenient definition is ${\omega_0}^2 = g/R$.



A. [**5** points] Draw all the forces (in the lab frame) on the bead when it is in an equilibrium position (for $0 < \theta < \pi$). Also show the net force (label it clearly).

B. [**5** points] Write the Lagrangian for the bead.

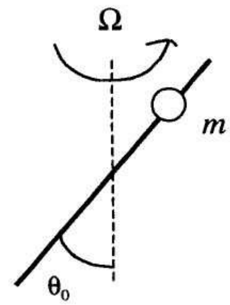
C. [**10** points] Derive the equation of motion from the Lagrangian.

D. [**10** points] Find the stable equilibrium position, θ_q, of the bead as a function of ω. There is a critical value of the angular velocity, ω_c, below which the nature of the equilibrium changes. Find ω_c. Describe the nature of the change.

E. [**15** points] Find the frequency of small oscillations of the mass about the equilibrium point θ_q. Assume that the angular velocity is above the critical value ω_c and $0 < \theta_q < \pi$.

1.5 Bead on Rotating Wire

A bead of mass m slides without friction along a straight wire at an angle θ_0 from vertical that is rotating with a constant angular velocity Ω about a vertical axis. A downward vertical gravitational force mg acts on the bead.



A. [**8** points] Show that the Lagrangian for the bead, using as a generalized coordinate the displacement s measured along the wire from the point of intersection with the rotation axis, is

$$L = \tfrac{1}{2}m\left(\dot{s}^2 + s^2\Omega^2\sin^2\theta_0\right) - mgs\,\cos\theta_0\,.$$

B. [**8** points] Obtain the equations of motion from the Lagrangian, and show that the condition for equilibrium at constant position on the wire is $s = s_0 \equiv g\cos\theta_0/(\Omega\sin\theta_0)^2$.

C. [**10** points] Derive the above result for s_0 by directly applying Newton's second law of motion to the bead.

D. [**10** points] Discuss the stability of this orbit against small displacements along the wire by finding an equation for the deviation $\eta(t) \equiv s(t) - s_0$.

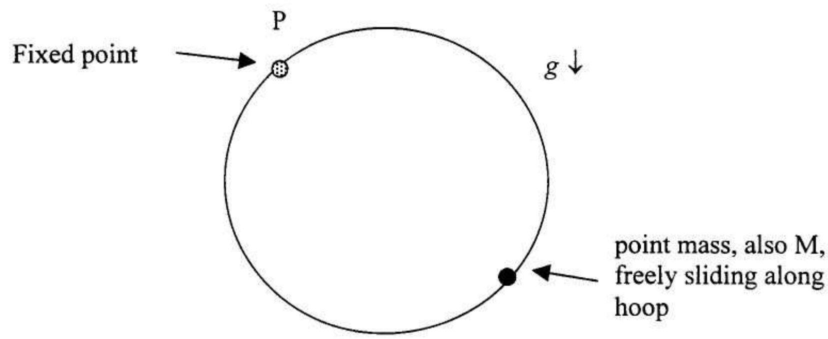
E. [**8** points] Find the constraint force which keeps the bead moving with uniform angular velocity in the $\hat{\phi}$-direction as the displacement s varies. You may use Lagrangian methods or Newton's Second Law.

F. [**9** points] Find H, the Hamiltonian of the system, in terms of a suitable coordinate and momentum.

G. [**12** points] Show (i) whether H is conserved, (ii) whether H is equal to the sum of the kinetic and potential energies, and if not, (iii) whether the energy of the bead is conserved. Explain the physics behind your answers.

1.6 Bead on Swinging Hoop

A thin hoop of radius R and mass M is free to oscillate in its plane around a fixed point P. On the hoop there is a point mass, also M, which can slide freely along the hoop. The system is in a uniform gravitational field $\boldsymbol{g}$.



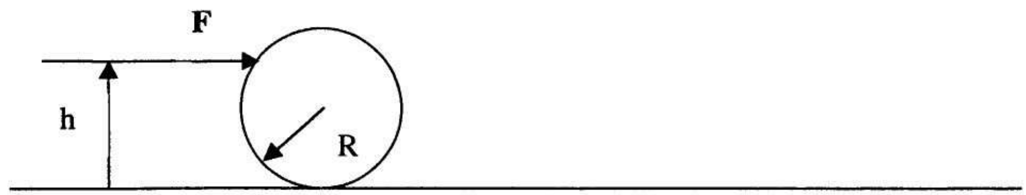
A. [**20** points] Introduce appropriate coordinates describing the combined motion of the hoop and point mass, showing them on the diagram above. How many unconstrained coordinates are required?

B. [**50** points] Consider small oscillations. Derive the Lagrangian, and the Lagrange equations. Find the normal mode eigenfrequencies.

C. [**30** points] Find the normal mode eigenfunctions and sketch the motion associated with each eigenfunction.

1.7 Billiard Ball

A spherical billiard ball with mass M and radius R is initially at rest on a pool table. At $t = 0$, a cue stick strikes the ball at a point a height h above table, exerting a horizontal impulsive force $\boldsymbol{F}$ on the ball. Immediately after this, the ball is observed to have a horizontal velocity $\boldsymbol{v}_0$. The coefficient of sliding friction between the ball and the table is μ. The moment of inertia of a sphere about an axis through its center of mass is $\frac{2}{5}MR^2$.



In terms of the given quantities:

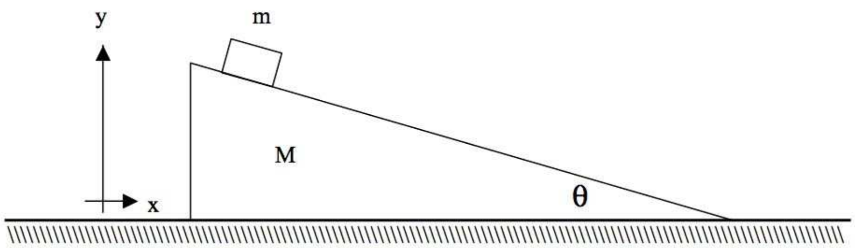
A. [**8** points] What is the angular velocity of the ball about its center of mass immediately after the cue stick hits it?

B. [**10** points] At what time does the ball start rolling without sliding along the table?

C. [**5** points] What is the final translational velocity of the ball?

D. [**12** points] Calculate the final translational kinetic energy, rotational kinetic energy, and total kinetic energy of the ball. Physically interpret any changes in these quantities, and explain the sign of changes.

1.8 Block on Wedge

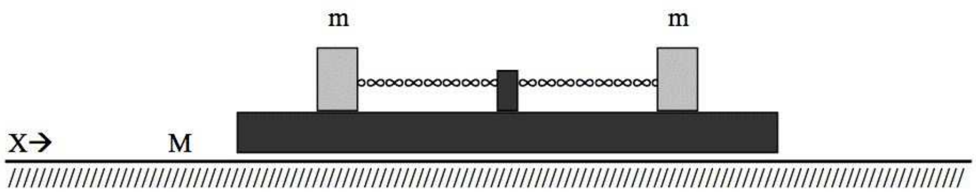


A wedge of mass M sits on a frictionless table. A block of mass m slides on the frictionless slope of the wedge. The angle of the wedge with respect to the table is θ. Take x positive to the right. Let the coordinates for the block be (x_1, y_1), and those of the point of the wedge $(x_2, 0)$.

A. [**8** points] Write the constraint equation for the block sliding on the wedge and the Lagrangian for the block and wedge.

B. [**22** points] Derive the equations of motion for the block and wedge, using the method of Lagrange Multipliers to incorporate the constraint.

1.9 Blocks on Slider



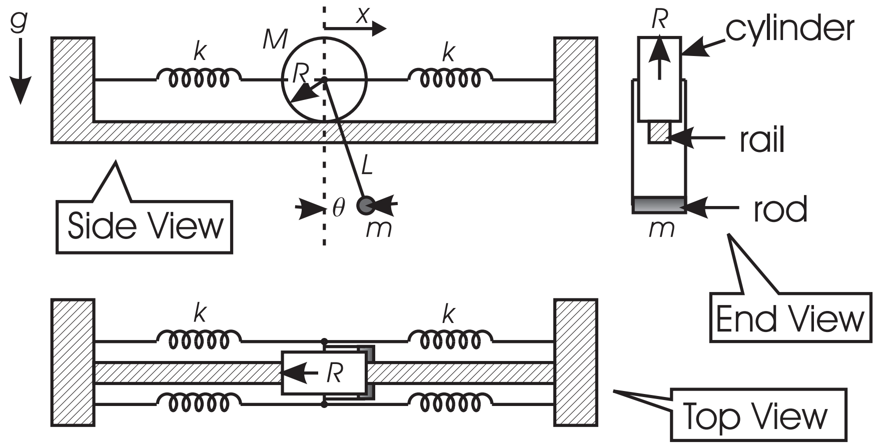
A platform of mass M sits on a frictionless table. Two identical blocks of mass m are attached with identical springs to a post fixed to the platform. The springs are massless and have force constant k. The blocks move on the frictionless surface of the platform and are constrained to move along the x axis (parallel to the table surface and in the plane of this paper).

A. [**10** points] Give the Lagrangian of the system of masses and springs.

B. [**25** points] Calculate the normal frequencies of the system.

C. [**5** points] Describe the normal modes of vibration corresponding to these frequencies.

1.10 Cart and Pendulum



A solid cylinder of radius R and mass M rolls along a horizontal rail without slipping. The center of the cylinder is attached to the walls by four massless springs, each having a spring constant k. A rod of negligible radius and mass m is hung from a frictionless and massless axle at the center of the cylinder by means of rigid massless wires of length L. The rod is free to swing in response to gravity as shown in the figure. The moment of inertia of the cylinder about the axle is $I = \frac{1}{2}MR^2$.

A. [**8** points] Write down the kinetic energy of this system in terms of the parameters given.

B. [**7** points] Write down the potential energy of this system in terms of the parameters given.

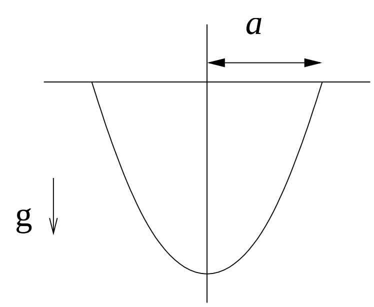
C. [**10** points] Write down the coupled equations of motion describing the cylinder's displacement from equilibrium along the rail, x, and the hanging rod's angle from equilibrium, θ. Do not solve or simplify the equations at this point.

D. [**15** points] After making *both* of the following two simplifying assumptions to your answer in part C, determine the frequencies of the normal vibrational modes of this system:

- let $M = 2m$ and $kL = mg$, where g is the acceleration of gravity;
- assume both the pendulum and cylinder have small oscillations.

E. [**10** points] Assume that $x(t{=}t_0) = x_{\max}$, where $x_{\max}$ is the maximum positive displacement of the cylinder from equilibrium. For *each* of the normal vibrational modes, sketch both the normalized cylinder displacement, $x/x_{\max}$, and the normalized rod swing angle, $\theta/\theta_{\max}$, as a function of time using the same time axis for both quantities. Sketch only for one full period T of the system's motion (from time t_0 to $t_0 + T$).

1.11 Catenary



An unstretchable cord of uniform density ρ is suspended from two points of equal height (see Figure). The gravitational acceleration is taken to be a constant g in the negative z direction.

A. [**10** points] Write the expression for the potential energy U and the length l for a given curve $z = z(x)$.

B. [**10** points] Write down the Euler-Lagrange equation which determines the shape of the hanging cord in equilibrium for fixed length l.

C. [**10** points] Treating x as the analogue of time, write an expression for the analogue of the conserved energy.

D. [**10** points] Show that the equilibrium configuration is given by $z = A\cosh(x/A) + B$, where A and B are constants.

E. [**10** points] Someone grabs the middle point of the cord and pulls it down. Some part of the cord goes up, the other part goes down. Does the center of mass of the cord move up or down? The cord is not stretchable. (You should be able to solve this part without extensive calculations.)

1.12 Central Force

A particle of mass m moves in a circle of fixed radius under the influence of an attractive central force $\boldsymbol{F}(\boldsymbol{r}) = f(r)\,\hat{\boldsymbol{e}}_r$, with $\hat{\boldsymbol{e}}_r$ an outward radial unit vector and

$$f(r) = -\frac{k}{r^2}\,e^{-r/a}\,,$$

where k and a are positive constants (and r is the radial distance from the origin).

A. [**20** points] Determine under what conditions the circular orbit is stable, and for such an orbit, compute the frequency of small oscillations.

Now consider the scattering of a particle by a central force. Suppose that trajectories with impact parameter b are deflected through an angle θ.

B. [**10** points] Define the differential cross section, and show that it is given by

$$\frac{d\sigma}{d\Omega} = -\frac{b\,db}{\sin\theta\,d\theta}\,.$$

C. [**10** points] For the force given above, and a particle of mass m and energy E, derive an expression for the differential scattering cross section in the limit of small θ. Evaluate the expression to first order in b/a.

D. [**10** points] Can the limits of small scattering angle, $\theta \ll 1$ (implying b large), and $b \ll a$ be mutually compatible? If so, characterize the regime where both approximations are valid.

1.13 Central Potential (1)

[**40** points] A particle moves in three dimensions subject to the attractive central force

$$\boldsymbol{F}(\boldsymbol{r}) = -\left(\frac{\kappa}{r^2} + \frac{\eta}{r^4}\right)\hat{\boldsymbol{r}}\,,$$

with large angular momentum. ("Large" means greater than anything that is relevant to compare it to in the following.) The constants κ and η are positive; $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction.

A. Find, and sketch, the effective radial potential. Show that it has a local maximum at a radius r_1 and a local minimum at a radius r_2. What are r_1 and r_2?

B. For what range of energy and initial position can the particle reach the origin but not reach infinity?

C. For what range of energy and initial position will the motion of the particle remain bound, but never reach the origin?

D. Describe, and sketch, the different possible types of trajectories for which the particle initially comes in from infinity.

E. Describe, and sketch, the different possible types of trajectories for which the particle never reaches infinity as $t \to \pm\infty$.

1.14 Central Potential (2)

A particle of mass m moves in a central potential $U(r)$.

A. [**4** points] Show that the angular momentum $\boldsymbol{L}$ about the force center ($r = 0$) is conserved.

B. [**2** points] Show that the motion of the particle must lie in a plane perpendicular to $\boldsymbol{L}$.

C. [**5** points] Show that the total energy may be written as $E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r)$, where $U_{\text{eff}}(r) = L^2/(2mr^2) + U(r)$ and $L = |\boldsymbol{L}|$.

D. [**5** points] Assume a circular orbit of radius R exists; show that $E = U_{\text{eff}}(R)$ and $dU_{\text{eff}}(r)/dr|_{r=R} = 0$.

E. [**4** points] State the condition for which the circular orbits specified in part D are stable.

F. [**10** points] Let the central force be $\boldsymbol{F}(\boldsymbol{r}) = -(b/r^2 - c/r^4)\,\hat{\boldsymbol{r}}$, where $b > 0$, $c > 0$, and $\hat{\boldsymbol{r}}$ is the unit vector in the radial direction. Calculate the values of R that give rise to stable orbits as a function of L, b and c.

1.15 Central Potential (3)

A particle of mass m is moving in the r, θ plane subject to a central force. The potential energy of the particle is of the form

$$V(r) = -\frac{\zeta}{r^6} + \tfrac{1}{2}kr^2\,,$$

where $\zeta > 0$ and $k > 0$.

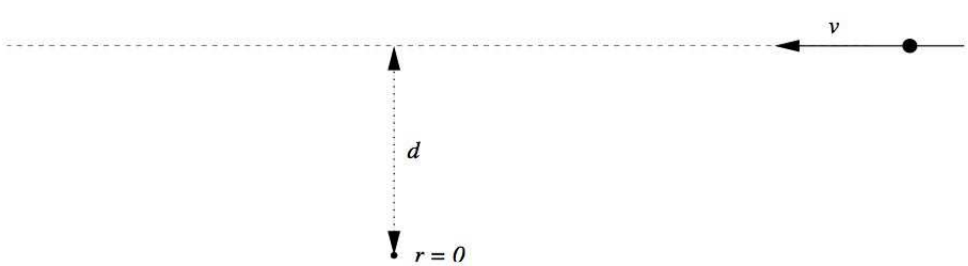
A. [**6** points] What are Hamilton's equations for this particle? User r and θ as your coordinates.

B. [**10** points] For what range of values of the particle's angular momentum L are there circular orbits?

C. [**6** points] For an L in the range you found in (B), how many circular orbits are there, and at what radii? Which ones are stable?

D. [**14** points] Consider orbits which are not circular, and which neither pass through $r = 0$ nor reach $r = \infty$. Such orbits will have a minimum radius r_- and a maximum radius r_+. For a particle of angular momentum L, what is the smallest r_- that such an orbit can have? Find an equation (which you should not attempt to solve) for the corresponding r_+, and compute r_+ approximately in the limit that k is very small.

E. [**14** points] Suppose that $k = 0$. Then the particle may approach from spatial infinity, with an initial velocity $\boldsymbol{v}$ and an initial impact parameter d, as shown below. Under what condition or conditions will the particle be unable to avoid hitting the point at $r = 0$? State the conditions as mathematical relationships involving $\boldsymbol{v}$ and d.

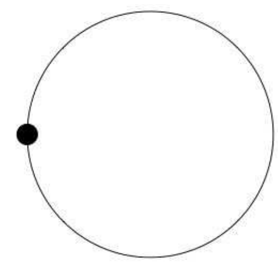


1.16 Central Potential (4)

A particle moves in a central potential of the form $V(\boldsymbol{r}) = -C/|\boldsymbol{r}|^{\alpha}$, where C and α are constants.

A. [**20** points] For which values of α are there stable circular orbits?

B. [**20** points] It is known that during a portion of its trajectory, the particle moves along a circle of radius R that goes through the point $\boldsymbol{r} = 0$. Let r and θ denote polar coordinates of the particle. Using Kepler's second law, give the expressions for the time derivatives of the radius and polar angle, $\dot{r}$ and $\dot{\theta}$, as functions of the polar angle θ and constants of motion.



A. [**20** points] Find α (hint: use energy conservation). Are there stable circular orbits in this potential?

1.17 Closed Orbits

The equation for radial motion in a central potential, $U(r)$, is identical to the motion of a particle in one dimension in the effective potential $U_{\text{eff}}(r) = U(r) + L^2/2mr^2$, where m and L are the particle's mass and angular momentum respectively. Consider L as well as m to be given. Let the potential $U(r)$ be written $U(r) = Ar^n/n$ with n a non-zero integer and $A > 0$ ensuring a stable orbit for $n > -2$.

A. [**5** points] If the particle is in a circular orbit, what is the angular frequency Ω of the motion as a function of L, m, and n?

B. [**20** points] If the circular motion is perturbed slightly in the radial direction, the particle will oscillate in this direction with some angular frequency ω. Calculate this frequency, and show that the ratio ω/Ω is a function only of n. If this ratio is an integer, then the particle's orbit will be closed. The condition that the ratio be an integer tells you what potentials permit closed orbits.

C. [**5** points] State a physical example of a power-law potential which you know to give closed orbits. Check whether your result in part B passes the test of correctness provided by your example.

D. [**20** points] For the case of a particle of energy $E = -|E| < 0$ moving in a non-circular, closed orbit in the Coulomb potential $U(r) = -e^2/(4\pi\epsilon_0\, r)$, calculate the two turning points (that is $\dot{r} = 0$) of the orbit in terms of L and $|E|$.

1.18 Diatomic & Triatomic Vibrations

In the simplest mechanical model, the carbon dioxide molecule is a chain O-C-O, and the carbon monoxide molecule is a chain C-O. The mass of the carbon atom is $m_1 = 12$ atomic units, and that of the oxygen atom is $m_2 = 16$ atomic units. The chemical bonds between the carbon and the oxygen atoms are modeled as springs with (identical) spring constant k.

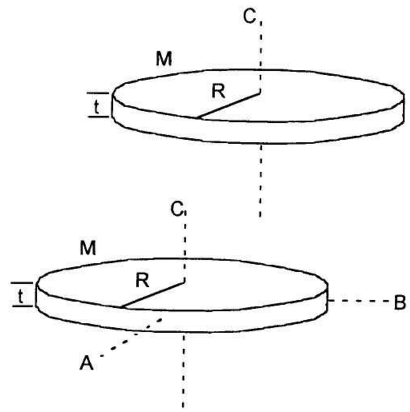
A. [**5** points] Restrict yourself to longitudinal motion (in which each atom moves only along the direction of the line connecting the atoms in the molecules). How many modes of oscillation exist for each molecule?

B. [**20** points] Find the frequencies of the longitudinal oscillations of the two molecules.

C. [**15** points] Graphically sketch the longitudinal oscillation modes of the CO_2 molecule.

1.19 Disk Dynamics

Consider a uniform density disk of radius R, thickness t, and mass M.

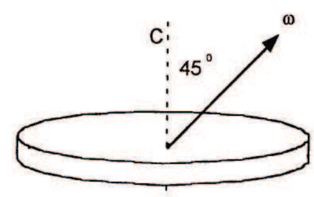


A. [**10** points] Calculate the moment of inertia, I_C, about the symmetry axis C (dotted line in diagram).

B. [**15** points] For a lamina, the perpendicular axis theorem states that the sum of the two moments of inertia about perpendicular axes in the lamina is equal to the moment of inertia about an axis through the intersection of the two axes and perpendicular to the lamina. Prove this theorem and use it to calculate the moments of inertia, I_A and I_B about axes A and B in the limit $t \ll R$.

C. [**10** points] Axes A, B, and C are the principal axes of the moment of inertia tensor referred to the center of mass. Explain why this is so.

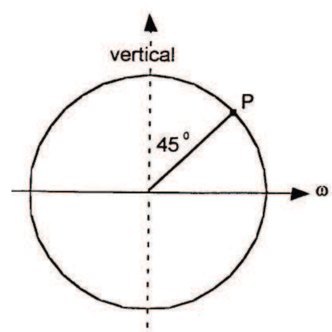
The disk is set spinning with angular velocity ω about an axis through the center of mass that makes an angle 45° with respect to the symmetry axis C. The axis is vertical.



D. [**15** points] Calculate the angular momentum L of the disk.

E. [**20** points] Describe the subsequent motion. Neglect gravity.

The disk is now flipped vertically into the air spinning with a horizontal angular velocity ω about axis B. At the top of its trajectory it strikes an object that instantaneously brings to rest a point P on its rim. At this moment the plane surface of the disk is vertical (as shown) with a radius to the point P making an angle of 45° with respect to the vertical.



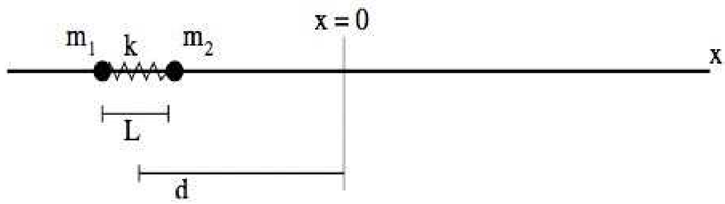
F. [**10** points] Show that the change in angular momentum is equal to the impulsive torque and calculate the impulsive torque about P.

G. [**20** points] Using the results of part A, calculate the angular velocity ω' and velocity of the center of mass $\boldsymbol{v}_{\rm cm}$ immediately after impact.

1.20 Driven Masses (A)

Effect of an External Force on particles connected by a spring

Two point masses m_1 and m_2 are connected by a spring with spring constant k and equilibrium length L. The masses are constrained to move (without friction) only on the x-axis.



Before $t = 0$, the masses are at rest, with $x_1 = -d - L/2$ and $x_2 = -d + L/2$ (see figure). External forces act on the masses for a time interval of length of T. For $0 < t < T$, the force on mass 1 is $F_0\,\hat{x}$, while the force on mass 2 is $4F_0\,\hat{x}$, where F_0 is a positive constant. At all other times the forces are zero.

A. [**5** points] warmup: if $k = 0$ (no spring), what are x_1(t) and $x_2(t)$ for $t > 0$?

B. [**7** points] warmup: if $k = \infty$ (rigid rod), what are $x_1(t)$ and $x_2(t)$ for $t > 0$?

C. [**13** points] Now suppose k is finite but very small. Also take $m_1 = m_2 = m$. Approximately, what are $x_1(t)$ and $x_2(t)$ for $t > T$? [Hint: use the approximation that k is very small to break the problem into two simple parts.]

1.21 Driven Masses (B)

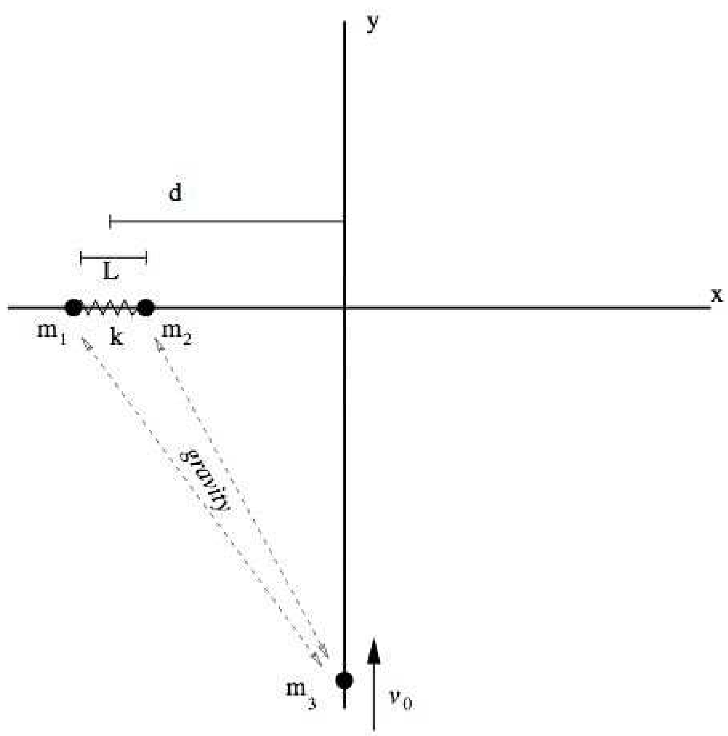
General Questions

Now we add particle 3 into the system. Particle 3 is *constrained* to move (without friction) only on the y-axis; particles 1 and 2 are similarly *constrained* to move (without friction) only on the x-axis. Particle 3 interacts with particles 1 and 2 **via ordinary gravity**:

$$|\boldsymbol{F}_{13}| = -G\,\frac{m_1 m_3}{r_{13}^2}\,, \qquad |\boldsymbol{F}_{23}| = -G\,\frac{m_2 m_3}{r_{23}^2}\,,$$

where G is Newton's constant.

Particles 1 and 2 are still bound by a spring. [Ignore all gravitational interactions between particles 1 and 2; they are small compared to the forces exerted by the spring.]



A. [**7** points] Write the exact Lagrangian for this system, as a function of x_1, x_2, y_3.

B. [**6** points] Write the exact equations of motion for object 2.

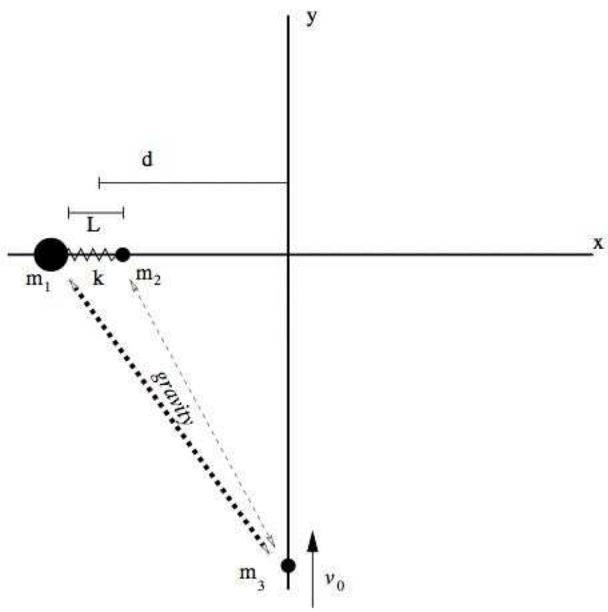
C. [**12** points] Are there any conserved quantities in this system, and if so, what are they? [Hint: The motion is constrained.] Be sure that you justify your answer.

1.22 Driven Masses (C)

A Special Case

The forces are as in part (B).

Now let $m_1 \gg m_2 = m_3$. In this case x_1 is nearly constant, but x_2 may oscillate.
At $t = -\infty$, particles 1 and 2 are at rest with $x_1 = -d - L/2, x_2 = -d + L/2$, where $L \ll d$.
At $t = -\infty$, particle 3 is at $y = -\infty$ and has velocity $v_0\,\hat{y}$.



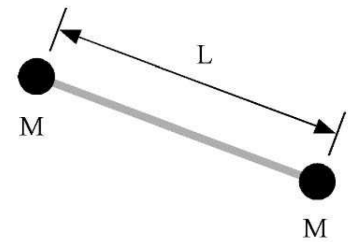
A. [**8** points] The motion of mass 1 is small, as is the force of particle 2 on particle 3; but the force of mass 1 on mass 3 is large and causes a large $\ddot{y_3}$. In the approximation that you **only** compute the effect of mass 1 on the motion of mass 3, show that the maximum value of $\dot{y_3}$(t) is $\dot{y_3}(t)|_{\max} \approx \sqrt{v_0{}^2 + 2Gm_1/d}$.

B. [**8** points] Now include the gravitational interactions between masses 2 and 3. This is a very small effect on particle 3 (on which the dominant force is that from particle 1.) However, gravity has an important effect on particle 2: it causes x_2 to oscillate. (Meanwhile, $\dot{x_1}$ is very small; it can be ignored.)
After particle 3 has moved well past $y_3 = 0$, particle 2 will generally continue to oscillate. For very small k, estimate [do not calculate exactly] the energy stored in the oscillations of x_2 after particle 3 has gone by. You may assume the amplitude of oscillation is small compared to L.

C. [**9** points] Suppose $v_0 \to 0$ (so that the initial motion of particle 3 is extremely slow, though not quite exactly zero.) In this limit, including all interactions (but with k still small), make a qualitative graph of $\dot{y_3}$(t) versus t. Make sure your graph captures both the early-time and late-time behavior of $\dot{y_3}$(t). Briefly justify your answer [two or three sentences.]

1.23 Dumbbell Dynamics

Two identical disks of mass M are connected by a massless rigid rod of length L. They are resting on a frictionless, horizontal table.



A. [**5** points] Choose a system of generalized coordinates, including any coordinate(s) that you will need for describing the constraint(s). You must carefully describe your coordinates both with a sketch and with a brief written description.

You are strongly advised to read the entire problem and think carefully about your choice of coordinates, as some systems of coordinates may greatly simplify your work while others may make subsequent calculations intractable.

B. [**5** points] Write the equation(s) of constraint needed to enforce the condition that the rod have length L.

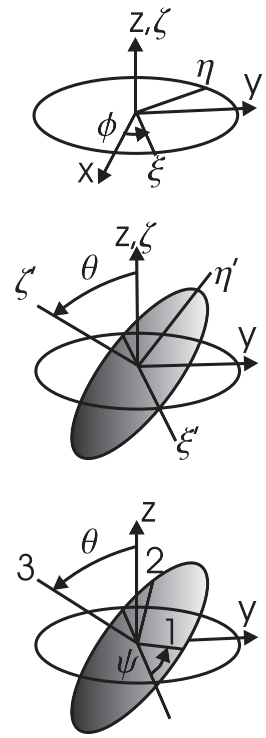
C. [**10** points] Write the Lagrangian for this system.

D. [**10** points] Write Lagrange's equations of motion.

E. [**10** points] Determine the tension in the rod.

1.24 Earth Wobble

Consider the motion of a solid as described by the Euler angles, $\phi,\ \theta,\ \psi$, defined as shown in the figure. We use 1, 2, 3 to label the principal axes of the solid.



A. [**5** points] Show that the components of the angular velocity with respect to the principal axes are

$$\omega_1 = \dot\phi\ \sin\theta \sin\psi + \dot\theta\ \cos\psi\,,$$
$$\omega_2 = \dot\phi\ \sin\theta \cos\psi - \dot\theta\ \sin\psi\,,$$
$$\omega_3 = \dot\phi\ \cos\theta + \dot\psi\,.$$

B. [**10** points] Start with the Lagrangian for a force-free motion of a solid and show that

$$I_1\,\dot\omega_1 = \omega_2\,\omega_3\,(I_2 - I_3)\ ,$$
$$I_2\,\dot\omega_2 = \omega_3\,\omega_1\,(I_3 - I_1)\ ,$$
$$I_3\,\dot\omega_3 = \omega_1\,\omega_2\,(I_1 - I_2)\ ,$$

where $I_1,\ I_2,\ I_3$ are the principal moments of inertia.

C. [**10** points] Consider now a body with symmetry around its 3–axis (*i.e.*, $I_2 = I_1$) and show that the angular velocity satisfies:

$$\omega_3(t) = \omega_z\,,$$
$$\omega_1(t) = \omega_{xy}\cos\Omega t\,,$$
$$\omega_2(t) = \omega_{xy}\sin\Omega t\,,$$

with $\omega_{xy},\ \omega_z$ constants, and $\Omega = \frac{I_3 - I_1}{I_1}\,\omega_z$.

D. [**5** points] Express the angular momentum vector in the principal axes and the kinetic energy in terms of the moments of inertia and the constants ω_{xy}, ω_z used above.

E. [**5** points] Noticing that for the Earth $\omega_z \sim 2\pi$ day^{-1}, and that the Earth can be considered as a slightly non-spherical top, say, $I_2 = I_1 = (1-\epsilon)I_3$ with $\epsilon \approx 3\times 10^{-3}$, calculate the period of wobbling, *i.e.*. of precession of the angular velocity vector around the axis of symmetry.

F. [**15** points] Consider an inertial frame of reference external to the rotating top and make a sketch showing the angular momentum vector, the angular frequency, and the body 3-axis, with circles indicating the precession trajectories. Choose the direction of the axes at your convenience. Calculate the precession frequency of the body axis around the angular momentum. Assume $\omega_{xy} \ll \omega_z$.

1.25 Elastic Rod

A. [**25** points] Consider a function $F(\ddot{y}(s), \dot{y}(s), y(s), s)$ and its integral,

$$J \equiv \int_a^b F(\ddot{y}, \dot{y}, y, s)\; ds\,.$$

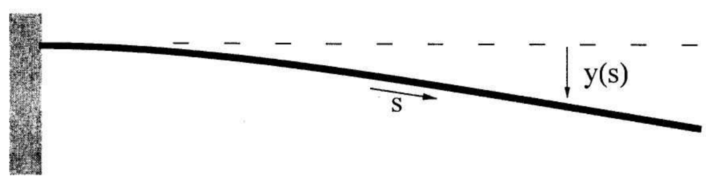
The various derivatives $\dot{y}(s) \equiv dy/ds$, *etc.*, of $y(s)$ are continuous. Prove that the solution to the extremal condition $\delta J = 0$ with respect to functional variations of y [*i.e.*, $y(s) \to y(s) + \phi(s)$] is given by the solution to the differential equation

$$\frac{d}{ds^2}\left(\frac{\partial F}{\partial \ddot{y}}\right) - \frac{d}{ds}\left(\frac{\partial F}{\partial \dot{y}}\right) + \frac{\partial F}{\partial y} = 0\,,$$

subject to the boundary condition

$$\left[\frac{\partial F}{\partial \ddot{y}}\,\dot{\phi} + \left(\frac{\partial F}{\partial \dot{y}} - \frac{d}{ds}\frac{\partial F}{\partial \ddot{y}}\right)\phi\right]\Bigg|_{s=a}^{s=b} = 0\,.$$

B. Consider a thin elastic rod of length L with one end clamped horizontally and the other end free. Let s denote the length along the rod and let $y(s)$ denote the vertical displacement from horizontal of the rod at position s, as illustrated in the figure.



There are two contributions to the potential energy, one from the curvature of the rod, and one from gravity, so that

$$V = \int_0^L \left[\tfrac{1}{2}k\,\ddot{y}^{\,2} - gy(s)\,\rho(s)\right]\,ds\,,$$

where k is related to the Young's modulus of the rod material, $g > 0$ is the acceleration due to gravity, and $\rho(s)$ is the linear mass density of the rod at position s.

i. [**5** points] Making use of the results of part A, show that

$$k\,\ddddot{y} = g\rho(s)\,. \qquad (*)$$

ii. [**10** points] Equation $(*)$ is a fourth-order differential equation. State the four boundary conditions which can be used to integrate Eq. $(*)$, and briefly explain their origin. Your explanation for the boundary conditions can be physical or mathematical.

iii. [**10** points] Consider the special case of a uniformly-loaded rod, *i.e.*, $\rho(s) = \rho_0$. Solve Eq. $(*)$ using your specified boundary conditions. If you were not able to determine all four boundary conditions, then work toward a partial answer, clearly labeling any undetermined integration constants.

1.26 Elliptical Orbit

You send a spacecraft from Earth to Jupiter via an elliptical orbit around the Sun tangent to both Earth's orbit and to Jupiter's orbit. Assume that the orbits of the Earth and of Jupiter are circular and coplanar, and ignore other planets as well as the gravitational attraction of the spacecraft to Earth and Jupiter. Call the radius of Jupiter's orbit R_{J} and the radius of Earth's orbit $R_{\mathrm{e}} = 1$ AU. For general information (you do not need this number) $R_{\mathrm{J}} \approx 5.2$ AU. The speed of the Earth in its orbit is $v_{\mathrm{e}} = 2\pi$ AU/year.

A. [**15** points] For a general elliptical orbit around the Sun, derive the following relationship between the semi-major axis a of the ellipse and the total energy E,

$$a = \frac{GMm}{2|E|} ,$$

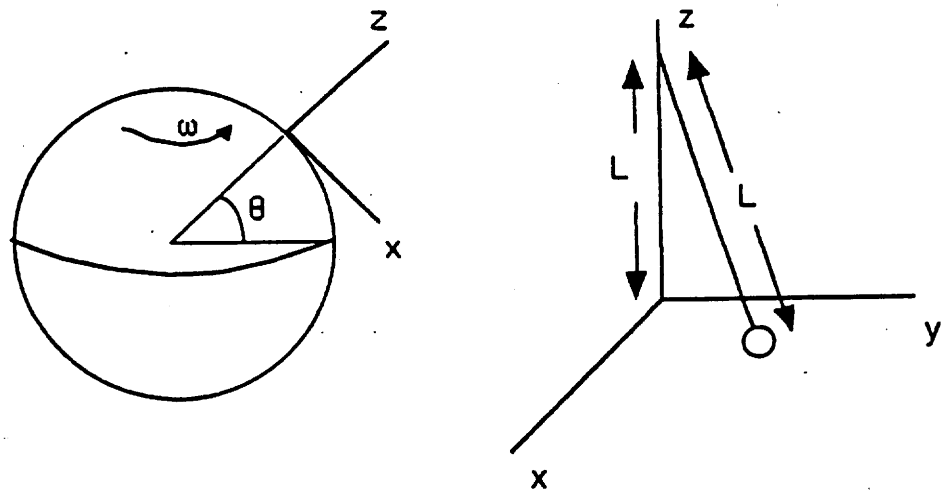
where G is Newton's constant, M the mass of the Sun and m is the mass of the orbiting object. Assume $M \gg m$.

B. [**15** points] What speed does the spacecraft have when it reaches the orbit of Jupiter? Give your answer in terms of the speed of Jupiter in its orbit v_{J}, and the orbit radii, R_{J} and R_{e}. Is this spacecraft speed larger or smaller than the speed of Jupiter in its orbit?

C. [**5** points] When the spacecraft reaches Jupiter's orbit, how does its speed compare with its speed when it started its elliptical path at Earth's orbit (give the answer in terms of orbit radii)?

D. [**15** points] Show that the time-average of the potential energy of the spacecraft over one orbital period is $2E$. [Hint: consider the time average of dS/dt where $S = \boldsymbol{r} \cdot \boldsymbol{p}$.]

1.27 Foucault Pendulum (1)



Analyze the motion of a Foucault pendulum with a ball of mass m supported by an inextensible massless wire of length L (on the order of a few meters). The pendulum is at latitude θ (measured from the equator) on the surface of the Earth. Treat the Earth as a sphere of radius $R = 6400\,\text{km}$. Work in coordinates fixed to the surface of the Earth, with the x-axis pointing south, the y-axis pointing east, and the z-axis pointing vertically upward. The pendulum is suspended from the point $z = L$, $x = y = 0$. The Earth revolves with angular frequency ω.

A. [**23** points] Find the exact equations of motion.[1]

B. [**27** points] Make appropriate approximations to the equations assuming small displacements from equilibrium. Take into account the magnitude of the Earth's angular velocity, the frequency of the pendulum, and the magnitude of the centrifugal "force" relative to the gravitational force. Clearly state the justifications for your approximations.

C. [**20** points] Solve the approximate equations for the motion of the pendulum bob. Describe the motion with initial conditions $x = a$, $y = 0$, $\dot{x} = \dot{y} = 0$.

D. [**16** points] What Lagrangian gives rise to the (approximate) equations of motion? Find the canonical momenta to the x and y coordinate of the pendulum bob.

E. [**14** points] Find the corresponding Hamiltonian. Is the Hamiltonian a constant of the motion? Explain.

[1] The formulas relating velocities and accelerations in rotating and inertial frames are:

$$\begin{aligned}\boldsymbol{v}_{\text{inertial}} &= \boldsymbol{v}_{\text{rotating}} + \boldsymbol{\omega}\times\boldsymbol{r}\\ \boldsymbol{a}_{\text{inertial}} &= \boldsymbol{a}_{\text{rotating}} - 2\boldsymbol{\omega}\times\boldsymbol{v}_{\text{rotating}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\,.\end{aligned}$$

1.28 Foucault Pendulum

An observer stationed at a point in a coordinate system M whose origin is at O and with Cartesian unit-vectors $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, and $\hat{\boldsymbol{e}}_z$ observes a vector $\boldsymbol{A}$ and its time-derivative $\frac{d\boldsymbol{A}}{dt}\big|_{\mathrm{M}}$:

$$\boldsymbol{A} = A_1\,\hat{\boldsymbol{e}}_x + A_2\,\hat{\boldsymbol{e}}_y + A_3\,\hat{\boldsymbol{e}}_z\,, \qquad \frac{d\boldsymbol{A}}{dt} = \frac{dA_1}{dt}\,\hat{\boldsymbol{e}}_x + \frac{dA_2}{dt}\,\hat{\boldsymbol{e}}_y + \frac{dA_3}{dt}\,\hat{\boldsymbol{e}}_z\,.$$

Subsequently, the observer notices that M is rotating with respect to a coordinate system F that is fixed in space, and whose origin coincides with O.

A. [**12** points] Show that the time-derivative of **A** as measured in the fixed coordinate system F, $\frac{d\boldsymbol{A}}{dt}\big|_{\mathrm{F}}$ can be written as

$$\left.\frac{d\boldsymbol{A}}{dt}\right|_{\mathrm{F}} = \left.\frac{d\boldsymbol{A}}{dt}\right|_{\mathrm{M}} + \omega \times \boldsymbol{A}\,.$$

B. [**12** points] Show that the acceleration of a particle measured in frame F is related to that measured in M by

$$\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{\mathrm{F}} = \left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_{\mathrm{M}} + 2\omega \times \left.\frac{d\boldsymbol{r}}{dt}\right|_{\mathrm{M}} + \omega \times (\omega \times \boldsymbol{r}_{\mathrm{M}}) + \frac{d\omega}{dt} \times \boldsymbol{r}_{\mathrm{M}}\,.$$

Consider a simple pendulum formed by a mass attached to the end of a string of length L, mounted to the ceiling of the physics building. In spherical coordinates defined with respect to the axis of rotation of the Earth, the physics building is located at the coordinates θ (the polar angle) and ϕ (the azimuthal angle). Assume that the period of the Earth's rotation about its axis is much longer than the period of the pendulum when in an inertial frame.

C. [**13** points] Show that for small displacements from vertical defined by coordinates x and y in the horizontal plane of the physics building, the position of the mass satisfies

$$\ddot{x} - 2\omega\,\dot{y}\cos\theta + \frac{g}{L}\,x = 0\,, \quad \ddot{y} + 2\omega\,\dot{x}\cos\theta + \frac{g}{L}\,y = 0\,.$$

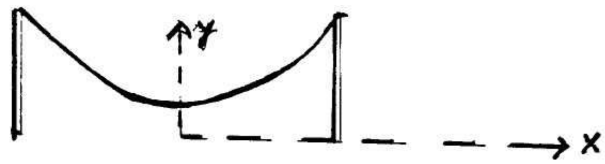
where ω is the angular frequency of the Earth's rotation.

D. [**13** points] Solve these coupled differential equations subject to these boundary conditions at $t = 0$:

$$x = 0\,, \quad \dot{x} = 0\,, \quad y = A\,, \quad \dot{y} = 0\,.$$

1.29 Hanging Cable

A flexible cable hangs under its own weight from two towers of equal height H, as sketched below. Choose the x-axis such that the height of the cable is an even function of x. Because of this, one need only consider the half of the cable for which $x \geq 0$. Let the mass per unit length of the cable be ρ.



A. [**10** points] Let the equation for the shape of the cable be $x = x(y)$. [This will be easier than writing the shape as $y = y(x)$.] Show that the potential energy of the cable (*i.e.*, the half that we are dealing with) can be written as

$$U = \rho g \int_{y_0}^{H} y \, [1 + (x')^2]^{1/2} \, dy \,,$$

where $x' \equiv dx/dy$, g is the acceleration of gravity, and y_0 is the lowest point of the cable.

B. [**15** points] Minimize this expression subject to the constraint that the length of the cable is fixed and thereby obtain an equation for $dy/dx = 1/x'$ in terms of y. (The value of the cable length is not important.)

You are not being asked to solve this equation, but one may show that the solution which is even in x can be written $y = x_0 \, [\cosh(x/x_0) - 1] + y_0$.

Now do the same problem using forces. Let the tension in the cable be T, with T_x and T_y denoting its x and y components.

C. [**5** points] The derivative dT_x/dx is zero. Why is this? As a consequence, T_x is constant.

D. [**10** points] The derivative dT_y/dx is not zero. Write down an equation for it.

E. [**5** points] The components of the tension are related to the slope of the cable according to $T_y/T_x = dy/dx \equiv y'$. Why is this?

F. [**5** points] Using the statement (E) in your equation (D), obtain an equation for dy'/dx in terms of y' and T_x.

G. [**5** points] Show that the solution for y' which is odd in x can be written $y' = \sinh(x/x_0)$. Give an expression for the characteristic length x_0 in terms of T_x. It follows that $y = x_0 \, [\cosh(x/x_0) - 1] + y_0$ as stated above.

1.30 Inertia Tensor

The inertia tensor of an object composed of point masses $\{m_p\}$ is given by

$$I_{ij} = \sum_p m_p \Big[\delta_{ij} \sum_{k=1}^{3} (x_p^k)^2 - x_p^i x_p^j\Big].$$

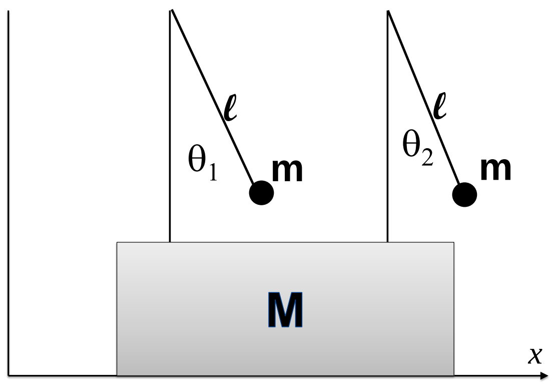
A. [**10** points] Show that $I_{\hat{\boldsymbol{n}}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{I} \cdot \hat{\boldsymbol{n}}$ is the moment of inertia about an axis specified by an arbitrary unit vector $\hat{\boldsymbol{n}}$.

B. [**10** points] If the inertia tensor is diagonal, $I_{ij} = I_i\, \delta_{ij}$, and the angular velocity $\boldsymbol{\omega}$ points in an arbitrary direction, derive an explicit expression for the components of the angular momentum $\boldsymbol{L}$ in terms of the components of $\boldsymbol{\omega}$ and I_i.

C. [**5** points] Under what conditions will the angular momentum point in the same direction as the angular velocity?

1.31 Inertially Coupled Pendula

Two pendula are attached to a cart of mass M that can move without friction along a straight line. The two pendula are constrained to swing only in the plane that contains the straight line along which the cart moves. The two pendula are identical: they both have a mass m attached to a rigid rod of length l whose mass can be neglected, with the rod pivoted on a frictionless pivot. The force of gravity mg acts downward on both of the pendula masses. Denote the position of the cart along the line by x and the angles that the two pendula rods make with the vertical by θ_1 and θ_1.

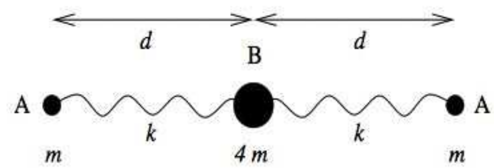


A. [**15** points] Write down the Lagrangian of this system. Simplify your final expression by making the small angle approximations for both θ_1 and θ_2.

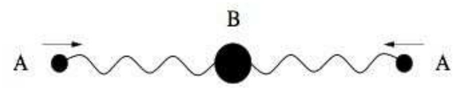
B. [**15** points] Derive the Lagrange equations of motion.

C. [**20** points] Combine the equations of motion to eliminate the x coordinate. Then solve the resulting expressions to obtain the two non-trivial normal modes of the system.

1.32 Linear Triatomic



Consider a linear triatomic molecule, with two atoms of type A at its ends and an atom of type B at the middle. At equilibrium, the distance between each A atom and the central B atom is d, as shown in the figure at right, and the effective spring constants for small oscillations are both equal to k. The mass of an A atom is m, the mass of a B atom is $4m$.



This molecule has various modes of vibration. For instance, one simple mode of vibration, sketched at the right, involves equal and opposite motion by the two A atoms, with the B atom at rest.

A. [**6** points] What is the frequency of the mode illustrated, in terms of m and k?

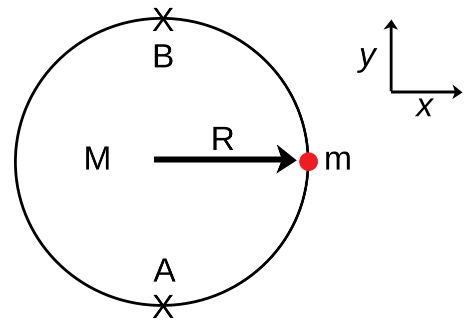
B. [**10** points] There is another oscillation mode with a nonzero frequency in which each atom moves only horizontally. What is the eigenvector of this mode, and what is its eigenfrequency?

C. [**6** points] Since the molecule has three atoms and modes in three spatial dimensions, it has nine eigenmodes of motion. Clearly and unambiguously, describe the nine eigenmodes. When necessary, you may wish to draw pictures analogous to the figure above to show how the three atoms move.

D. [**14** points] At time $t < 0$, the molecule is at rest. At time $t = 0$, the A atom on the left-hand end of the molecule is struck by another atom. The collision occurs rapidly, so the impulse approximation is valid: the A atom's position barely moves, but it acquires a horizontal velocity v to the right. What is the subsequent motion, as a function of time, of the three atoms?

E. [**14** points] Once again, at time $t < 0$, the molecule is at rest. At time zero, the A atom on the left-hand end of the molecule is struck by another atom. As before, the collision occurs rapidly, so the impulse approximation is valid: the A atom's position barely moves, but this time it acquires a *vertical* velocity v upward. Which modes of the system are excited? (You may describe them in words or in pictures, but be unambiguous!) Estimate as best you can (do not calculate) the energy imparted to each of the excited modes. *Hint: You may wish to consider these questions in a different reference frame.*

1.33 Mass on Disk



A thin disk of radius R and mass M in the xy-plane has a point mass $m = 5M/4$ attached to its edge, as shown in the figure. The moment of inertia tensor of the disk *without* the mass m about its center of mass is (the z-axis is out of the page):

$$I = \frac{MR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

in the basis defined by the unit vectors $\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y$ and $\hat{\boldsymbol{e}}_z$.

A. [**10** points] Identify and draw the principal axes of the disk alone about the point A. Identify and draw the principal axes of the point mass, m, alone about the point A. Clearly state which general principles you use at arriving at your answers.

B. [**5** points] Show that the moment of inertia tensor of the combination of disk and point mass about the point A, in the coordinate system shown, is

$$I = \frac{MR^2}{4}\begin{pmatrix} 10 & -5 & 0 \\ -5 & 6 & 0 \\ 0 & 0 & 16 \end{pmatrix}.$$

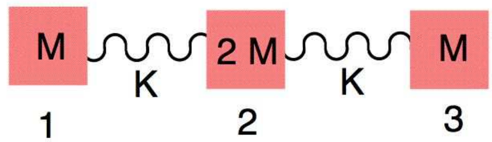
C. [**10** points] Find the principal moments and the principal axes about the point A. Give numerical values.

D. [**10** points] The disk is constrained to rotate about the y-axis with angular velocity ω by pivots at the points A and B. Give the angular momentum about A as a function of time and find the force provided by the pivot at B (ignore gravity).

At some point in time, when the disk is in the xy-plane and rotating with angular velocity $\omega\,\hat{\boldsymbol{e}}_y$, the pivots at A and B are instantaneously removed and the combined disk-mass system is allowed to free-fall under the influence of gravity, which acts in the y-direction.

E. [**10** points] Describe qualitatively the subsequent motion of the combined disk+mass system.

1.34 Normal Modes and Driven Systems



A system is comprised of three masses joined by two springs, as shown in the figure. The two springs have equal spring constants. The outer masses have mass M, while the middle mass has mass $2M$. The system is constrained to move in one dimension. Let η_i denote the displacement of the i'th mass from its equilibrium position.

A. [**10** points] Find the Lagrangian for the system.

B. [**10** points] Find the Euler-Lagrange equations of motion, and show that they can be written as

$$T\,\ddot{\boldsymbol{\eta}} + V\boldsymbol{\eta} = 0\,,$$

where T and V are matrices and $\boldsymbol{\eta}$ is a column vector. Find T and V.

C. [**10** points] Find the normal modes of this isolated system and their frequencies, *i.e.*, find the eigenvalues ω_j and eigenvectors a_j of this system.

D. [**10** points] For an arbitrary system with non-degenerate eigenvalues (such as this) show that the eigenvectors can be normalized so that $a_i^\dagger\,T a_j = \delta_{ij}$, where T is defined as in part B.

Consider the situation where this isolated system is initially at rest with all masses in their equilibrium positions. Starting at time $t = 0$, a force $F(t) = F_0\,\cos(\Omega t)$ is applied to mass 1.

E. [**10** points] By writing the displacement in terms of the eigenvectors found above, $\boldsymbol{\eta}(t) = c_1(t)\,a_1 + c_2(t)\,a_2 + c_3(t)\,a_3$, find the displacement of mass 3 as a function of time.

1.35 Orbit Perturbations

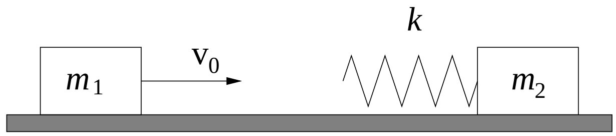
A point mass m moves in a circular orbit of radius r under the influence of a central force with potential $-K/r^n$.

A. [**20** points] Find the conditions on n such that the circular orbit is stable under small perturbations (*i.e.*, the mass will oscillate about the circular orbit).

B. [**10** points] Find the frequency of small oscillations about stable circular orbits and express this frequency in terms of the angular velocity of the mass moving in these orbits.

1.36 Mass-spring system

A mass m_1, with initial velocity v_0, strikes a mass-spring system consisting of a massless spring of spring constant k, attached to a mass m_2. The mass-spring system is initially at rest. There is no friction.

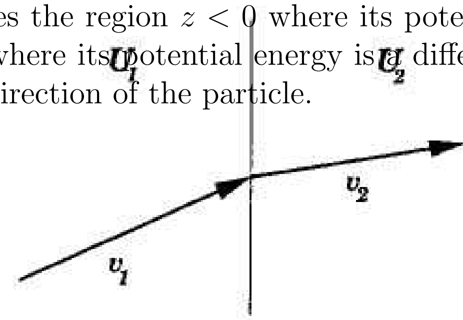


A. [**25** points] What is the maximal compression of the spring during the collision? (Assume the spring to be long enough so that at maximal compression it still has positive length).

B. [**25** points] Find the velocity v_1 and v_2 of the masses m_1 and m_2 long after the collision.

1.37 Particle Refraction

[**20** points] A particle of mass m and velocity $\boldsymbol{v}_1$ leaves the region $z < 0$ where its potential energy is a constant U_1, and enters the region $z > 0$ where its potential energy is a different constant U_2, as illustrated. Determine the change in direction of the particle.

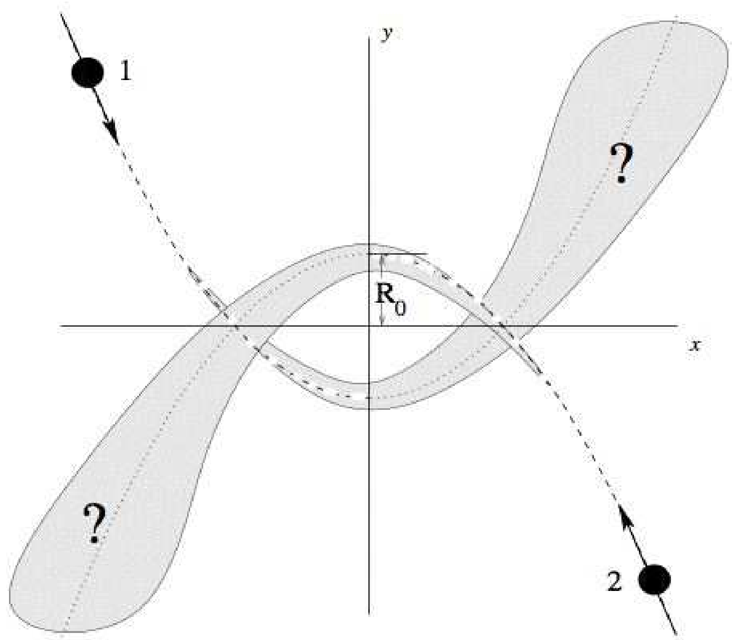


1.38 Passing Stars

Two identical stars — spheres of gaseous material of radius R_s and mass M — approach each other along initially parabolic orbits.[1] In the center-of-mass system, where $x_1 = -x_2, y_1 = -y_2$, their paths for t large and negative are given by

$$y_1 = {x_2}^2/r_0 - R_0; \quad y_2 = -{x_2}^2/r_0 + R_0\,,$$

where r_0 and R_0 are the constants which define the particular orbit. Both r_0 and R_0 are several times larger than R_s.



A. [**10** points] In what crucial and remarkable way does the motion differ from the corresponding Newtonian two-body problem with point masses?

B. [**8** points] Is the condition that the orbits be parabolic necessary for the feature that you have identified as "remarkable" in part A? Why or why not?

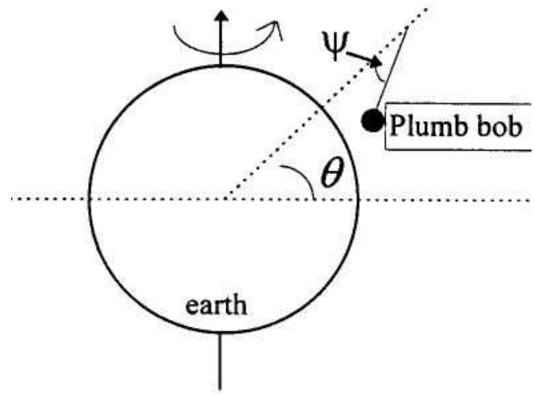
C. [**7** points] Without calculating anything, and without trying to be precise, make a rough plot, in the xy-plane, of the motion of star number 1. Make sure the "remarkable" feature is clearly evident.

D. [**Extra Credit**] Can you suggest any observable consequence that this phenomenon might have? [Note: in our region of the galaxy, stars almost never pass at such close range.]

[1] Problems 1.20–1.22 may be helpful prerequisites for this problem.

1.39 Plumb Bob

A plumb-bob is a small mass suspended from a flexible string, at rest relative to the Earth. The line of the plumb-bob string defines locally what we call "vertical". Due to the Earth's rotation, the plumb-bob line, if extended, will not necessarily pass through the Earth's center. The latitude, θ, of Seattle is 47.3° and you may consider the Earth to be a sphere with radius 6400 km.



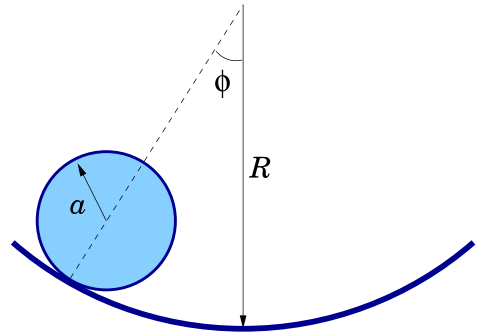
A. [**6** points] Draw on the diagram the vector, $\boldsymbol{a}$, that shows the acceleration of the plumb-bob relative to an inertial (non-rotating) frame of reference.

B. [**4** points] At what latitude(s) will the plumb-bob line, if extended, pass through the center of the Earth?

C. [**15** points] For a plumb-bob at Seattle, at what angle, ψ, does the plumb-bob hang relative to the radius vector from the Earth's center? (You may safely assume that the plumb-bob length $\ll$ Earth's radius.)

1.40 Rolling Cylinder

A homogeneous cylinder of radius a, and total mass μ, rolls without slipping inside a cylindrical surface with radius of curvature R, as shown. Assume uniform gravitational acceleration downward.



A. [**20** points] Derive the Lagrangian for this system.

B. [**15** points] Determine the frequency for small oscillations about the equilibrium at the bottom of the surface.

C. [**15** points] As the amplitude of oscillation increases, does the frequency increase or decrease? First make a qualitative argument. Then derive an expression that would allow determination of the frequency as a function of amplitude, $\phi_{\max}$, and verify its small amplitude limit.

1.41 Rotating Frame

Consider a particle of mass m moving on a two dimensional plane (x, y), in a central gravitational potential of a massive body $V(r) = -GMm/r$ where $r = \sqrt{x^2 + y^2}$. Assume $M \gg m$.

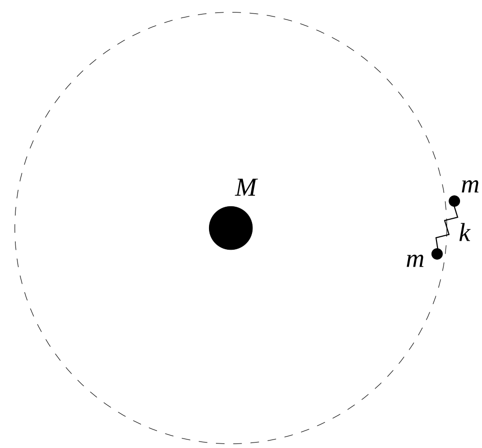
A. [**10** points] Write down the Lagrangian for this particle in the coordinate system rotating with angular velocity $\boldsymbol{\omega}$ around the center of gravity.

B. [**10** points] Show that the Euler-Lagrange equation which follows from the Lagrangian derived above can be written as

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}_{\text{gravity}} + \boldsymbol{F}_{\text{centripetal}} + \boldsymbol{F}_{\text{Coriolis}},$$

where $\boldsymbol{F}_{\text{Coriolis}} = 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega}$.

Consider now a system of two particles with equal mass m, connected by a spring with a spring constant k. The separation between the two particles is much smaller than the distance from the two to the massive body, but cannot be neglected for this problem. The two bodies move approximately along a circular orbit. Consider the motion only in the plane of that orbit.



C. [**10** points] Write down the system of linear differential equations describing small oscillations around the equilibrium (two particles on top of each other moving on a circle) in an appropriate rotating frame.

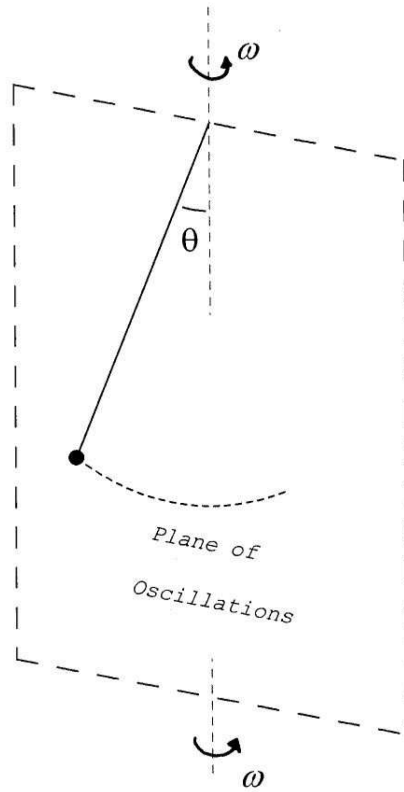
D. [**5** points] Show that the normal modes can be divided into oscillations of the center of mass of the two-particle system and the relative motion.

E. [**10** points] Find all normal frequencies of the oscillations. Show that when k is smaller than some critical value k_c, there exists an unstable mode, and find k_c.

F. [**5** points] Describe qualitatively the long-term evolution of the system when $k < k_c$ and the separation between particles is small in the beginning.

1.42 Rotating Pendulum

A pendulum is constructed from a point mass M suspended from a rigid massless rod of length l. The rod is attached with a hinge (requiring the oscillations to be in the plane normal to the hinge axis) to a vertical shaft which rotates with a constant angular speed ω. Hence, as a consequence of the hinge, the pendulum is constrained to move in a vertical plane which rotates at a constant angular speed ω, as illustrated at right. In all questions below, ignore any effect of the rotation of the Earth.



A. [**10** points] Working in the fixed (non-rotating) frame, derive the kinetic and potential energy, determine the Lagrangian for the system, and show that the equation of motion for mass M is

$$\ddot{\theta} + \left(\frac{g}{l} - \omega^2 \cos\theta\right) \sin\theta = 0\,.$$

B. [**20** points] Depending on the value of ω, the equation of motion allows for up to two equilibrium positions. Determine which equilibrium position(s) are stable, and find the period of small oscillations about each equilibrium position.

C. [**10** points] Determine the power supplied by the motor which rotates the vertical shaft as a function of θ and $\dot{\theta}$.

D. [**5** points] Viewed in the fixed (non-rotating) frame, is the total mechanical energy of the mass M conserved? Is the Hamiltonian conserved? Assume arbitrary initial conditions (not necessarily at equilibrium). Explain your answers.

E. [**5** points] Viewed in the rotating frame, is the total mechanical energy of the mass M conserved? Is the Hamiltonian conserved? Assume arbitrary initial conditions (not necessarily at equilibrium). Explain your answers.

1.43 Safe Driving

Consider some motions of an automobile which has a wheelbase (the distance between front and rear axles) l, and whose center-of-mass is half way between the front and rear axles and a distance h above the ground.

A. [**20** points] Suppose that the driver is braking the car. Sketch and label all the forces acting on the car and where they act. Suppose that the braking results in a de-acceleration b of the car. Let x be the fraction of the weight of the car carried by the front wheels so $1-x$ is the fraction of the weight carried by the rear wheels. Prove that:

$$x = \frac{1}{2} + \frac{b}{g}\frac{h}{l}\,,$$

where g is the acceleration of gravity at the surface of the Earth.

B. [**15** points] Suppose that the car is going around an (un-banked) curve of radius R (with $R \gg l$) at speed v. There is a maximum speed $v_{\max}$ that the car can go around this curve before the tires slip. Show that $v_{\max}$ is given by

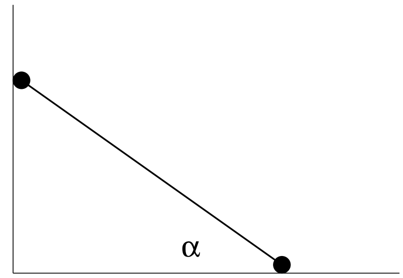
$$v_{\max} = \mu g R\,,$$

where μ is the coefficient of static friction of the tires. In view of the fact that the car has a finite moment of inertia about the vertical axis through the center of mass, show that this result does not require that the center of mass be exactly half way between the front and rear axles. Be sure to explain your reasoning.

C. [**15** points] Now suppose that the car is going around the curve at a speed that is just a little less than $v_{\max}$ and the driver becomes nervous and applies the brakes fairly hard. What happens? Do not give precise quantitative results, but be sure to explain your reasoning.

1.44 Sliding dumbbell

A dumbbell of mass m and length $2l$ leans against a frictionless vertical wall while standing on a frictionless horizontal floor. The two (equal) weights at the two ends of the dumbbell have negligible size and carry all the mass. The dumbbell is initially at rest and then released with initial angle α_0. At some point in the subsequent motion, the dumbbell may separate from the wall.



A. [**10** points] Assume that the dumbbell has not separated from the wall and the floor. Taking the angle α as the generalized coordinate, write the Lagrangian for the system and the Euler-Lagrange equation of motion.

B. [**10** points] Find the forces acting on the ladder at the two ends as a function of the angle α.

C. [**15** points] Find the angle α at the moment when the ladder separates from the wall

D. [**15** points] Describe the subsequent motion of the dumbbell after its separation from the wall.

1.45 Sliding rod

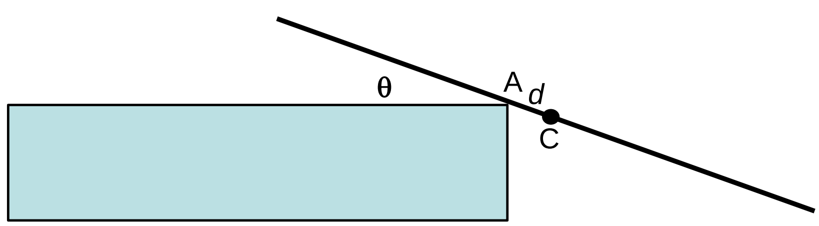
A uniform rod of mass M and length L is placed at right angles to an edge of a horizontal table. The center of mass C of the rod projects a distance d beyond the edge at point A. The coefficient of static friction equals μ. The rod, flat on the table, is released at rest. It starts to rotate about A and eventually slides off the table. The figure below shows the rod at time when it is rotating, but not yet sliding.

A. [**8** points] Calculate the moments of inertia of the rod, I_C about point C, and I_A, about point A. You may express the answers for parts (b) - (d) below in terms of I_C and I_A.

B. [**12** points] Calculate the angular velocity ω of the rod as a function of the rotation angle θ before sliding occurs.

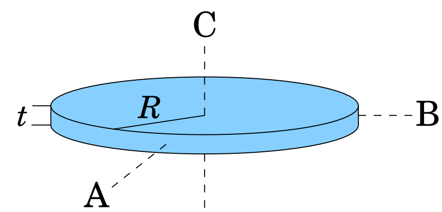
C. [**15** points] The force acting on the rod by the table edge has a component in a direction perpendicular to the rod. Calculate this component N as a function of θ before sliding begins.

D. [**15** points] Calculate the angle θ when sliding begins.



1.46 Spinning Disk

Consider a uniform density circular disk of radius R, thickness t, and mass M.

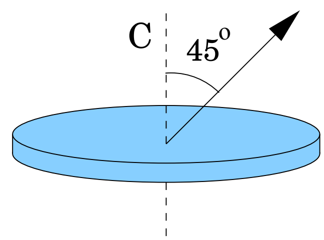


A. [**10** points] Calculate the moment of inertia, I_C, about the symmetry axis C (the vertical dotted line in diagram).

B. [**10** points] For any thin plate, the perpendicular axis theorem states that the sum of the two moments of inertia about perpendicular axes in the plate is equal to the moment of inertia about an axis through the intersection of the two axes and perpendicular to the plate. Prove this theorem and use it to calculate the moments of inertia I_A and I_B about axes A and B, in the limit $t \ll R$.

C. [**10** points] Axes A, B, and C are the principal axes of the moment of inertia tensor referred to the center of mass. Explain why this is so.

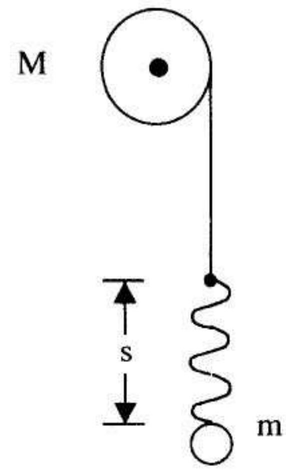
D. [**10** points] The disk is set spinning so that at time $t = 0$ its angular velocity ω points along an axis through the center of mass that makes an angle 45° with respect to the symmetry axis C. Calculate the angular momentum $\boldsymbol{L}$ of the disk at time zero.



E. [**10** points] Describe the subsequent motion, by specifying the directions of the angular momentum and angular velocity vectors. No external forces act on the disk at positive times.

1.47 Spring & Pulley

A uniform density solid cylinder of mass M and radius R is free to rotate about its axis, which is horizontal. The moment of inertia of the cylinder is $I = \frac{1}{2}MR^2$. Part of a long cable of negligible mass is wound around the cylinder with the remainder of the cable hanging vertically. A massless spring, with spring constant k, is attached to the hanging end of the cable, and a mass m is attached to the end of the spring. A vertical gravitational force mg acts downward on the mass m. Describe the system using as coordinates the extension s of the spring and the vertical position y of the mass m (with the axis of the cylinder at $y = 0$).



A. [**6** points] Find an expression for the kinetic energy T of the system.

B. [**4** points] Find an expression for the potential energy V of the system.

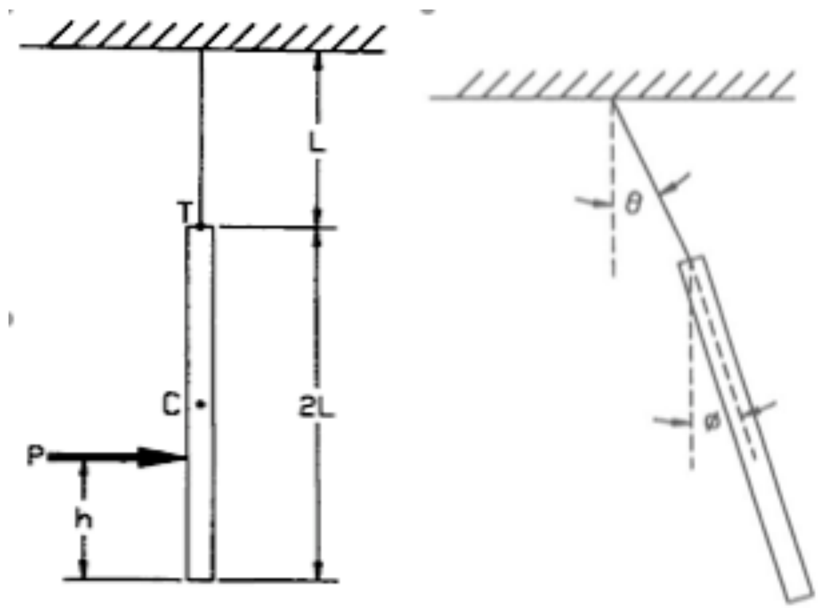
C. [**8** points] Write down the Lagrangian for the system and find the Lagrange equations of motion.

D. [**4** points] Find the canonical momenta that are conjugate to s and y.

E. [**6** points] Find the Hamiltonian in terms of the canonical variables.

F. [**7** points] If $s = \dot{s} = 0$ and $\dot{y} = 0$ at $t = 0$, find the subsequent motion of the mass m.

1.48 Sweet Spot



A uniform rod of mass M and length $2L$ is suspended at one end from the ceiling by a massless string of length L. An impulse of magnitude P is applied to the rod in a horizontal direction at a distance h above the bottom end of the rod.

A. [**5** points] Calculate the speed of the center of mass C of the rod immediately after the impulse.

B. [**5** points] Calculate the speed of the top end of the rod (point T on the diagram) immediately after the impulse. Give your answer in the form $v_{\mathrm{T}} = AP/M$, where A is an expression that depends on h and L only.

For the rest of the problem assume that the impulse is small, so that the string remains taut and the angles θ and ϕ defined in the figure remain small at all times, so that you can use a small angle approximation.

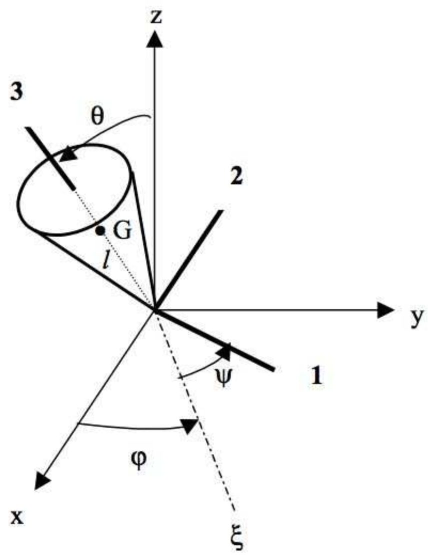
C. [**10** points] Write down expressions for the kinetic energy T and potential energy V in terms of θ and ϕ and their time derivatives.

D. [**20** points] Calculate the oscillation frequencies of the normal modes of the system.

E. [**10** points] For a particular value of h (a "sweet spot") the motion resulting from the impulse occurs at only one of the two frequencies determined in part D. Calculate the dimensionless ratio h/L and identify the corresponding frequency.

1.49 Symmetric Top

A symmetric top of mass M is in a uniform gravitational field along the z-axis and has one fixed point at the tip. The principal axes of the top are labeled 1, 2 and 3. The diagonal components of the inertia tensor in the principal axes are I_1, I_2 and I_3, with $I_1 = I_2$ because of axial symmetry. The center of mass is located at the point labeled G, a distance of l from the tip.



A. [**3** points] Show that the moment of inertia of the top about the 1-axis in the above figure is

$$I = I_1 + Ml^2 \,.$$

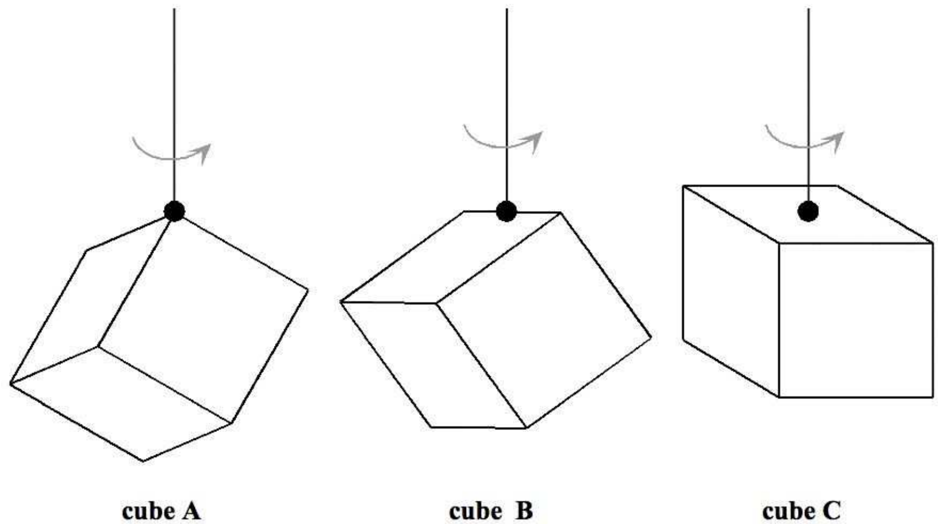
B. [**17** points] Show that the Lagrangian can be written in terms of the Euler angles, θ, ϕ and ψ as

$$L = \tfrac{1}{2}\, I \left(\phi^2 \sin^2\theta + \dot{\theta}^2\right) + \tfrac{1}{2}\, I_3 \left(\phi \cos\theta + \dot{\psi}\right)^2 - Mgl\cos\theta \,.$$

C. [**10** points] Show that components of the angular momentum about two axes are conserved.

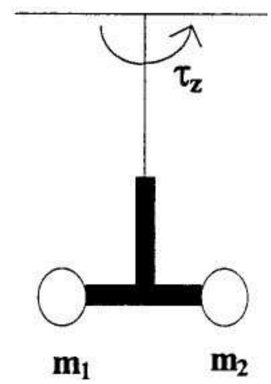
1.50 Three Cubes

[**40** points] A torsion pendulum consists of a vertical wire attached to a mass which may rotate about the vertical. Consider three torsion pendulums which consist of identical wires from which identical homogeneous solid cubes are hung. Cube A is hung from a corner, cube B from midway along an edge, and cube C from the middle of a face. What are the ratios of periods of the three pendulums, $T_A : T_B : T_C$? Briefly explain or derive your answer.



1.51 Torsion Pendulum

A torsion pendulum consists of two masses, m_1 and m_2, supported by a holder which is, in turn, suspended from a long elastic fiber. Torques along the fiber $\hat{z}$ axis cause the pendulum to twist in the xy plane by an angle θ. The fiber provides a restoring torque given by $\tau_z = -k\theta$, where k is the torsion constant of the fiber. The moment of inertia of the pendulum about the $\hat{z}$ axis is I.



For the following questions you may assume that $m_1 = m_2$, although it is not necessary to do so.

A. [**5** points] The torsion pendulum can undergo motions other than its torsional oscillation. Describe in words or by sketches two of the normal modes (other than the torsional mode) of the torsion pendulum. (*You may consider the fiber to be massless.*)

B. [**10** points] Write the Lagrangian and Lagrange equation of motion for the torsional mode of the pendulum, and a solution (other than $\theta = \dot{\theta} = 0$) to this equation.

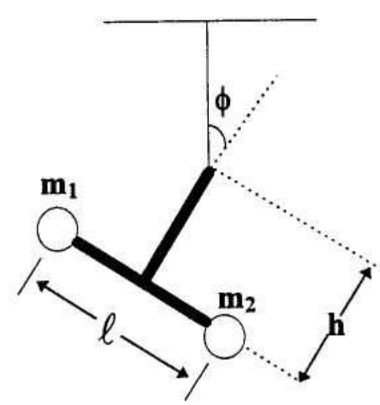
C. [**5** points] Find the momentum that is *canonical* to the coordinate θ.

D. [**5** points] In general, there is also a viscous (frictional) torque proportional to the angular velocity, $\dot{\theta}$, about the $\hat{z}$ axis: $\tau_{\text{fric},z} = -\gamma\dot{\theta}$. Derive the equation: $I\ddot{\theta} + \gamma\dot{\theta} + k\theta = \tau_{\text{ext},z}$, where $\boldsymbol{\tau}_{\text{ext}}$ is the torque on the pendulum due to any external forces.

E. [**5** points] For $\tau_{\text{ext},z} = 0$, what is the resonant (natural) frequency of the torsional mode of oscillation when $\gamma < \sqrt{kI}$?

F. [**10** points] For $\tau_{\text{ext},z} = \tau_0 \cos\omega t$, derive an expression for the steady state motion of the torsion pendulum.

G. [**10** points] In general, it is not possible to make the masses m_1 and m_2 identical, so the torsion pendulum, at rest, will hang tipped by an angle ϕ, relative to the fiber, as shown in the figure. Derive an expression for the tip angle ϕ in terms of m_1, m_2, h and ℓ. You should assume that the fiber is massless and flexible, and that the holder is rigid and has negligible mass.



1.52 Vibrating Spring

A. [**8** points] Derive from first principles the Lagrangian describing small transverse vibrations of a string with mass per unit length $\mu(x)$ and tension $T(x)$. Assume that the ends of the string satisfy either fixed-end or free-end boundary conditions.

B. [**7** points] Show how the Euler-Lagrange equations for the transverse displacement y of the string follow from Hamilton's principle and lead to a wave equation for the string displacement.

C. [**7** points] Find the canonical momentum density and construct the Hamiltonian density for the string. Show that the Hamiltonian is conserved.

D. [**6** points] Deduce an expression for the energy flux along the string.

E. [**12** points] Let $x = 0$ denote the midpoint of the string. Now assume that the mass density along the string changes from a uniform value $\mu = \mu_1$ for $x < 0$ to a uniform value $\mu = \mu_2$ for $x > 0$. The tension T is uniform everywhere. Find the transmission and reflection coefficients for a harmonic wave with angular frequency ω incident from the left, and demonstrate that these are consistent with conservation of energy.

1.53 Waves on a String

A thin stretched string, fixed at $x = a$ and $x = b$, has mass per unit length $\sigma(x)$ and tension $\tau(x)$. In the following, let $u(x,t)$ represent transverse small-displacement solutions to the wave equation. (Note that varying tension would require a longitudinal external force, gravity for example, but the source of the external force is not relevant here.) Assume that no external force acts transverse to the string, so that its equilibrium configuration is straight.

A. [**15** points] Using a labeled free-body diagram, derive the wave equation by applying Newton's law. State all assumptions.

B. [**15** points] Construct the Lagrangian density and use Hamilton's principle (*i.e.*, the principle of stationary action) to derive the equation of motion.

C. [**5** points] Assuming normal-mode solutions of the form $u(x,t) = \rho(x)\cos(\omega t + \phi)$, find the differential equation for $\rho(x)$.

D. [**15** points] Show that setting to zero the variation of the functional

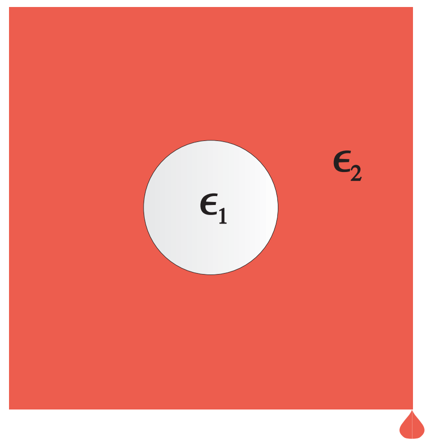
$$\Lambda[\rho] \equiv \int_a^b dx \left[\tau(x)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \Bigg/ \int_a^b dx \ \left[\sigma(x)\,\rho^2\right]$$

with respect to ρ also gives the equation of motion for $\rho(x)$ of the previous part. Comment on why this works, interpreting the value of the functional. Your discussion may specialize to the case of constant σ and τ.

Chapter 2

Electromagnetism

2.1 Blood Bath



Model a sample of blood as insulating spheres, representing cells, immersed in a conductive medium. The aim is to obtain the behavior of a sample under low frequency ($\omega \lesssim 1$ GHz) AC fields.

First consider a sphere of radius a with permittivity ϵ_1 immersed in a homogeneous medium with permittivity ϵ_2 under a *static* field which, at very large distance from the sphere is $\boldsymbol{E} = (0,\, 0,\, E_0)$.

A. [**15** points] State the boundary conditions for the radial and tangential components of the electric field and calculate the electric potential in the entire space.

B. [**10** points] Show that the effective permittivity ϵ, of a dilute system of spheres of permittivity ϵ_1 and number density per unit volume n, within a bath of permittivity ϵ_2, is given by

$$\frac{\epsilon}{\epsilon_2} = 1 + 4\pi\, na^3 \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\,.$$

Assume that the concentration of spheres is low enough that the field in each sphere is not influenced by the others.

Now consider a sphere with permittivity ϵ_1 and zero conductivity immersed in a homogeneous medium with permittivity ϵ_2 and conductivity σ_2 under a field which, at very large distance from the sphere is $\boldsymbol{E} = (0,\, 0,\, E_0\, e^{i\omega t})$. Assume that ω is such that magnetic fields and radiation effects can be neglected.

C. [**5** points] Use Maxwell's equations to show that the boundary conditions are now given by:

$$\epsilon_1\, E_1^{\perp} = (\epsilon_2 + \sigma_2/i\omega)\, E_2^{\perp}\,, \qquad \text{and} \qquad E_1^{\parallel} = E_2^{\parallel}\,.$$

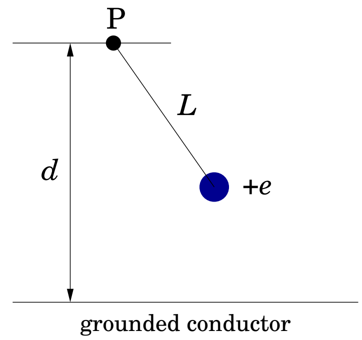
D. [**20** points] Obtain an expression for the complex permittivity of a solution with n spheres per unit volume at frequency ω. Find the real part of the permittivity at both very low and very high frequencies. In which of these regimes is the real part of the permittivity higher. Explain why and determine an approximate value for the frequency that characterizes the transition from the low-frequency to the high-frequency behavior.

2.2 Charge Density Wave

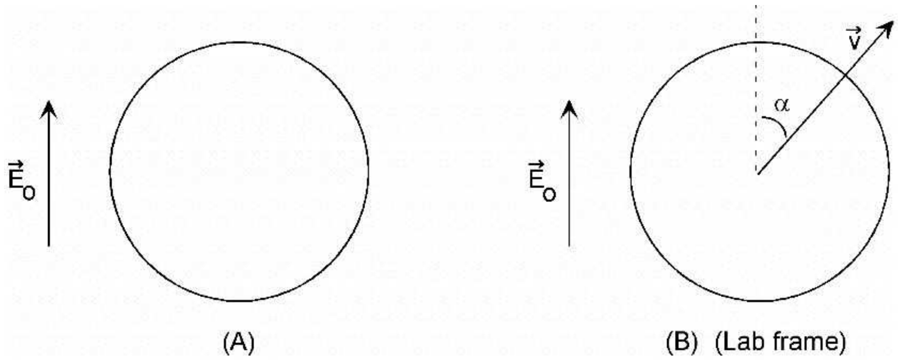
[**25** points] An infinite charge sheet lies in the xy-plane and carries a periodic surface charge density, $\sigma = \sigma_0 \cos(kx)$. Calculate the electric potential produced by this charge distribution everywhere in space. Check that your answer reduces to an expected result in the limit $k \to 0$.

2.3 Charged Pendulum

[**20** points] A particle of mass m and charge e is suspended on a string of length L fixed at point P. At a distance d under the point of suspension there is an infinite plane grounded conductor. Compute the frequency of the oscillations of the pendulum if the amplitude of the oscillations is small. Neglect gravity.



2.4 Conducting Sphere



A. An isolated, uncharged conducting sphere of radius R is placed in a uniform electric field with magnitude E_0 pointing in the z-direction, as illustrated in figure (A).

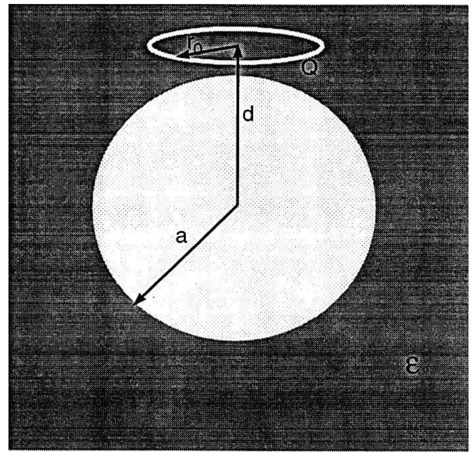
  i. [**15** points] Derive an expression for the electric potential everywhere outside the sphere.

  ii. [**5** points] Derive an expression for the angular distribution of the induced surface charge on the sphere.

B. [**10** points] Suppose that the sphere is now forced to move with constant speed v in the xz-plane at some angle α with respect to the z-axis, as illustrated in figure (B). Assume that α is greater than zero and less than $\pi/2$. When v is relativistic, explain (qualitatively or quantitatively) how the induced surface charge distribution *viewed in the reference frame of the sphere* will differ from your answer in part A-ii. How do the magnitude, angular orientation, and angular distribution of the induced surface charge differ?

2.5 Conductors in Dielectric

A loop of radius r_0 and charge Q is located above a grounded, conducting sphere of radius a, as shown in the figure. The plane of the loop is displaced vertically from the center of the sphere by a distance d. The entire system is embedded in a dielectric of infinite extent with dielectric constant ε.[1]



A. [**12** points] From Maxwell's equations, derive the boundary conditions satisfied by an electrostatic field at the interface between the dielectric and the conducting sphere.

The Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ for (minus) the Laplacian in the region *outside* the conducting sphere is given by

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{a}{|\boldsymbol{r}'|}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}''|}\right),$$

where $\boldsymbol{r}$ is the field position, $\boldsymbol{r}'$ is the source position, and $\boldsymbol{r}'' \equiv a^2\,\boldsymbol{r}'/|\boldsymbol{r}'|^2$ (with the origin chosen to be the center of the conducting sphere).

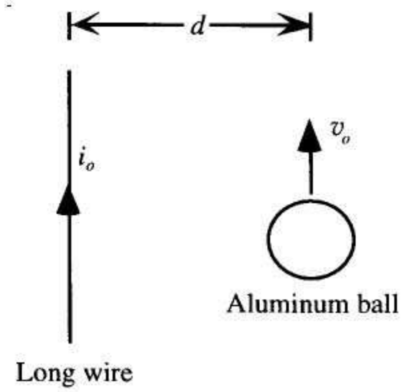
B. [**12** points] What properties does $G(\boldsymbol{r}, \boldsymbol{r}')$ possess to make it the appropriate Green's function. Draw a schematic to show the location of the sphere, a point charge, and any image charges that might arise.

C. [**14** points] Express the potential in the dielectric medium resulting from the charged loop as a sum of Legendre polynomials.

D. [**12** points] Find the charge induced on the conducting sphere.

[1]Possibly useful relations: $1/|r-r'| = 4\pi\sum_\ell \frac{1}{2\ell+1} Y^*_{\ell m}(\Omega')\, Y_{\ell m}(\Omega)\, r_<^\ell / r_>^{\ell+1}$, and $Y_{\ell 0} = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$, where $r_< \equiv \min(r, r')$ and $r_> \equiv \max(r, r')$.

2.6 Currents and Forces



An uncharged aluminum ball (with negligible magnetic susceptibility) is constrained to move with a constant velocity $\vec{v}_0$ (not necessarily non-relativistic) parallel to a long straight wire. The wire has no net charge but carries a current I_0 in the same direction as the motion of the ball. The distance between the wire and the center of the ball is d.

A. [**20** points] Give an argument to account for the *existence* of an attractive force in the lab frame (the frame in which the wire is at rest). Then describe how the force depends on d, in the limit where d is large compared to the size of the ball.

B. [**40** points] The current-carrying wire can be modeled as two long thin rods that have uniform linear charge density and are in relative motion. Suppose that in the lab frame the negatively charged rod is at rest while the positively charged rod moves toward the top of the page with velocity $m\vec{v}_0$. The charge densities of the rods as measured in the lab frame are $\lambda_{\pm}^{(\mathrm{lab})}$.

i. In the lab frame, determine the electric and magnetic fields due to the wire, $\boldsymbol{E}^{(\mathrm{lab})}$ and $\boldsymbol{B}^{(\mathrm{lab})}$, at a distance d from the wire. Express your answer in terms of I_0.

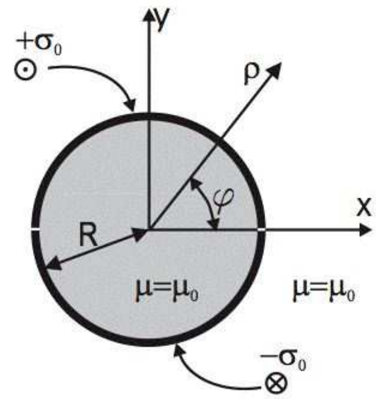
ii. In the rest frame of the ball, determine the electric and magnetic fields due to the wire, $\boldsymbol{E}^{(\mathrm{ball})}$ and $\boldsymbol{B}^{(\mathrm{ball})}$, at a distance d from the wire.

iii. Is there a frame (moving in any direction) in which the magnetic field due to the wire, at a distance d from the wire, vanishes? If so, determine the velocity of that frame relative to the lab frame. If not, explain why not.

iv. Denote the magnitudes of the electromagnetic forces exerted on the ball by the wire in the lab frame and in the rest frame of the ball by $F_{\mathrm{BW}}^{(\mathrm{lab})}$ and $F_{\mathrm{BW}}^{(\mathrm{ball})}$, respectively. Determine the non-relativistic limit of the ratio $F_{\mathrm{BW}}^{(\mathrm{lab})}/F_{\mathrm{BW}}^{(\mathrm{ball})}$. Show your work and explain your reasoning.

2.7 Current-Carrying Cable

A cable consists of a material of magnetic permeability $\mu = \mu_0$ with the shape of an infinitely long cylinder of radius R, covered by two infinitesimally thin conducting sheets as shown in the figure. The two conducting sheets are insulated from each other and carry surface current densities $\sigma = \sigma_0 \hat{\boldsymbol{e}}_z$ (top sheet, $0 \le \phi \le \pi$) and $\sigma = -\sigma_0 \hat{\boldsymbol{e}}_z$ (bottom sheet, $\pi \le \phi \le 2\pi$).



A. [**15** points] Given that the volume current density is zero, show that one can use a magnetic scalar potential ψ, such that $\boldsymbol{H} = \nabla\psi$ everywhere. Using Maxwell's equations, show that $\nabla^2\psi = 0$ inside and outside the cable, write down the general form of the solutions (inside and outside) in cylindrical coordinates, and specify the boundary conditions that should be used at the surface of the cable.

B. [**10** points] Use qualitative arguments to sketch the field lines.

C. [**15** points] Show that the magnetic field inside and outside the cable has the form

$$\boldsymbol{B}^{\text{in}}(\boldsymbol{x}) = A^{\text{int}} \sum_{n=1,3,5,\cdots} \frac{1}{n}\left(\frac{\rho}{R}\right)^{n-1} \left(\hat{\boldsymbol{e}}_\rho \cos n\phi - \hat{\boldsymbol{e}}_\phi \sin n\phi\right),$$

$$\boldsymbol{B}^{\text{out}}(\boldsymbol{x}) = A^{\text{out}} \sum_{n=1,3,5,\cdots} \frac{1}{n}\left(\frac{R}{\rho}\right)^{n+1} \left(-\hat{\boldsymbol{e}}_\rho \cos n\phi - \hat{\boldsymbol{e}}_\phi \sin n\phi\right),$$

with the constants A^{in} and A^{out} determined by the characteristics of the cable.

D. [**10** points] At large distances from the cable, the field has the form,

$$\boldsymbol{B}(\boldsymbol{x}) \sim \frac{C}{\rho^2}\left(-\hat{\boldsymbol{e}}_\rho \cos\phi - \hat{\boldsymbol{e}}_\phi \sin\phi\right).$$

This is the same form as the field produced by two infinitely thin wires located at $x = 0$, $y = \pm d/2$ and carrying currents $\pm I\,\hat{\boldsymbol{e}}_z$. Find a current I and separation d that would produce, at large distances, the same field as the cable of this problem.

2.8 Dipoles and Dielectrics

A. [**15** points] Consider a dielectric sphere of radius R and permittivity ϵ in empty space. Find the field inside the sphere when the electric field at large distances from the sphere is uniform and aligned in the z-direction.

B. [**20** points] Prove that for an arbitrary distribution of charges residing within a sphere of radius R, the volume integral of the electric field over the enclosing sphere is related to the electric dipole moment of the charge distribution,

$$\int_{r<R} d^3x\, \boldsymbol{E}(\boldsymbol{r}) = -\frac{\boldsymbol{p}}{3\epsilon_0}\,.$$

Here ϵ_0 is the vacuum permittivity, and $\boldsymbol{p}$ is the electric dipole moment of the charge distribution,

$$\boldsymbol{p} \equiv \int d^3r\, \boldsymbol{r}\, \rho(\boldsymbol{r})\,.$$

[Hint: at some point in your proof you may need the relationship:

$$\int_{r=R} d^2S\, \frac{\boldsymbol{n}}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{4\pi}{3}\,\boldsymbol{r}'\,,$$

where the integral is over the surface of a sphere with radius R, $\boldsymbol{n}$ is the outward unit normal to the surface, and $\boldsymbol{r}'$ is an arbitrary point within the sphere. If you use this relationship, first prove it.]

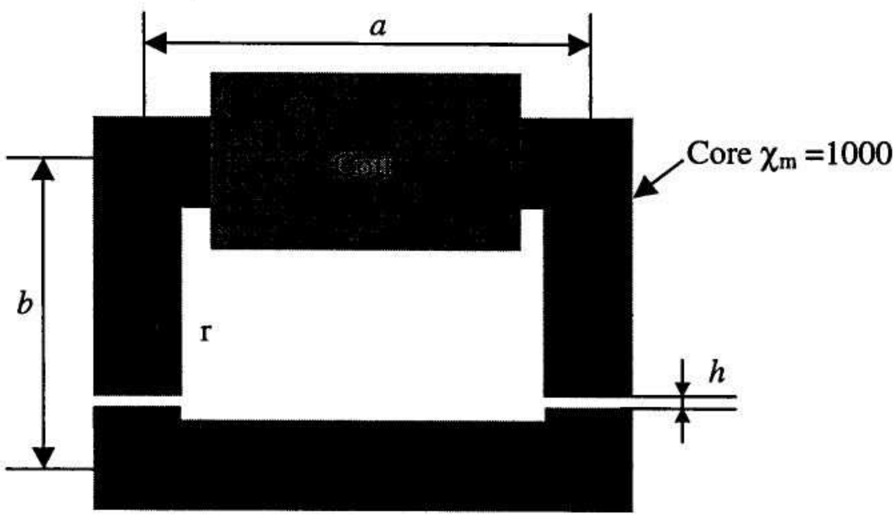
C. [**15** points] Consider an electric dipole located at the origin. The field, for $r > 0$, is given by

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\,\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3}\right]\,.$$

Show that the volume integral over a sphere with radius $R > 0$ (excluding the point at the origin) yields zero, seemingly contradicting what was proven in part B. Explain the origin of this apparent discrepancy.

2.9 Electromagnet Hoist

A U-shaped electromagnet designed for lifting metal objects is sketched below. The dimensions of the magnet core are shown in the figure. The core has circular cross-sectional area $A = \pi r^2$ and magnetic susceptibility $\chi_m = 1000$. Assume that the object being lifted has the same magnetic susceptibility and cross-sectional area. The coil has N turns and a constant current I flows through it. Your goal is to estimate the lifting force as a function of the gap h. Assume that the gap h is much much smaller than a, b, and r, but that $\chi_m h$ is comparable to a and b.



A. [**10** points] Write down the relationship between B and H in the core and in the gap.

B. [**10** points] Calculate the magnetic fields in the core and in the gap as a function of h. [Hint: use Ampere's law.]

C. [**10** points] Calculate the energy of the electromagnetic field. Assume that the field is concentrated in the core and the gaps.

D. [**10** points] Calculate the work done by the power supply of the coil when the gap distance is changed by an amount δh. Assume that the power supply maintains a constant current.

E. [**10** points] Calculate the lifting force.

2.10 Electrostatic Sheet

Consider a non-conducting sheet in empty space, located in the xy-plane. The electrostatic potential on the sheet is given by

$$\Phi(x, y, z{=}0) = C \sin(k_x x) \sin(k_y y),$$

with k_x, k_y, and C constants.

A. [**15** points] Find the potential $\Phi(x, y, z)$ everywhere in space.

B. [**10** points] Determine the charge per unit area $\sigma(x, y)$ on the sheet.

Replace the above sheet of charge by a sheet with an electric dipole moment per unit area given by $\boldsymbol{p}(x, y, 0) = C' \cos(\kappa x) \hat{\boldsymbol{e}}_x$, where C' and κ are constants.

C. [**10** points] Determine the potential $\Phi(x, y, z)$ everywhere in space.

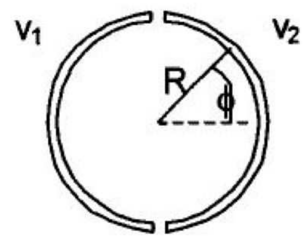
Finally, consider a non-conducting sheet located in the xy-plane with a periodic surface charge density given by $\sigma(x, y) = \sigma(x + x_0, y)$ together with

$$\sigma(x, y) = \begin{cases} \sigma_0, & 0 < x < x_0/2; \\ 0, & x_0/2 < x < x_0. \end{cases}$$

D. [**15** points] Calculate the potential $\Phi(x, y, z)$ everywhere in space. You may express the result as an infinite series if you wish.

2.11 Electrostatic Cylinder (1)

A long cylindrical conducting tube of radius R is separated into two equal parts by an infinitely fine cut parallel to the cylinder axis. The two parts are held at constant potentials V_1 and V_2.

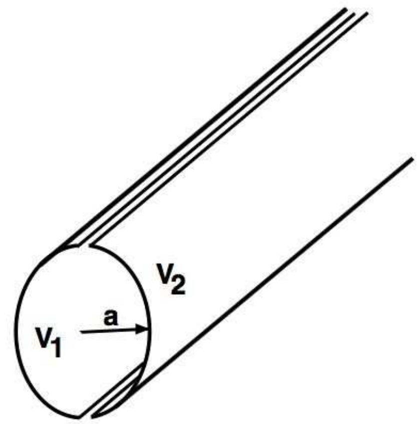


A. [**10** points] What is the differential equation satisfied by the potential V inside and outside the cylinder? What are the boundary conditions on the potential as the radial distance $r \to \infty$?

B. [**15** points] Determine the potential $\Phi(r, \theta)$ at all points inside the cylinder.

2.12 Electrostatic Cylinder (2)

An infinitely long hollow cylindrical conductor of radius a is divided into two parts by a plane through the axis, and the parts are separated by a negligibly small gap. The parts are kept at potentials V_1 and V_2.



A. [**15** points] Show that the most general form of the potential inside the conducting cylinder is[1]

$$\Phi(r,\theta) = \sum_{m=0}^{\infty} (r/a)^m \left[A_m \cos(m\theta) + B_m \sin(m\theta)\right].$$

B. [**15** points] Find the coefficients A_m and B_m.

[1]Possibly useful: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$.

2.13 Electrostatic Shells

The most general solution to Laplace's equation, $\nabla^2\Phi = 0$ with azimuthal symmetry may be expressed in the form

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \, .$$

A conducting sphere of radius a at a potential V_0 is surrounded by a concentric non-conducting spherical shell of radius b with a surface charge density $\sigma(\theta) = K\cos\theta$, where K is constant.

A. [**9** points] Write the allowed form of the potential in each of the two regions $a < r < b$ and $r > b$, without regard to the boundary conditions at $r = a$ and $r = b$.

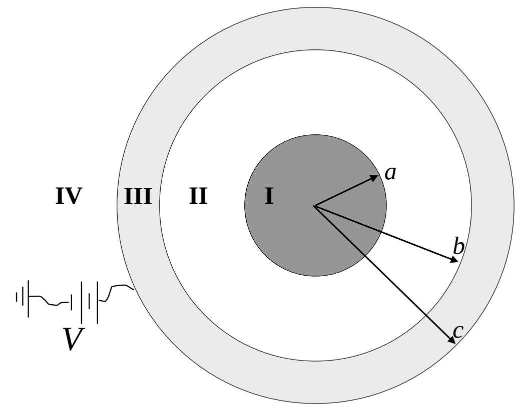
B. [**9** points] What is the discontinuity in the electric field across the non-conducting spherical shell? Explain the physical basis of your answer.

C. [**8** points] What are the boundary conditions that must be satisfied by the potential $\Phi(r,\theta)$?

D. [**14** points] Find the potential in both regions.

2.14 Electrostatic spheres

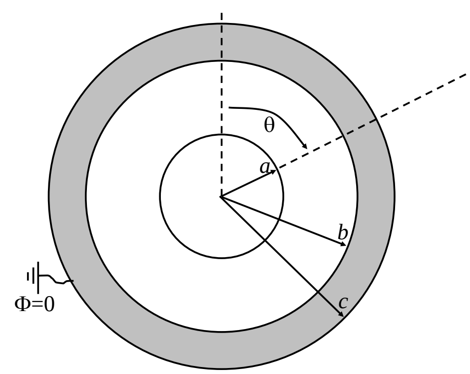
A uniform charge density sphere of radius a has total charge Q. It is surrounded by a spherical shell that is centered on the same origin. The shell is made of copper, having inner and outer radii b and c, respectively, and the shell is kept at potential V.



A. [**10** points] Find the potential Φ and the field $\vec{E}$ in the four marked regions: I (charged sphere); II (vacuum); III (copper shell); and IV (vacuum). Assume $\Phi(\infty) = 0$. Carefully sketch the radial potential $\Phi(r)$, label the axes, and indicate the values of the potential at $r = a, b$, and c.

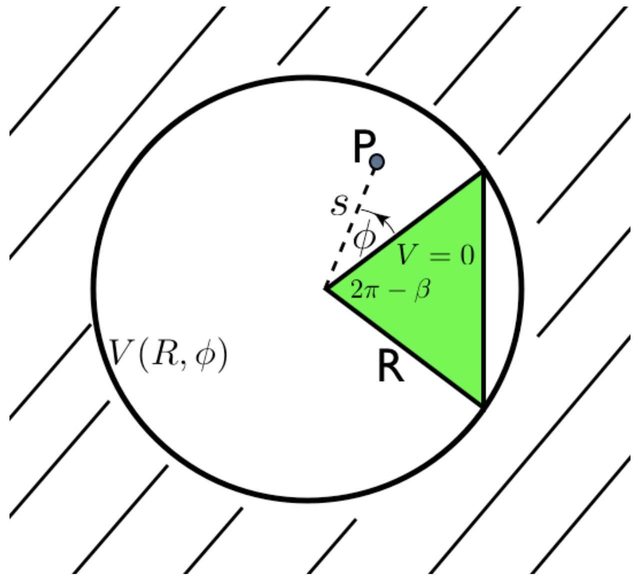
B. [**15** points] Find the total electrostatic energy inside radius b.

C. [**25** points] Next, consider the slightly modified geometry shown on the right. The copper shell that surrounds the sphere is now grounded and the inner sphere of radius a is different. It has a surface charge distribution arranged so that $\Phi(a, \theta, \phi) = V \cos^2 \theta$. Find $\Phi(r, \theta, \phi)$ for the region between the sphere and the shell; i.e., $a < r < b$.



2.15 A Conducting Wedge inside a Cylinder

A hollow cylinder of radius R has a potential $V(R,\phi)$ maintained on its surface, as shown in the figure. A conducting wedge with surfaces at $\phi = 0$ and $\phi = \beta$, and its apex at the symmetry axis of the cylinder (as shown by the shaded region), is placed inside the cylinder and held at a potential of $V = 0$ (s denotes the radial coordinate).



A. [**10** points] Derive the most general form for the potential inside the cylinder for $0 < \phi < \beta$ for a general boundary-condition $V(R,\phi)$. [1]

B. [**10** points] Determine the potential resulting from the boundary condition

$$V(R,\phi) = \overline{V}_1 \sin\left(\frac{\pi\phi}{\beta}\right) + \overline{V}_3 \sin\left(\frac{3\pi\phi}{\beta}\right) \quad ,$$

where $\overline{V}_{1,3}$ are constants.

C. [**20** points] Determine the electric field at each point inside the cylinder for $0 < \phi < \beta$, and determine the charge density on the $\phi = 0$ and $\phi = 2\pi - \beta$ surfaces of the wedge. Comment on the behavior of the electric field near the tip of the wedge as a function of the wedge apex-angle, $2\pi - \beta$.

D. [**10** points] For the situation where $\beta = \pi$ and $\overline{V}_3 = 0$, draw the electric field lines and equi-potential surfaces inside the cylinder.

[1] Possibly useful: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin\theta}\frac{\partial^2}{\partial \phi^2}$.

2.16 Energy and Momentum (1)

Maxwell's equations and the Lorentz force law are (where $\boldsymbol{D} = \epsilon \boldsymbol{E}$, $\boldsymbol{H} = \boldsymbol{B}/\mu$):

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho\,, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\tfrac{\partial}{\partial t}\boldsymbol{B}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0\,, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \tfrac{\partial}{\partial t}\boldsymbol{D}\,, \quad \boldsymbol{F} = q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\,.$$

A. [**15** points] Starting from the Lorentz force law show that the work per unit time done on all charged particles in a system is given by

$$\frac{dW}{dt} = \int d^3r\; \boldsymbol{E} \cdot \boldsymbol{J}\,,$$

where $\boldsymbol{J}$ is the current density of the charged particles.

B. [**20** points] A constant current I flows uniformly through a straight cylindrical wire of finite resistance R that has a potential difference V maintained between its two ends. The wire has length L and radius $a \ll L$. Compute the electric and magnetic fields at the surface of the wire, and compute the Poynting vector. Explicitly indicate the direction of the Poynting vector with respect to the wire.

C. [**15** points] Determine the electromagnetic energy per unit time entering the wire and compare this with naive expectations based on ohmic heating.

2.17 Energy and Momentum of Electromagnetic Fields

A. [**7** points] Starting from the Lorentz force law, show that the work per unit time done on all charged particles in a system is given by

$$\frac{dW}{dt} = \int d^3x\, \boldsymbol{E}(\boldsymbol{x}) \cdot \boldsymbol{J}(\boldsymbol{x})\,,$$

where $\boldsymbol{J}(\boldsymbol{x})$ is the current density of the charged particles.

B. [**7** points] A constant current I flows uniformly through a straight cylindrical wire of finite resistance R which has a potential difference V maintained between its two ends. The wire has length L and radius $a \ll L$. Compute the electric and magnetic fields at the surface of the wire, and compute the Poynting vector. Explicitly indicate the direction of the Poynting vector with respect to the wire.

C. [**6** points] Determine the electromagnetic energy per unit time entering the wire, and compare this with the ohmic heating.

Consider now an insulating sphere of radius a with charge Q distributed uniformly over its surface and uniform magnetization $\boldsymbol{M} = M\,\hat{\boldsymbol{e}}_z$ throughout its volume.

D. [**15** points] Calculate the angular momentum in the electromagnetic field.

E. [**15** points] Assume the magnetization is slowly brought to zero (for example, by heating up the sphere). Because of the varying magnetic field, an electric field will appear that will change the mechanical angular momentum of the sphere. Compute the net change of mechanical angular momentum and compare with the answer to the previous question. Assume that the rotation of the sphere is such that $v/c \ll 1$ for any point on the surface of the sphere.

2.18 Faraday Effect

A plane wave with frequency ω travels in a medium filled with molecules that can be regarded as simple harmonic oscillators with charge q, mass m, and spring constant κ.

A. [**20** points] Neglecting the effect of the magnetic field of the wave on the molecular oscillations, show that the index of refraction is

$$n = \sqrt{1 + \frac{Nq^2}{\epsilon_0\, m(\frac{\kappa}{m} - \omega^2)}}\,,$$

where N is the number of molecules per unit volume. [Hint: write the equations of motion for the harmonic oscillator in the presence of the electric field of the wave, and then use it to determine the polarization.]

B. [**20** points] Assume now that a static magnetic field in the direction of propagation of the wave is present in the medium. Show that the two states of circular polarization (R/L) of the wave have a different index of refraction given by

$$n = \sqrt{1 + \frac{Nq^2}{\epsilon_0\, m(\frac{\kappa}{m} - \omega^2 \pm \frac{q\boldsymbol{B}}{m})}}\,,$$

where the upper and lower signs correspond to the two states of circular polarization of the wave. As in part A, ignore the magnetic field generated by the wave.

C. [**5** points] Explain what would happen to an electro-magnetic wave initially plane-polarized as it travels through a medium as described in part B.

2.19 Gauging Away

[**30** points] Consider vector and scalar potentials given by:

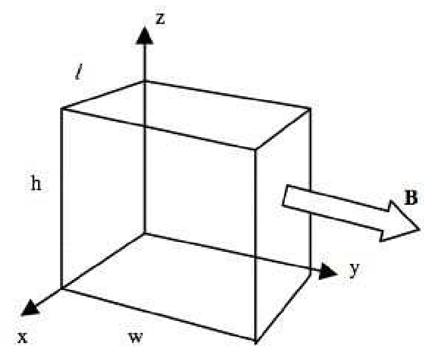
$$\boldsymbol{A}(t,\boldsymbol{r}) = -\frac{q\,t}{4\pi\epsilon_0\,|\boldsymbol{r}|^2}\,\hat{\boldsymbol{e}}_r\,, \qquad \Phi(t,\boldsymbol{r}) = 0\,.$$

A. Find the corresponding electric and magnetic fields, and the charge and current distributions.

B. Use the gauge function $\lambda(t,\boldsymbol{r}) = -qt/(4\pi\epsilon_0\,|\boldsymbol{r}|)$ to transform the potentials, and comment on the result.

2.20 Hall Probe

[**30** points] A Hall probe with dimensions as shown has conductivity σ and carries charge density ρ. The probe is placed in an unknown magnetic field B oriented in the $+y$ direction. An external potential V_{ext} is applied to two ends of the probe, producing an electric field in the $+z$ direction. Between which pair of ends is the equilibrium Hall voltage V_{Hall} observed? Derive an expression for the magnetic field B in terms of the above quantities and the dimensions of the probe.



2.21 Ionospheric Waves

The ionosphere is a plasma containing N free electrons per unit volume.

A. [**10** points] First, assume the external magnetic field is zero. By considering the motion of the electrons in the electric field of a plane wave, and neglecting the magnetic field of the wave, show that the index of refraction of the medium, as a function of the frequency ω, is

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

and determine ω_p. Explain why we can ignore the positively charged ions.

B. [**10** points] Describe in words what happens when a plane wave with frequency $\omega < \omega_p$ hits the plasma.

C. [**10** points] Now assume that the plasma is in a constant finite magnetic field $\boldsymbol{B}$ aligned with the z axis. A *circularly* polarized electromagnetic plane wave propagates along the $+z$-direction in the plasma with the electric field

$$\boldsymbol{E}^{(\sigma)}(z,t) = E_\sigma \, e^{i(k_\sigma z - \omega t)} \, (\hat{\boldsymbol{e}}_x + i\sigma \hat{\boldsymbol{e}}_y) \, ,$$

where $\sigma \equiv \pm 1$ determines the polarization of the wave. Show that the velocity of an electron in the ionosphere is $\boldsymbol{v}^{(\sigma)} = v_0^\sigma \, e^{i(k_\sigma z - \omega t)} \, (\hat{\boldsymbol{e}}_x + i\sigma \hat{\boldsymbol{e}}_y)$ with

$$v_0^\sigma = \frac{-i|e|E_\sigma}{m\,(\omega - \sigma\omega_0)} \, ,$$

where m is the electron mass, $\omega_0 \equiv |e|B/m$, and $e < 0$ is the electric charge of the electron.

D. [**10** points] Show that the refractive index for each circular polarization state may be written in the form

$$n_\sigma = \sqrt{1 - \frac{{\omega_p}^2}{\omega(\omega - \sigma\omega_0)}} \, ,$$

and define ω_p.

A linearly polarized electromagnetic plane wave is propagating in the $+z$-direction. At $z = 0$ the electric field of the wave is along the x-direction, $\boldsymbol{E}(z{=}0, t) = E_0 \, e^{-i\omega t} \, \hat{\boldsymbol{e}}_x$. Neglect the magnetic field associated with the electromagnetic wave compared to $\boldsymbol{B}$.

E. [**10** points] What is the angle between the plane of polarization of the electromagnetic wave at $z = z_0$ and $z = 0$?

2.22 Inverse Electrostatics

A static charge distribution produces a radial electric field,

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{A\, e^{-b|\boldsymbol{r}|}}{|\boldsymbol{r}|}\, \hat{\boldsymbol{e}}_r\,,$$

where A and $b > 0$ are constants.

A. [**15** points] What is the charge density? Make a sketch of the function.

B. [**5** points] What is the total charge Q?

2.23 Localized Current Density

A current distribution is confined to a region of space of size ℓ. At distances $r \gg \ell$ the magnetic field is

$$\boldsymbol{H}(\boldsymbol{r}, t) = \frac{\boldsymbol{v} \times \boldsymbol{r}}{r^3} (1 - ikr)\, e^{i(kr - \omega t)} \,,$$

with $\boldsymbol{v}$ a vector fixed in space (and $r \equiv |\boldsymbol{r}|$, $k \equiv \omega/c$).

A. [**10** points] Using words and pictures, describe the magnetic field and its associated electric field $\boldsymbol{E}(\boldsymbol{r}, t)$.

B. [**5** points] What type of current distribution might give rise to such fields?

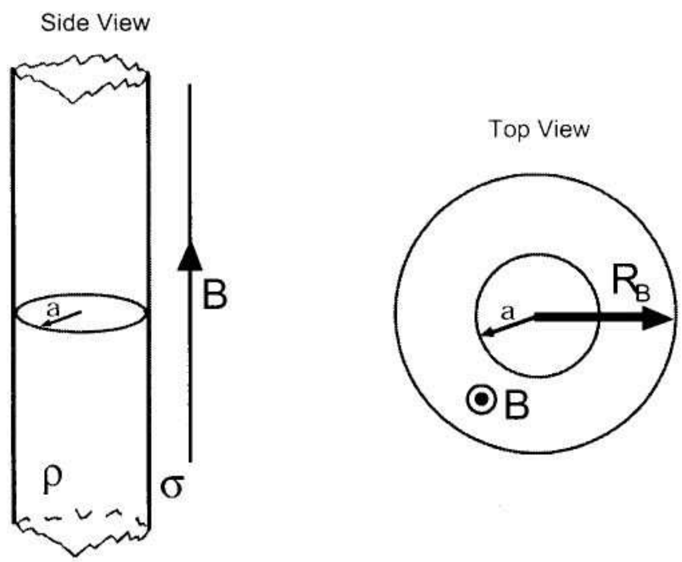
C. [**15** points] In the region of space where $r \gg \ell$ but $kr \ll 1$, determine the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ as completely as possible.

D. [**15** points] In the region of space $kr \gg 1$, determine the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ as completely as possible.

E. [**10** points] Determine the rate of energy flow away from the region containing the current source.

2.24 Magnetic Torque

An infinitely long cylinder of insulating material with radius a, permeability $\mu = \mu_0$ and permittivity $\epsilon = \epsilon_0$ has a uniform volume charge density $\rho > 0$, a surface charge density σ, and is electrically neutral (vanishing total charge per unit length). This cylinder is placed in a uniform magnetic field $\boldsymbol{B} = B\,\hat{\boldsymbol{e}}_z$ filling a cylindrically symmetric volume of radius R_{B} ($R_{\mathrm{B}} > a$) and of infinite extent in the z-direction. The symmetry axes of the insulating cylinder and the magnetic field coincide, as shown in the figure. The cylinder is free to rotate about its symmetry axis.



A. [**6** points] Find the electric field $\boldsymbol{E}$ everywhere.

B. [**9** points] Compute the Poynting vector and momentum density everywhere. Find the angular momentum per unit length of the system, $d\boldsymbol{L}/dz$, with the cylinder at rest.

C. [**10** points] The magnetic field is now turned off with time dependence $\boldsymbol{B}(t)$. Determine the electric field induced by the time-varying magnetic field.

D. [**10** points] Find the torque per unit length, $d\boldsymbol{\tau}/dz$, on the cylinder as a function of time and determine the mechanical angular momentum per unit length of the cylinder as a function of time. Assume that any velocity at any point in the charge distribution is small, so relativistic effects can be neglected.

2.25 Magnetostatics

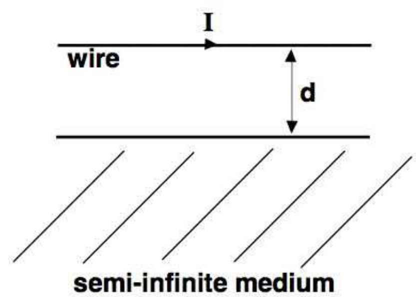
A. [**6** points] Write Maxwells equations for $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{B}$, in the presence of a free charge density ρ and free current density $\boldsymbol{j}$.

B. [**5** points] Define the field $\boldsymbol{H}$ in terms of the magnetic field $\boldsymbol{B}$, the magnetization $\boldsymbol{M}$, and the permeability of free space.

C. [**14** points] Consider an infinite straight wire, with negligible radius, in empty space carrying a constant current I, located at $x = y = 0$ and running in the z-direction. State Ampere's law and use it to find the magnetic field $\boldsymbol{B}$ a distance r from the wire. Show that everywhere except at the wire, the field $\boldsymbol{H}$ can be written in terms of a magnetic scalar potential, $\boldsymbol{H} = -\nabla\psi$, where (up to an additive constant)

$$\psi(\boldsymbol{x}) = -\frac{I}{2\pi}\,\mathrm{Im}\left[\log(x+iy)\right].$$

A long, thin straight wire carrying current I is placed a distance d above a semi-infinite magnetic medium of permeability μ.



D. [**10** points] Given that there are no free chagres or currents at the boundary of the magnetic medium, write down the relation between components of $\boldsymbol{B}$ just above the boundary and the components of $\boldsymbol{B}$ just below the boundary. Write this relation in terms of the magnetic scalar potential just above the boundary, $\psi_>$, and just below the boundary, $\psi_<$.

E. [**15** points] Find the force per unit length (including direction) on the wire using the method of images.

2.26 Maxwell's Equations

A. [**5** points] Write the differential form of Maxwell's equations describing electric and magnetic fields in the presence of arbitrary charge and current distributions.

B. [**5** points] How are the electric and magnetic fields related to the scalar potential $\Phi(\boldsymbol{r},t)$ and vector potential $\boldsymbol{A}(\boldsymbol{r},t)$?

C. [**5** points] If a given system is time-reversed, *i.e.*, $t \to -t$, what happens to the charge density, the current density, the electric field, and the magnetic field?

D. [**5** points] If a given system is spatially-inverted, *i.e.*, $\boldsymbol{x} \to -\boldsymbol{x}$, what happens to the charge density, the current density, the electric field, and the magnetic field?

Consider a system with vector and scalar potentials of the form

$$\boldsymbol{A}(x,y,z,t) = \frac{B_0}{a}\, xy\, \hat{\boldsymbol{e}}_z\,, \qquad \Phi(x,y,z,t) = 0\,,$$

where B_0 and a are time-independent constants.

E. [**6** points] Compute the electric and magnetic fields of this system, and sketch representative field lines in the $z=0$ plane.

F. [**6** points] Do the electric and magnetic fields satisfy the free-space Maxwell's equations?

G. [**8** points] An observer moves with velocity $\boldsymbol{v} = v\,\hat{\boldsymbol{e}}_z$ in the z-direction along a line passing through the point (x,y) in the xy-plane. What is the difference between the scalar potential at the point (x,y) and the point $(0,0)$ in the z=0 plane, as measured by the moving observer?

2.27 Maxwell Stress Tensor

[**40** points] Recall the definition of the Maxwell stress tensor,

$$T_{ij} = \epsilon_0 \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2 \right) + \mu_0 \left(B_i B_j - \tfrac{1}{2}\delta_{ij} B^2 \right) .$$

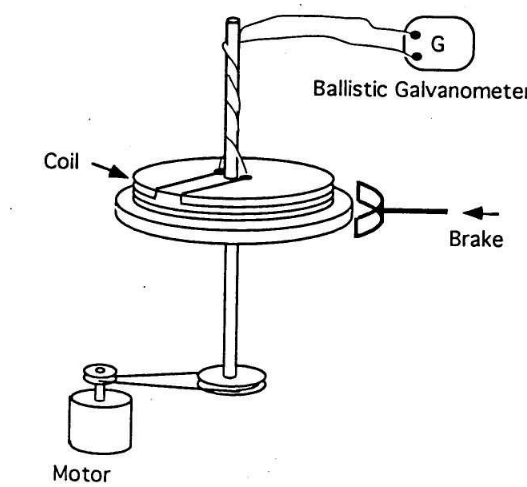
A. Show that the trace of the stress tensor is a scalar, and identify the corresponding invariant in this case.

B. Show that the total force exerted on charges in a volume V is, in the static case, given by

$$F_i = \oint_S T_{ij} \, d\Sigma_j \, ,$$

were the surface S is the boundary of the volume V, and $d\mathbf{\Sigma}$ is the outward directed surface area element. (There is an implied sum over repeated indices). It is sufficient to show this when there is a single point charge within the volume V.

2.28 Metallic Conductors



[**40** points] In the early years of this century, Richard Tolman conducted a series of experiments designed to determine the properties of the charge carriers that make electric current in liquid electrolytes and in conducting metals. He (and others) showed that the charge carriers in liquids were heavier than a hydrogen atom. The experiment in metallic conductors proved more difficult. The apparatus used in his first successful experiments is shown schematically in the figure. A coil of copper wire wound about a non-conducting disk was rotated rapidly about the axle by means of an electric motor. After the disk reached a steady speed, the motor was disconnected from the axle and brought to rest. With the disk now spinning at a known rate, a brake was applied to the edge of the disk bringing it rapidly to rest. A current pulse was detected with the ballistic galvanometer. (The thin wires connecting the ends of the copper coil to the galvanometer were allowed to twist about the axle.)

A. Using a simplified model in which the conductor is isolated from the external circuit, write the equation of motion for the "free" charge carriers during the very short stopping time. You may neglect collisions which turn out to have a very small effect in this experiment.

B. Using a result of part A, derive an expression for the charge to mass ratio of the charge carriers in terms of the angular velocity ω of the disk, the disk radius r, and the length L of the copper coil, and the charge Q measured by the ballistic galvanometer, and whatever properties of the galvanometer circuit you need. (Neglect effects due to the Earth's or other sources of extraneous electromagnetic fields.)

C. The experimenters found it necessary to surround the apparatus with compensating coils to cancel both vertical and horizontal components of the Earth's magnetic field. Why? Assume that the axle is vertical, the coil is perpendicular to the axle, and it does not "wobble".

2.29 Metallic Reflection

Consider ultraviolet-rays with angular frequency ω incident on a plane metal surface. The density of conduction electrons in the metal is n_e.

A. [**5** points] Comment on why metals are shiny (reflective to visible light), and the difference in color between silver and gold or copper.

B. [**25** points] Determine the critical angle of incidence θ_c beyond which the incident rays are totally reflected.

C. [**10** points] For normal incidence, what is the reflection coefficient?

2.30 Multipoles and Capacitance

A. [**10** points] In a static system, the energy stored in the electric field is

$$W \;=\; \frac{\varepsilon_0}{2} \int d^3r \; |E(r)|^2 \quad .$$

Show that this can be written as

$$W \;=\; \frac{1}{2} \int d^3r \; \rho(r) \; \Phi(r) \quad .$$

B. [**10** points] By performing a multipole expansion of the potential, $\Phi(r)$, far from a localized charge distribution, find the contributions to $\Phi(r)$ from the charge, electric dipole moment and electric quadrupole moments of the distribution.

C. [**10** points] Consider a localized charge distribution that is axially-symmetric about the z-axis and for which all electrostatic moments vanish except for the quadrupole moments. This charge distribution has a quadrupole moment Q_{zz} and is centered at the origin of a Cartesian coordinate system. A point charge, q_0, is located far from the distribution at $r = (r, \theta, \phi)$ in spherical coordinates, where θ is the angle measured from the z-axis. The force on the point charge can be written as

$$F \;=\; \alpha \; \hat{e}_r \;+\; \beta \; \hat{e}_z \quad ,$$

where $\hat{e}_r$ is the unit vector pointing toward the charge and $\hat{e}_z$ is the unit vector along the z-axis. What are α and β in terms of r, θ and ϕ?

D. [**20** points] Consider an isolated system comprised of two spherical conducting shells of radii a_1 and a_2, separated by a distance $d \gg a_{1,2}$. Show that if shell-2 is grounded, and shell-1 is held at a potential V_1, the charge on shell-1 is approximately

$$q_1 \;=\; 4\pi\varepsilon_0 \; V_1 \; \frac{a_1 d^2}{d^2 - a_1 a_2} \quad .$$

Find the charge on shell-2. Discuss the corrections to these expressions.

2.31 Planar Waveguide

[**40** points] A waveguide is formed by two infinite parallel perfectly conducting planes separated by a distance a. Choose the z-direction to be normal to the planes. Consider the guided plane wave modes in which the field strengths are independent of the y coordinate. For a given wavelength λ, find the allowed frequencies ω. For each such mode, find the phase velocity v_p and group velocity v_g.

2.32 Plasma Dispersion

[**40** points] A beam of plane-polarized electromagnetic radiation with (angular) frequency ω and electric field amplitude E_0 is normally incident on a region of space containing a low density, neutral plasma containing n_0 electrons/unit volume.

A. Determine the frequency-dependent conductivity, $\sigma(\omega)$.

B. Show that the dispersion relation for electromagnetic radiation has the form, $\omega^2 = k^2 c^2 + {\omega_{\rm pl}}^2$, for some plasma frequency $\omega_{\rm pl}$. What is the (frequency-dependent) index of refraction?

C. A pulsar emits a pulse of broadband electromagnetic radiation which is 1 ms in duration. The pulse propagates 1000 light years ($\approx 10^{19}$ m) through interstellar space to reach radio astronomers on Earth. Assume that the interstellar medium contains a low density plasma with plasma frequency $\omega_{\rm pl} = 5000\ {\rm s}^{-1}$. Estimate the difference in measured pulse arrival times for radio telescopes operating at 400 MHz and 1000 MHz.

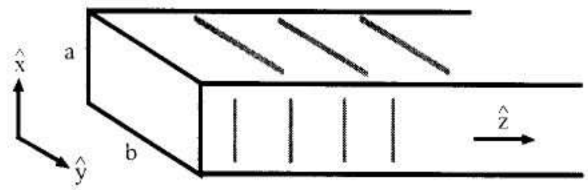
2.33 Point Charge and Conducting Sphere

[**30** points] Consider an isolated spherical conductor of radius a, together with a point charge q located a distance $r > a$ from the center of the conductor.

A. Show that the effect of the conductor is equivalent to the presence of two point charges: $+qa/r$ at the center of the conductor, and $-qa/r$ at a distance a^2/r along the line from the center of the conductor to the outside point charge.

B. What is the smallest positive charge that can be applied to the sphere which will cause the resulting surface charge density on the sphere to be everywhere positive?

2.34 Rectangular Waveguide



A waveguide is formed from a rectangular cavity inside a perfect conductor. The cavity has sides of length a and b in the x and y directions and has infinite extent in the z-direction. Further, the cavity is filled with a linear, homogeneous dielectric with permeability $\mu = \mu_0$ and permittivity ε. A *transverse magnetic* (TM) traveling wave exists in the wave guide of the form

$$\boldsymbol{E}(x, y, z, t) = [E_{0x}(x, y)\,\hat{\boldsymbol{e}}_x + E_{0y}(x, y)\,\hat{\boldsymbol{e}}_y + E_{0z}(x, y)\,\hat{\boldsymbol{e}}_z]\; e^{i(kz-\omega t)}\,.$$

A. [**10** points] Write down Maxwell's equations for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields inside the dielectric-filled waveguide. Specify the boundary conditions that $\boldsymbol{E}$ and $\boldsymbol{B}$ must satisfy at the conductor-dielectric interface.

B. [**10** points] Use Maxwell's equations to write E_{0x}, E_{0y}, B_{0x}, and B_{0y} in terms of E_{0z}.

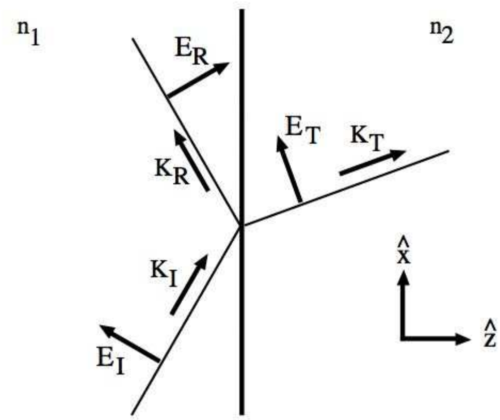
C. [**10** points] Find the second-order partial differential equation that $E_{0z}(x, y)$ satisfies in the wave guide.

D. [**10** points] Solve the equation in part C to find solutions for $E_{0z}(x, y)$ in the wave guide which satisfy the boundary conditions of part A.

E. [**10** points] Find the cut-off frequency of the TM_{mn} mode.

2.35 Reflection and Refraction

A monochromatic plane-wave polarized in the plane of incidence is incident upon a boundary between two linear, homogeneous media at an angle θ_I to the normal. Medium 1 has refractive index n_1 and medium 2 has refractive index n_2. Both media have permeabilities equal to that of the vacuum, $\mu_1 = \mu_2 = \mu_0$.



A. [**7** points] Write down the boundary conditions that relate arbitrary electric and magnetic fields on either side of the boundary.

B. [**8** points] Write down general expressions for the incident, reflected and transmitted electric and magnetic fields. Indicate all spatial and time dependences.

C. [**8** points] Derive relations between the reflected scattering angle θ_R and the incident angle θ_I, and between the transmitted angle θ_T and the incident angle θ_I.

D. [**9** points] Relate the reflected electric field to the incident electric field.

E. [**9** points] Find an expression for Brewsters angle, θ_B, the incident angle for which in-plane polarized light is perfectly transmitted. What is θ_B for media with $n_1 = 1$ and $n_2 = 2.4$?

F. [**9** points] Find the reflection and transmission coefficients.

2.36 Retarded Fields

[**60** points] The "retarded fields" produced by a point charge moving with arbitrary velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ can be written in the form

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{e/(4\pi\epsilon_0)}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}}_{\rm ret})^3}\left\{\frac{\hat{\boldsymbol{n}}_{\rm ret}-\boldsymbol{\beta}}{\gamma^2\,|\boldsymbol{R}_{\rm ret}|^2} + \frac{\hat{\boldsymbol{n}}_{\rm ret}\times[(\hat{\boldsymbol{n}}_{\rm ret}-\boldsymbol{\beta})\times\boldsymbol{a}]}{c^2|\boldsymbol{R}_{\rm ret}|}\right\}, \qquad \boldsymbol{B}(\boldsymbol{r}) = \hat{\boldsymbol{n}}_{\rm ret}\times\frac{\boldsymbol{E}(\boldsymbol{r})}{c}\,.$$

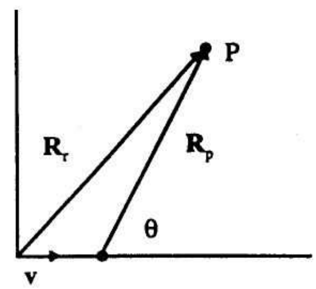
Here $\boldsymbol{R}_{\rm ret} \equiv \boldsymbol{r} - \boldsymbol{r}_{\rm ret}$ is the displacement vector of the point $\boldsymbol{r}$, where the field is evaluated, from the position $\boldsymbol{r}_{\rm ret}$ of the charge at retarded time $t' \equiv t - |\boldsymbol{r}-\boldsymbol{r}_{\rm ret}|/c$. The unit vector $\hat{\boldsymbol{n}}_{\rm ret} \equiv \boldsymbol{R}_{\rm ret}/|\boldsymbol{R}_{\rm ret}|$, the velocity, and the acceleration are evaluated at the retarded time t' and, as usual, $\boldsymbol{\beta} \equiv \boldsymbol{v}/c$ and $\gamma \equiv 1/\sqrt{1-\boldsymbol{\beta}^2}$.

A. Consider a charged particle moving with constant velocity $\boldsymbol{v}$.

i. Show from the expressions above that the electric field produced by the particle is radial from the *present* position (not the retarded position) of the particle, *i.e.*,

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{e\,\hat{\boldsymbol{n}}_p}{4\pi\epsilon_0\,\gamma^2\,\boldsymbol{R}_p{}^2(1-\beta^2\sin\theta^2)^{3/2}}\,,$$

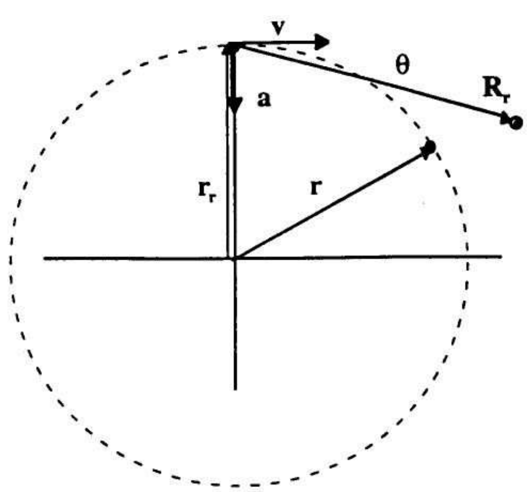
where $\boldsymbol{R}_p$ is the displacement of the point $\boldsymbol{r}$ from the present position of the charge, θ is the angle between $\boldsymbol{R}_p$ and $\boldsymbol{v}$, and $\hat{\boldsymbol{n}}_p \equiv \boldsymbol{R}_p/|\boldsymbol{R}_p|$.



ii. How does the electric field for a relativistic particle differ from that of a slowly moving particle?

iii. What is the intensity of the radiation emitted by this charge if it is moving with relativistic speed?

B. Now suppose the charged particle moves in a circular path at constant speed. Derive an expression for the radiation intensity emitted by this particle *in the plane of the orbit.*



2.37 Retarded Potentials

The retarded potentials

$$\Phi(\boldsymbol{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\,\frac{\rho(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,, \qquad \boldsymbol{A}(\boldsymbol{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\boldsymbol{J}(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|}\,, \tag{1}$$

with $t_r \equiv t - |\boldsymbol{x}-\boldsymbol{x}'|/c$, are solutions to the (Lorentz gauge) wave equations:

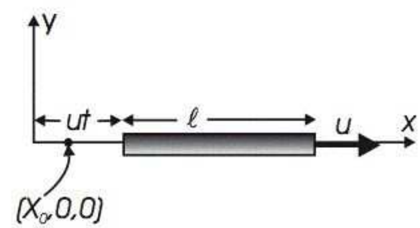
$$\Box\Phi(\boldsymbol{x},t) = -\rho(\boldsymbol{x},t)/\epsilon_0\,, \qquad \Box\boldsymbol{A}(\boldsymbol{x},t) = -\mu_0\,\boldsymbol{J}(\boldsymbol{x},t)\,,$$

(with $\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$). In the simple case of a point particle with charge q moving with constant velocity $\boldsymbol{v}$, Eq. (1) yields the (Lienard-Wiechert) potentials,

$$\Phi(\boldsymbol{x},t) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\boldsymbol{r}| - \boldsymbol{r}\cdot\boldsymbol{v}/c}\,, \qquad \boldsymbol{A}(\boldsymbol{x},t) = \frac{\boldsymbol{v}}{c^2}\,\Phi(\boldsymbol{x},t)\,. \tag{2}$$

where $\boldsymbol{r}$ is the displacement vector from the retarded position of the charge to the point where the potential is evaluated.

A bar is moving at constant speed $\boldsymbol{u} = u\,\hat{\boldsymbol{e}}_x$, as shown in the figure. The length of the bar, in the lab frame, is ℓ. The bar has negligible diameter, and has charge Q uniformly distributed over its length.



A. [**15** points] Use Eq. (1) to calculate the potentials at point $P = (x_0, 0, 0)$ as a function of time. Assume the bar is to the right of the point P, as shown in the figure.

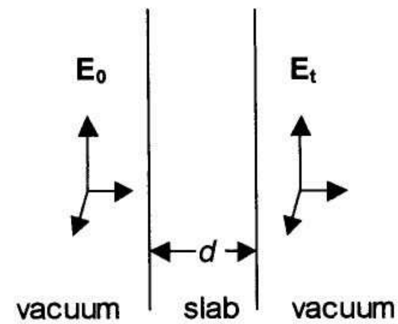
B. [**10** points] Show that your expressions reduce to the Lienard-Wiechert potentials of Eq. (2) in the limit $\ell \to 0$ with u/c fixed.

C. [**15** points] In the $\ell \to 0$ limit, calculate the electric and magnetic fields as a function of time at the origin. Start either with the potentials you found in part B, or with Eq. (2).

D. [**10** points] Re-calculate the fields of part C, starting with Coulomb's law in the rest frame of the charge and then applying the appropriate Lorentz transformation.

2.38 Slab Transmission

A plane electromagnetic wave with wavelength λ is normally incident on an infinite slab of non-conducting material with thickness d, constant permittivity ε, and permeability μ. The incident electric field amplitude is $\boldsymbol{E}_0$, and the transmitted electric field amplitude is $\boldsymbol{E}_{\mathrm{t}}$. The reflected wave is not shown. There is vacuum on either side of the slab.



A. [**5** points] State carefully the boundary conditions satisfied by the fields at the interface between material and vacuum.

B. [**20** points] Find the equations relating incident, transmitted and reflected waves, and use them to calculate the transmission coefficient $T \equiv |\boldsymbol{E}_{\mathrm{t}}|^2/|\boldsymbol{E}_0|^2$.

C. [**10** points] For what particular ratios of slab thickness to wavelength in the slab will maximum transmission occur?

2.39 Superconductors

A superconductor can be regarded as a perfect conductor, which is at the same time a perfect diagmagnetic material with zero magnetic permeability $\mu = 0$.

A. [**10** points] Consider a superconductor in contact with vacuum. Formulate the boundary conditions for the electric and magnetic fields, $\boldsymbol{E}$, and $\boldsymbol{B}$, on the surface the superconductor.

B. [**10** points] Consider a uniform magnetic field. Insert a superconducting sphere with radius R into this field. Determine the magnetic field outside the sphere (in any system of coordinates).

Consider a planar waveguide of two parallel superconducting slabs, separated by a distance d. The space between the two slabs is vacuum.

C. [**10** points] Are the modes of this waveguide the same or different than the modes in the waveguide where the superconductors are replaced by conductors with infinite conductivity, but with the magnetic permeability of vacuum? Explain your answer in words.

D. [**20** points] Find the velocity of propagation of the mode with the lowest frequency at given wave number q.

2.40 Small Loop, Big Loop

A closed circular loop of thin wire of radius a, lying in the xy-plane and centered at the origin, carries a clockwise persistent current I when viewed down the z-axis toward the ring.

A. [**10** points] Using words and pictures, describe the magnetic field due to the loop.

B. [**15** points] Give explicit expressions for the magnetic field $\boldsymbol{H}(z)$ everywhere on the z-axis, and also find the leading asymptotic behavior of $\boldsymbol{H}(\boldsymbol{r})$ at large distance, $|\boldsymbol{r}| \gg a$.

C. [**20** points] A second wire loop of radius $b \gg a$ and electrical resistance R is placed, concentric and coplanar, around the small loop described above. The small loop is then flipped $180°$ about the x-axis, causing a momentary current to flow in the large loop. Determine the direction of this current and how much charge it carries during the flipping.

2.41 Time Dependent Fields (1)

The scalar and vector potentials produced by an arbitrary localized source may be expressed as

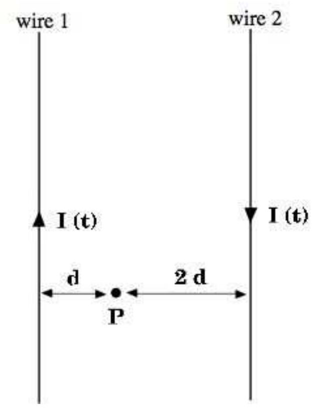
$$\Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} \int d\tau \, \frac{\rho(\boldsymbol{r}', t_r)}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int d\tau \, \frac{\boldsymbol{j}(\boldsymbol{r}', t_r)}{|\boldsymbol{r} - \boldsymbol{r}'|}.$$

A. [**10** points] Carefully define the quantities μ_0, ϵ_0, ρ, $\boldsymbol{j}$, $d\tau$, $\boldsymbol{r}'$, and t_r.

Two infinite, parallel, straight wires (wire 1 and wire 2) separated by a distance $3d$, run parallel to the z-axis. One carries a current $I(t)$ in the $+z$-direction, while the other carries current $I(t)$ in the $-z$-direction, where

$$I(t) = q_0 \, \delta(t),$$

(with $\delta(t)$ a Dirac delta function). Point P lies in the plane of the wires and in the $z = 0$ plane. It is located a distance d from wire 1, and a distance $2d$ from wire 2, as illustrated at right.



B. [**8** points] If an event occurs on wire 1 at a distance z along the z-axis above the point P, how much time must pass before this event can influence physics at point P? What is this time difference for an event on wire 2 with the same z-coordinate as the event on wire 1?

C. [**18** points] Determine the vector potential $\boldsymbol{A}(t)$ and scalar potential $\Phi(t)$ at the point P, due to the current $I(t)$, as a function of time.

D. [**12** points] Determine the leading behavior of the electric field at the point P for large times, $t \gg 2d/c$.

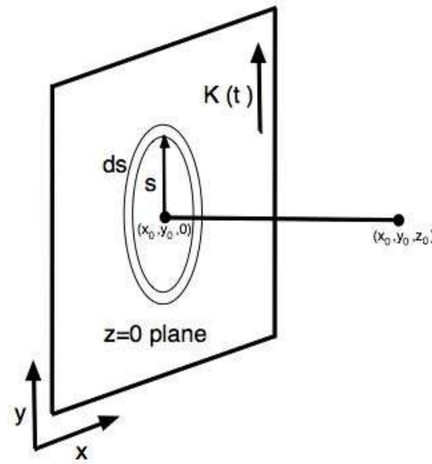
2.42 Time Dependent Fields (2)

The scalar and vector potentials resulting from an arbitrary localized source may be expressed as

$$\Phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,, \qquad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\boldsymbol{j}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,.$$

A. [**5** points] Define ρ, $\boldsymbol{j}$, $\boldsymbol{r}$, and $\boldsymbol{r}'$. Define t_r in terms of $\boldsymbol{r}$, $\boldsymbol{r}'$, t, and the speed of light c.

After time $t = 0$, a time-dependent, uniform current density $\boldsymbol{K}(t)$ flows in the infinite xy-plane at $z = 0$, so $\boldsymbol{K}(t) = \hat{\boldsymbol{e}}_y\,K_0(t)\,\delta(z)$, with $\delta(z)$ a Dirac delta function and $K_0(t)$ vanishing for $t < 0$. There are no free charges on the sheet, or anywhere in space, at any time.



B. [**10** points] For a point located at (x_0, y_0, z_0), give the contributions to $\Phi(x_0, y_0, z_0, t)$ and $\boldsymbol{A}(x_0, y_0, z_0, t)$ from the current in a ring of radius s and width ds in the xy-plane centered on $x = x_0$, $y = y_0$, $z = 0$ (as shown in the figure).

C. [**5** points] At any given time t, what is the maximum value of s that can contribute to $\Phi(x_0, y_0, z_0, t)$ and $\boldsymbol{A}(x_0, y_0, z_0, t)$?

D. [**15** points] Show that the vector potential may be expressed in the form

$$\boldsymbol{A}(x_0, y_0, z_0, t) = \hat{\boldsymbol{e}}_y\,\Theta(t - z_0/c)\int_0^{t-z_0/c} d\eta\, f(\eta)\,,$$

where $\Theta(x)$ is a unit step-function. Find $f(\eta)$.

E. [**7** points] Find the electric and magnetic fields for $z > 0$.

F. [**8** points] What is the intensity of radiation emitted by this sheet at a distance d from the sheet as a function of time?

2.43 Transverse Waves

Consider electromagnetic waves in free space of the form

$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}_0(x,y)\, e^{i(kz-\omega t)}\,,$$
$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{B}_0(x,y)\, e^{i(kz-\omega t)}\,,$$

were $\boldsymbol{E}_0(x,y)$ and $\boldsymbol{B}_0(x,y)$ are orthogonal to $\hat{\boldsymbol{e}}_z$.

A. [**20** points] Using Maxwell's equations, derive the relation between k and ω, as well as the relation between $\boldsymbol{E}_0(x,y)$ and $\boldsymbol{B}_0(x,y)$. Show that they satisfy the equations for *static* electric and magnetic fields in free space.

B. [**10** points] Derive the boundary conditions for $\boldsymbol{E}$ and $\boldsymbol{B}$ on the surface of a perfect conductor, taking the surface charges and currents into account.

XXX

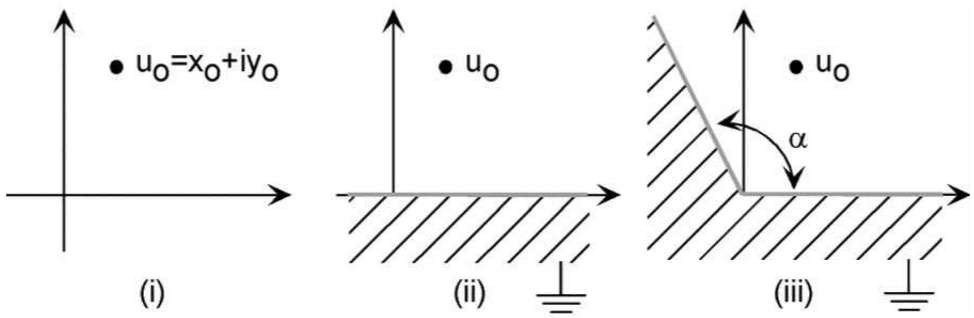
C. [**10** points] Consider a wave of the above type propagating along a coaxial cable, a cross section of which is shown at right. Assume that the central conductor and outer sheath are perfect conductors. Make a sketch of the electromagnetic field pattern, within the cross-section, for non-zero fields. Take the z-axis to point out of the page. Indicate the signs of the charges, the field directions, and the directions of the currents. Justify in words each of your choices on the sketch.

D. [**10** points] For a particular value of z, derive expressions for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the local charge per unit length λ and the current i in the central conductor.

2.44 Two-dimensional Electrostatics

A. [**6** points] Consider a uniformly charged line with charge density λ aligned parallel to the z-axis and passing through the point $(x_0, y_0, 0)$. Calculate the electric potential $\phi(x, y)$ for all points in the xy-plane.

B. Three-dimensional problems with continuous translational symmetry along one direction can be treated as two-dimensional. In such cases, methods from complex analysis are often helpful. Let the position of the line charge from part A be denoted as $u_0 = x_0 + iy_0$ (see Figure (i)).



i. [**6** points] Prove that the complex potential $w(u)$ in the xy-plane is

$$w(u) = -\frac{\lambda}{2\pi\epsilon_0} \ln \frac{u - u_0}{R},$$

where R is an arbitrary real constant. What is the relationship between $w(u)$ and $\phi(x, y)$? What basic properties must $w(u)$ satisfy?

ii. [**6** points] Find $w(u)$ in the upper half-plane, Im $u > 0$, for the same line charge, if the lower half-plane is filled with a grounded conductor (see Figure (ii)).

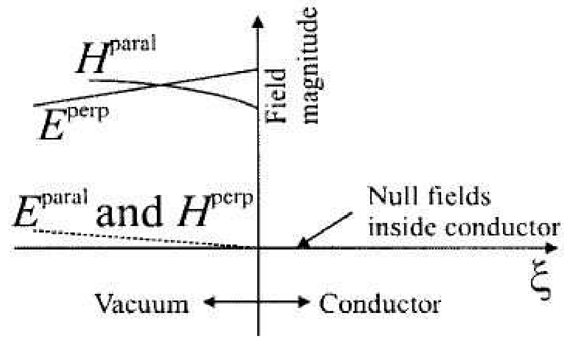
Now suppose that all space outside of a wedge of angle α is filled with a grounded conductor (see Figure (iii)).

iii. [**6** points] Explain why, for arbitrary values of α and u_0, the method of images is not helpful for this problem. Give a specific example for which the method of images would lead to great computational difficulties. Explain.

iv. [**6** points] What complex function $f(u)$ will map the flat boundary depicted in Figure (ii) onto the bent boundary in Figure (iii)? Find $w(u)$ in the region outside the conductor. If you cannot determine the function $f(u)$ then describe how you would calculate $w(u)$ if $f(u)$ were known.

2.45 Waveguides

A. [**25** points] Consider the fields near the surface of a conductor. If the conductor is perfect (*i.e.*, its conductivity $\sigma \to \infty$), then the fields will show behavior like that sketched at the right.



Now assume the conductor to be a material with magnetic permeability μ and large, but finite, conductivity σ.

i. [**20** points] Show that, if one neglects the displacement current and the variations of the fields in the direction parallel to the surface, the fields in the conductor should satisfy

$$\boldsymbol{E} \approx \frac{1}{\sigma}\,\boldsymbol{n}_\xi \times \frac{\partial \boldsymbol{H}}{\partial \xi}\,, \qquad \boldsymbol{H} \approx \frac{-i}{\mu\omega}\,\boldsymbol{n}_\xi \times \frac{\partial \boldsymbol{E}}{\partial \xi}\,,$$

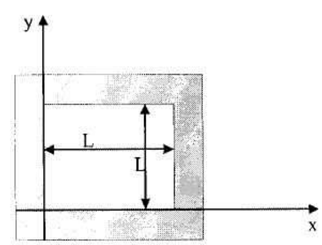
where ξ is a coordinate normal to the surface which vanishes at the surface and increases *inward* into the conductor, and $\boldsymbol{n}_\xi = \nabla\xi$ is a unit normal pointing into the conductor. Next, show that a solution of the above relations is given by

$$\boldsymbol{H} = \boldsymbol{H}_0\, e^{-(1-i)\xi/\delta}\,,$$

with $\boldsymbol{n}_\xi \cdot \boldsymbol{H}_0 = 0$ and $\delta \equiv C/\sqrt{\omega\sigma}$, with C some constant.

ii. [**5** points] Make a sketch of the magnitude of the fields inside the conductor, similar to the figure above, for the case when σ is high but not infinite.

B. [**25** points] Consider now a square wave guide with inner dimensions $L\times L$, as shown in the figure. The inside of the guide is under vacuum.



i. [**10** points] Assume the guide to be made of a perfect conductor and express the boundary conditions to compute $\boldsymbol{E}$ and $\boldsymbol{B}$ and give all solutions for $E_z(x,y,z,t)$ such that $\hat{\boldsymbol{e}}_z \cdot \boldsymbol{B} = 0$. Assume the waves are traveling in the positive z direction with angular frequency ω.

ii. [**15** points] Now treat the conductor as having large but finite conductivity. For the modes you just found, calculate the Poynting vector in the conductor and show that this implies that there is attenuation and that the magnitudes of the fields in the vacuum *inside* the waveguide will decrease as

$$|\boldsymbol{H}(z)| = |\boldsymbol{H}(0)|\,\exp\left(-\frac{f(\omega)z}{\sqrt{\sigma}}\right),$$

where $f(\omega)$ is independent of σ.

Chapter 3

Quantum Mechanics

3.1 Angular Momentum & Angular Distributions

Particle A, whose spin J is less than 2, decays into two identical spin-1/2 particles of type B.

A. [**14** points] What are the allowed values of the orbital angular momentum $\boldsymbol{L}$, the combined spin $\boldsymbol{S} = \boldsymbol{s}_1 + \boldsymbol{s}_2$ (where $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the spin vectors of the B particles), and the total angular momentum $\boldsymbol{J}$ of the decay products? Give the results from two cases: first if particle A has odd parity and then if A has even parity.

B. [**6** points] Now assume that particle A has even parity and is at rest. Suppose that you observe that one of the B particles from a given decay has its spin in the x-direction. What would a second observer find when they measure the x-component of the spin of the other B particle? What would they find if they measure the z-component of the other B particle's spin?

A negative parity spin-1 particle C decays into a spinless particle α and a spin-1 particle δ. Both α and δ have even parity, and the decay process conserves parity.

C. [**5** points] What is the angular distribution $W(\theta, \phi)$ of δ particles from an unpolarized sample of C's? (θ and ϕ are the usual polar and azimuthal angles relative to the z-axis.)

D. [**12** points] Spin polarized C particles are prepared at rest with spin projections along $\hat{z}$ of $+1$. Find the angular distribution $W(\theta, \phi)$ of the resulting δ particles.

E. [**8** points] Find the expectation value for the z-component of the spin of the δ particles emitted in the decay of these spin-polarized C particles.

3.2 Angular Momentum and Measurements

A. [**10** points] Show that a state with position wavefunction $(y - iz)^k$ is an eigenstate of the orbital angular momentum operator $\hat{L}_x$, and find its eigenvalue.

B. [**10** points] A particle moves in a three-dimensional central potential with a wavefunction of the form

$$\Psi(r, \theta, \phi) = R(r) \left[\tfrac{1}{2} \cos(4\phi) \;+\; 2\cos(2\phi)\right] e^{-2i\phi} \,. \tag{$*$}$$

A measurement of $\hat{L}_z$ performed on this system observes a value of $-6\hbar$, and a subsequent measurement of $\hat{L}_z$ is later performed on the same system. What are the possible outcomes of this second measurement and what are the probabilities of the possible outcomes?

C. [**10** points] Another system is prepared with the wavefunction in Eq. ($*$). A measurement of $\hat{L}_z$ is performed on this system. What are the possible outcomes of this measurement, and what are the probabilities of each possible outcome? What is the expectation value of $\hat{L}_z$, and what is its standard deviation?

3.3 Atom in a Harmonic Trap

An atom of mass M has a ground state with energy E_0 and a first-excited state with energy $E_1 = E_0 + \hbar\omega_0$. The atom is placed in a harmonic potential, $V(x, y, z) = \frac{1}{2}M\Omega^2(x^2 + y^2 + z^2)$. Assume that $\omega_0 \gg \Omega$, implying that the motion of the atom as a whole in the potential, and the dynamics of electronic degrees of freedom, are not strongly coupled. For times $t < 0$, the atom is in the ground electronic state, and the ground state of the atomic center-of-mass motion. Starting at time $t = 0$, an electromagnetic plane wave propagating in the z-direction with wave number $\boldsymbol{k}$ is incident on the atom. The interaction is described by:

$$\hat{H}_{\text{int}} = \epsilon\hat{F}(k)\left[e^{i(k\hat{Z}-\omega t)} + e^{-i(k\hat{Z}-\omega t)}\right],$$

where $\epsilon \ll \hbar\Omega$ has units of energy, $\hat{F}(k)$ is an operator that acts only on the internal electronic coordinates, and $\hat{Z}$ is the position operator which measures the z-coordinate of the atomic center of mass.

A. [**10** points] Show that at some much later time t, to lowest order in ϵ, the probability for finding the system in the state of oscillation $|0, 0, n\rangle$, with energy $(n + \frac{3}{2})\hbar\Omega$, and an excited electronic state with energy E_1, is maximal when $\omega = \pm(\omega_0 + n\Omega)$.

B. [**11** points] Define conventional lowering and raising operators,

$$a_Z = \left(\hat{Z} + \frac{i\,\hat{p}}{M\Omega}\right)\sqrt{\frac{M\Omega}{2\hbar}}, \qquad a_Z^\dagger = \left(\hat{Z} - \frac{i\,\hat{p}}{M\Omega}\right)\sqrt{\frac{M\Omega}{2\hbar}},$$

where $\hat{p}_Z$ is the momentum operator conjugate to $\hat{Z}$. Calculate the commutator $[a_Z, a_Z^\dagger]$, and express the matrix element $C(k) \equiv \langle 0, 0, n|e^{ik\hat{Z}}|0, 0, 0\rangle$ in terms of $a_Z, a_Z^\dagger$.

C. [**15** points] Calculate the matrix element $C(k)$ defined in part B. What criterion must Ω, M, and ω_0 satisfy for $C(k)$ to be near unity? Does this condition contradict the assumption $\omega_0 \gg \Omega$ made above?

3.4 Bound States in One-dimensional Wells (2)

A. [**15 pts** points] A spinless particle of mass m moves freely and non-relativistically in one dimension subject to a rectangular potential energy well ($V_0 > 0$),

$$V(x) = \begin{cases} -V_0\,, & x_0 \le x \le x_0 + L\,; \\ 0\,, & x < x_0 \text{ or } x > x_0 + L\,. \end{cases}$$

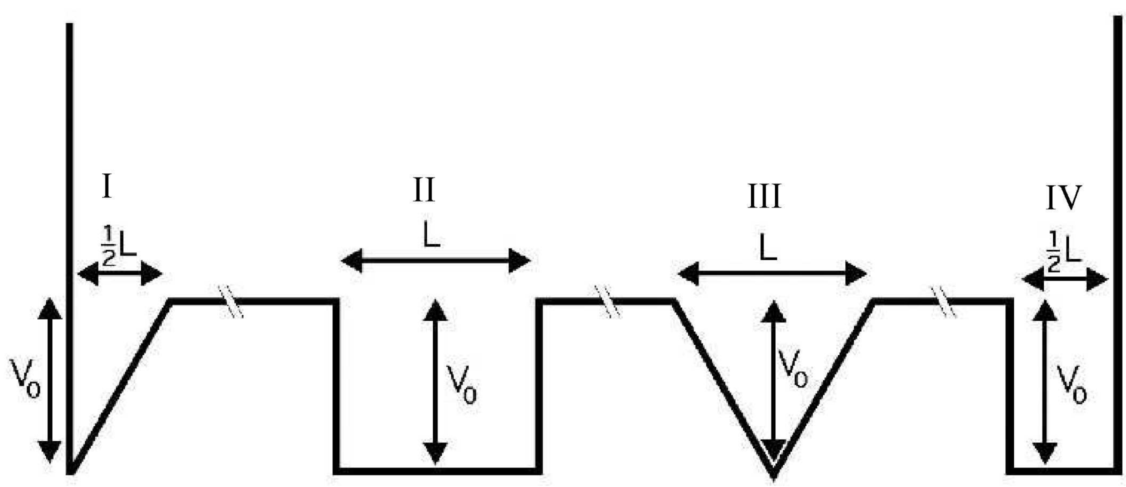
First consider the limit $V_0 \to \infty$. Make a sketch of the magnitude of the wave function, $|\Psi(x)|$, as function of x for both the ground state and the first excited state in the range $x_0/2 \le x \le 3x_0/2 + L$, assuming both are bound states. Find an expression for the energy (measured with respect to the bottom of the well) of the nth bound state in this infinite rectangular well ($n = 1$ is the ground state).

Describe in words and with a sketch how the form of the magnitude of the wave function, $|\Psi(x)|$, changes for a finite potential, $V_0 < \infty$.

B. [**10 pts** points] Define a length parameter l_e characterizing the (finite) depth of the potential well via

$$V_0 = \frac{h^2}{2m}\frac{1}{l_e^2}\,,$$

where m is the mass of the particle. Show, based on the intuition gained in part A, that for $l_e = L/n$ there are at least n bound states.



C. [**10 pts** points] The figure above shows four potential wells. All are *very* far apart from each other. Two are at the "edge", beyond which $V(x) = +\infty$. Which of the 4 wells has the lowest bound state energy? Explain your answer with a qualitative argument.

D. [**15 pts** points] Derive an exact relation between the bound state energy levels of the triangular well in the middle, III, and the semi-triangular well at the left edge, I.

3.5 Breaking Degeneracy

Consider an electron in a spherically symmetric potential. The spatial part of the wavefunction of the electron in an eigenstate of the Hamiltonian with energy E_1 has the form

$$\psi_1(\boldsymbol{r}) = x\, f(|\boldsymbol{r}|)\,,$$

and satisfies the normalization condition

$$\int d^3r\, |\psi_1(\boldsymbol{r})|^2 = 1\,.$$

(Here x is a Cartesian component of the position vector $\boldsymbol{r}$.)

A. [**10** points] Is $\psi_1(\boldsymbol{r})$ an eigenfunction of the orbital angular momentum operator L_z, the generator of rotations around the z-axis? Show your work.

B. [**10** points] Write down a complete set of linearly-independent wavefunctions that have the same energy eigenvalue E_1 and are related by rotational symmetry.

C. [**10** points] A spin-orbit interaction of the form

$$V_{\mathrm{SO}} = U_0\, \boldsymbol{S}\cdot\boldsymbol{L}/\hbar^2\,,$$

perturbs the system. [U_0 is independent of $\boldsymbol{r}$, and $\boldsymbol{L}$ and $\boldsymbol{S}$ are the orbital and spin angular momentum operators, respectively.] For both $j = \ell \pm 1/2$, find the first order energy splitting of the E_1 level. j here denotes a quantum number labeling the eigenvectors of total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

D. [**30** points] Including the spin-orbit interaction of part C, consider an electron initially in the state $|\psi_1\rangle$ with spin up along the z-axis.

- **i.** [**15** points] Explicitly calculate the Clebsch-Gordon Coefficients that allow you to rewrite $|\psi_1\rangle$ in terms of eigenvectors of $\boldsymbol{J}^2$. You may wish to recall that

$$J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle.$$

- **ii.** [**15** points] Find the probability that the electron remains in the state $|\psi_1\rangle$ with spin up after a time t.

3.6 Charged Oscillator

A particle with mass m and charge q moves in one dimension under the influence of a harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$ and, in addition, a uniform electric field $\boldsymbol{E} = E\,\hat{e}_x$.

A. [**10** points] What is the Hamiltonian for the particle?

B. [**15** points] Perform a coordinate transformation of the form $y = \alpha x + \beta$, where α and β are constants, such that the resulting Hamiltonian, expressed in terms of y, is equivalent to that of a charge-free harmonic oscillator. What are the required values of α and β?

C. [**10** points] Find the energy levels and corresponding eigenstates of the system.

3.7 Clebsch-Gordon Coefficients

A. [**20** points] Determine, from first principles, the Clebsh-Gordon coefficients for combining $j_1 = 1$ and $j_2 = 1$ states to produce angular momentum eigenstates with $m = 0$. Hint: Don't try to remember complicated recursion formulas — use simple reasoning based on elementary concepts (normalizations, orthogonality, ...) plus

$$J_{\pm}|jm\rangle \equiv (J_x \pm iJ_y)\,|jm\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle\,.$$

B. [**15** points] Express the total angular momentum operator $\boldsymbol{J}^2$ in terms of $\boldsymbol{J}^2_{(i)}$, $J_{(i)z}$, and $J_{(i)\pm}$ of the two $j = 1$ subsystems (for example: $J_z = J_{(1)z} + J_{(2)z}$), and use this to show that the state

$$\frac{1}{\sqrt{6}}\Big[|1,-1\rangle \otimes |1,1\rangle + 2|1,0\rangle \otimes |1,0\rangle + |1,1\rangle \otimes |1,-1\rangle\Big]$$

is an eigenstate of $\boldsymbol{J}^2$.

3.8 Commutator Consequences

[**20** points] Consider a system with Hamiltonian $\hat{H}$. Two operators, $\hat{A}$ and $\hat{B}$, commute with the Hamiltonian, $[\hat{H}, \hat{A}] = [\hat{H}, \hat{B}] = 0$, but do not commute with each other, $[\hat{A}, \hat{B}] \neq 0$. Show that the system has degenerate energy levels.

3.9 Cycling Around

The three mutually orthogonal states $|\psi_r\rangle$, $|\psi_g\rangle$, and $|\psi_b\rangle$ form a complete basis for the states of a quantum mechanical particle. Let R denote an operator which cyclically permutes the states, so that $R|\psi_b\rangle = |\psi_g\rangle$, $R|\psi_g\rangle = |\psi_r\rangle$, and $R|\psi_r\rangle = |\psi_b\rangle$. The particle Hamiltonian is

$$H = -\hbar\,\omega\,(R + R^\dagger)\,.$$

A. [**6** points] Is R an observable? Explain why or why not.

B. [**18** points] Find the energy eigenvalues and eigenstates of the particle.

3.10 Density Matrices

A. [**10** points] Explain briefly in what circumstances one should describe a physical system using a density matrix rather than a wavefunction.

B. [**10** points] Explain why the density matrix for an ensemble of spin-1/2 particles can always be written in the form

$$\rho = \tfrac{1}{2}\left(I + \boldsymbol{a}\cdot\boldsymbol{\sigma}\right),$$

with, as usual, $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, and $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

C. [**5** points] Show that the ensemble average of the spin is given by $\langle \boldsymbol{S} \rangle = \frac{1}{2}\hbar\,\boldsymbol{a}$.

D. [**15** points] The spins are placed in a magnetic field aligned in the z-direction, such that the Hamiltonian is $H = \omega S_z$, and are kept at temperature T. The density operator describing this situation is

$$\rho = Z^{-1}\, e^{-H/(k_{\mathrm{B}}T)}\,,$$

with $Z \equiv \mathrm{tr}(e^{-H/(k_{\mathrm{B}}T)})$. Calculate $\langle \boldsymbol{S} \rangle$ as a function of T, and comment on the result in the limits $T \to 0$ and $T \to \infty$.

3.11 Delta-function Potential

Consider a particle of mass M and charge Q moving in one dimension in an attractive potential of the form $V(x) = -\alpha\,\delta(x)$.

A. [**6** points] What are the boundary conditions at $x = 0$ that the wavefunction must satisfy?

B. [**7** points] Show that a bound state $|\phi\rangle$ exists for any value of the coupling $\alpha > 0$, and has the form

$$\langle x|\phi\rangle = \sqrt{\kappa}\, e^{-\kappa|x|}\,.$$

Express κ in terms of α, M and Q.

C. [**7** points] Show that the eigenstate of the Hamiltonian describing the scattering of a particle of momentum $p = \hbar k$ incident from $x = -\infty$ has a wavefunction of the form

$$\langle x|\psi_k\rangle = \frac{1}{\sqrt{2\pi}}\left[e^{ikx} - \frac{1}{1+iYk}\,e^{-ikx}\right]\theta(-x) + \frac{1}{\sqrt{2\pi}}\,\frac{iYk}{1+iYk}\,e^{ikx}\,\theta(x)\,,$$

where $\theta(x)$ is a unit step function. Determine Y. For a given positive energy $E = \frac{1}{2}\hbar^2k^2/M$, how many independent eigenstates are there?

D. [**6** points] Calculate $\langle\psi_{k'}|\psi_k\rangle$. [Note that $\int_{-\infty}^{0} dk\, e^{ikx} = \pi\delta(x) - i\,\mathcal{P.V.}\big(\frac{1}{x}\big)$, where $\mathcal{P.V.}$ denotes principal value.]

E. [**12** points] Consider a quantum system with a discrete spectrum of energy eigenvalues. The system is initially in its ground state $|0\rangle$, and at time t=0 a small, sinusoidally time-dependent perturbation to the Hamiltonian of the form

$$\hat{V}(t) = \hat{V}_0\, e^{-i\omega t}\,.$$

is applied. Show that the probability $P(t)$ of finding the system at some later time in an excited state $|k\rangle$ is

$$P_k(t) = \frac{1}{\hbar^2}\left|\big\langle k\big|\hat{V}_0\big|0\big\rangle\right|^2 F(t\,\Delta\omega_k)\,,$$

where $F(x) = 4\sin^2(x/2)/x^2$ and $\hbar\Delta\omega_k$ is the excitation energy of state $|k\rangle$. Further show that the total transition rate to a continuum of final states may be expressed as

$$W_{i\to\sum f} = \frac{2\pi}{\hbar^2}\left|\big\langle E_f\big|\hat{V}_0\big|E_i\big\rangle\right|^2 \rho(E_f)\,,$$

and define all the quantities in this expression.

F. [**12** points] Returning to the case of a particle moving in the presence of a one dimensional δ-function potential, suppose the particle is initially in the bound state, and a perturbation of the form of $\hat{V}(t) = q\,\mathcal{E}\,\hat{x}\,e^{-i\omega t}$ is applied to the system, where $\mathcal{E}$ is the magnitude of an electric field. What is the rate for excitation into the continuum?

3.12 Discrete or Continuous

Consider the two-dimensional potential

$$V(x,y) = \tfrac{1}{2}\,\lambda^2\,x^2\,y^2\,.$$

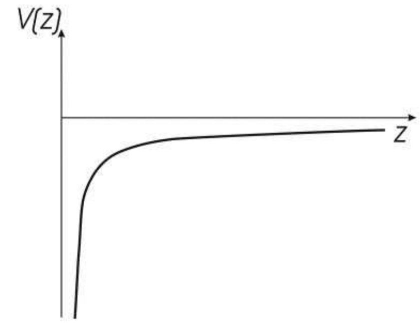
A. [**20** points] Does this potential have discrete or continous energy levels? Justify your answer in convincing detail.

B. [**20** points] Estimate (semi-quantitatively) the lowest energy level.

3.13 Electron near Insulator

Consider a non-relativistic electron that is trapped in a state above a semi-infinite flat insulator. Due to the attractive interaction between the electron and the insulator, the electron experiences a potential,

$$V(z) = \begin{cases} -A/z\,, & z > 0\,; \\ +\infty\,, & z < 0\,. \end{cases}$$



where A is a positive constant, and z is the distance between the electron and the surface of the insulator. The form of the potential suggests that the system is a one-dimensional analogue of a hydrogen atom. Assume the electron cannot penetrate into the insulator.

A. [**10** points] Sketch the wave functions in the z direction for the ground and first-excited states and justify the boundary conditions.

B. [**10** points] Determine the spatial part of the wave function for the ground state up to a normalization constant. [Hint: would $z\,e^{-az}$ work?]

C. [**10** points] What is the ground state binding energy in terms of A, $\hbar$, and m_e?

D. [**10** points] State all the quantum numbers for all the states bound in the z direction and write down the bound-state energies in terms of those quantum numbers. Take into account electron spin, and the fact that the electron can move in all three directions.

E. [**10** points] Is the total energy necessarily smaller than zero for a state bound in the z direction? Explain.

3.14 Electron near Liquid Helium

The potential energy of an electron of charge $-e$ and mass m, in the neighborhood of a horizontal surface of a dielectric liquid with dielectric constant $\epsilon > 1$, can be taken to be

$$V(x,y,z) = \begin{cases} -\frac{\epsilon-1}{\epsilon+1}\frac{e^2}{z}\,, & z > 0\,; \\ +\infty\,, & z < 0\,. \end{cases} \tag{1}$$

The electron is confined by an infinite potential barrier inside the region

$$0 < x < L_1, \qquad 0 < y < L_2\,. \tag{2}$$

A. [**15** points] Show that the time-independent Schrödinger equation, with a potential that is a function only of z, has stationary solutions, separable in Cartesian coordinates, of the form

$$\psi(x,y,z) = A\,f(x)\,g(y)\,h(z)\,,$$

and find the possible forms of $f(x)$ and $g(y)$ when the electron is confined in the region (2).

B. [**18** points] For the potential (1), show that one solution of the equation for the z-component of the wavefunction has the form

$$h(z) = z^{\alpha}\,e^{-z/l_0}\,. \tag{3}$$

Find expressions for the index α, the length l_0, and the energy E of the corresponding solutions $\psi(x,y,z)$.

C. [**10** points] What is the mean distance of an electron from the surface $z = 0$ in terms of the length l_0?

D. [**12** points] The dielectric constant of liquid helium is 1.057. Assuming that L_1 and L_2 are large, and using the facts that the binding energy of the electron in a hydrogen atom is $E_H = me^4/2\hbar^2 = 13.6$ eV, and the Bohr radius $a_0 \equiv \hbar^2/me^2 = 0.053$ nm, deduce the binding energy of an electron to the surface of liquid helium, and its mean distance from the surface. You can assume that the solution (3) gives the lowest possible energy.

3.15 Electromagnetic Transitions

Consider a spinless charged particle bound in a system by the Coulomb interaction, with Hamiltonian

$$\hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2M} \; + \; e\,\hat{V}_{\mathrm{C}}(|\hat{\boldsymbol{r}}|)\,.$$

[A caret (ˆ) over an object indicates that it is an operator, not a unit vector.] At time $t_0 = 0$, a sinusoidal perturbation is applied to the system, described by a Hamiltonian of the form

$$\hat{H}_1 = 2\,\hat{v}_1\,\cos\left(\boldsymbol{k}\cdot\hat{\boldsymbol{r}} - \omega t\right)\,, \tag{$*$}$$

where $\hat{v}_1$ is some not-yet-specified operator.

A. [**20** points] If the system is initially in its ground state, derive the long-time average transition rate to any one of the bound states of the system, assuming that the probability of the system being in any excited state remains small.[1]

Now consider the specific situation where the perturbation results from the system being immersed in a weak time-dependent electromagnetic field with wavelength $\lambda \gg R$, where R is the characteristic size of the bound states. The leading interaction produces an operator $\hat{v}_1$ (in Eq. ($*$)) given by

$$\hat{v}_1 = -\frac{eA}{M}\,\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{p}}\,,$$

where $\boldsymbol{\epsilon}$ defines the direction of the electromagnetic vector potential, and A denotes its magnitude.

B. [**10** points] Show that transition rates to excited states of the system are proportional to

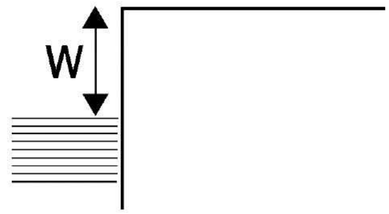
$$|\mathcal{M}_{fi}|^2 \propto |\langle f|\hat{\boldsymbol{r}}|i\rangle|^2\,.$$

C. [**10** points] State the Wigner-Eckhart theorem, and use it to derive selection rules for transitions between states with angular momentum $|l, m_l\rangle$ and $|l', m_l'\rangle$ induced by the electromagnetic field.

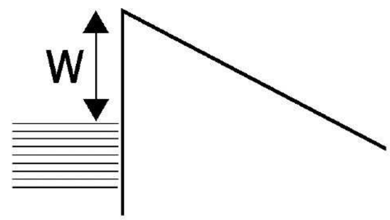
D. [**10** points] Consider two spherical tensors, $\hat{T}_q^{(k)}$ and $\hat{\theta}_u^{(s)}$, of rank k and s, respectively. How does the tensor product $\hat{T}_q^{(k)} \otimes \hat{\theta}_u^{(s)}$ transform under rotations? Write your answer in terms of the rotation matrices $\mathcal{D}^{(J)}_{mm'}(\alpha\beta\gamma)$, where α, β, and γ are the usual Euler angles.

[1] A possibly useful fact: $\lim_{t\to\infty} \frac{\sin^2(xt)}{x^2 t} = \pi\,\delta(x)$.

3.16 Field Emission



The energy levels near the surface of a metal appear roughly as shown in the figure to the right. The energy gap between the Fermi surface of the metal and the vacuum outside is the work function W. Photons with energies greater than W, incident on the surface of the metal, can free electrons from the metal surface.



In the presence of an electric field $\mathcal{E}$, the electron emission mechanism is modified. Electrons can now tunnel through the potential barrier, illustrated at right, in a process known as "field emission."

A. [**20** points] The major factor controlling the field emission rate (for sufficiently small $\mathcal{E}$) is the WKB barrier penetration factor $|T|^2$. Show that

$$|T|^2 = \exp\left(-\frac{4}{3}\frac{W}{\hbar}\frac{\sqrt{2m_e W}}{e\,\mathcal{E}}\right),$$

where m_e is the electron mass.

B. [**10** points] Consider a metal for which the work function $W = 2$ eV, corresponding to a light frequency $\nu = 5 \times 10^{14}$ Hz. Estimate the magnitude of the electric field $\mathcal{E}$, in volts per cm, required to induce substantial field emission. [Recall that $m_e c^2 \simeq 5 \times 10^5\,$eV and $c \simeq 3 \times 10^8\,$m/s.]

3.17 Harmonic Trap

Raising ($\hat{b}_j^\dagger$) and lowering ($\hat{b}_j$) operators for the three modes of oscillation of a particle of mass M in a potential of the form

$$V(\hat{\boldsymbol{x}}) = V_0 + \tfrac{1}{2}\sum_{j=1}^{3} K_j\,\hat{x}_j^2\,,$$

satisfy the commutation relations:

$$[\hat{b}_j^\dagger, \hat{b}_{j'}^\dagger] = [\hat{b}_j, \hat{b}_{j'}] = 0\,, \qquad [\hat{b}_j, \hat{b}_{j'}^\dagger] = \delta_{jj'}\,.$$

The position and momentum operators are related to the raising and lowering operators by

$$\hat{x}_j = \sqrt{\frac{\hbar}{2M\Omega_j}}\,(\hat{b}_j^\dagger + \hat{b}_j)\,, \qquad \hat{p}_j = i\sqrt{\frac{\hbar M\Omega_j}{2}}\,(\hat{b}_j^\dagger - \hat{b}_j)\,, \tag{$*$}$$

where $\Omega_j \equiv \sqrt{K_j/M}$.

A. [**5** points] Show that the possible eigenvalues of $\hat{n}_j \equiv \hat{b}_j^\dagger\hat{b}_j$ are non-negative.

B. [**18** points] Show that $\hat{b}_j\hat{b}_j^\dagger = \hat{n}_j + 1$.

C. [**10** points] What are the commutators $[\hat{n}_j, \hat{b}_{j'}]$ and $[\hat{n}_j, \hat{b}_{j'}^\dagger]$, and why are the operators $\hat{b}_j^\dagger$ and $\hat{b}_j$ called raising and lowering operators, respectively?

D. [**15** points] Use equation ($*$) to deduce the expectation values of $\hat{x}_j^2$ and $\hat{p}_j^2$ in an eigenstate of $\hat{n}_j$ with eigenvalue N_j.

E. [**10** points] A set of noninteracting bosons is in the ground state of such a harmonic trap, with $K_1 < K_2 < K_3$. The trap is suddenly switched off, so that the potential is now uniform. In which direction will the particles spread out most rapidly, and why?

3.18 Hydrogen Atom in Electric Field

Consider a hydrogen atom in a uniform external electric field $\mathcal{E}$ pointing in the z-direction. Neglect the spins of the electron and proton so that the Hamiltonian can be approximated as $H = H_0 + H_1$, where $H_0 = \boldsymbol{p}^2/(2\mu) - e^2/|\boldsymbol{r}|$, and $H_1 = e\,\boldsymbol{\mathcal{E}} \cdot \boldsymbol{r}$. Assume that the external field is sufficiently weak so that $H_1 \ll H_0$. Let $|nlm\rangle$ denote the eigenstates of H_0, with corresponding eigenenergies E_{nlm}. You needn't evaluate explicitly matrix elements appearing in this problem, but may simply leave them in the form $\langle n'\,l'm'|\mathcal{O}|nlm\rangle$, where $\mathcal{O}$ is the relevant operator.

A. [**10** points] Give a first-order expression for the shift in the ground state energy ($n = 1$, $l = 0$, $m = 0$) due to the external electric field, and then explain why this "linear Stark effect" vanishes.

B. [**15** points] Derive an expression for the second-order shift in the energy of the ground state. State explicitly the n, l, and m values of all states that are mixed into the ground state with non-vanishing amplitudes.

Now consider the $n = 2$, $l = 0$ first excited state.

C. [**5** points] In the absence of an external electric field, explain why this state cannot decay by emitting a single photon in an electric dipole transition.

D. [**15** points] Consider the leading-order effect of a non-zero external field $\mathcal{E}$ on the $n = 2, l = 0$ state. State explicitly the n, l and m values of *all* states that are admixed into the $n = 2$, $l = 0$ level with non-vanishing amplitudes. Then restrict yourself to admixed $n = 2$ levels and find the perturbed eigenstates in the presence of the external field $\mathcal{E}$. (Neglect splitting of the unperturbed $n = 2$, $l = 0$ and $n = 2$, $l = 1$ states.) Give the energy shifts of the perturbed eigenstates.

E. [**5** points] Neglect multi-photon decay of the unperturbed ($\mathcal{E} = 0$) metastable $n = 2$, $l = 0$ state and let τ_1 denote the decay lifetime of the unperturbed $n = 2$, $l = 1$ state. Express the lifetimes of the perturbed energy eigenstates in terms of $|\mathcal{E}|$ and τ_1.

Possibly useful information:
Spherical harmonics $Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$, $Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_{1,-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$.
Clebsh-Gordoan coefficients $C\langle J_1M_1, J_2M_2|J_3M_3\rangle$: $C\langle 11, 10|11\rangle = +1/\sqrt{2}$, $C\langle 10, 11|11\rangle = -1/\sqrt{2}$.

3.19 Nonrelativistic Hydrogen Atom

Consider the nonrelativistic hydrogen atom where we treat the proton as being infinitely massive and the electron moving in a stationary Coulomb ($1/r$) potential. At time $t = 0$, the electron in our hydrogen atom is prepared in the state:

$$\psi(\vec{r}, t = 0) \;=\; N[2\phi_{211} \;+ 2\phi_{210} \;+\; \phi_{31-1}].$$

Here the $\phi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi)$ are the normalized energy eigenfunctions with energy eigenvalues

$$E_n = -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m_e}{2\hbar^2 n^2},$$

where m_e and e are the mass and charge of the electron.

A.) (5 pts) Determine the constant N in the expression for $\psi(\vec{r}, t = 0)$.

B.) (5 pts) Determine whether the state $\psi(\vec{r}, t)$ has definite parity ($P\psi(\vec{r}, t) = \psi(-\vec{r}, t)$), and, if so, determine its parity. Explain your reasoning.

C.) (5 pts) What is the expectation for the energy, $\langle\psi|H|\psi\rangle$ at time $t = 0$ (write your expression in terms of the E_n's)? Explain your work.

D.) (5 pts) Compute $\hat{L}^2\psi(\vec{r}, t = 0)$ where $\hat{L}$ is the orbital angular momentum operator.

E.) (10 pts) Compute the expectation value for the z-component of the angular momentum, $\langle\psi|L_z|\psi\rangle$, at time $t = 0$ and find its uncertainty, σ_{L_z}, (also at $t = 0$).

F.) (10 pts) What is the probability that a measurement of the energy, at time $t' > 0$ will yield the value E_3? E_5? Explain your work!

G.) (10 pts) Find the expectation value of the x-component of the angular momentum, $\langle L_x\rangle$, in the state $\psi(\vec{r}, t = 0)$. Explain your work. HINT: $L_\pm Y_{lm} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}Y_{lm\pm1}$

3.20 Hydrogenic States

Position space wavefunctions for eigenstates of the hydrogen atom obtained from nonrelativistic quantum mechanics have the form

$$\psi_{n\ell m}(\boldsymbol{x}) = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)\,,$$

where $n = \ell{+}1, \ell{+}2, \cdots$, and $Y_\ell^m(\theta,\phi)$ are spherical harmonics.[1] The corresponding energy levels are given by the Bohr formula

$$E_n = -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m}{2\,\hbar^2\, n^2}\,, \qquad (*)$$

where m and e are the mass and charge of the electron. (The proton is assumed to be infinitely heavy.)

A. [**5** points] Show that the states with $\ell = 1$ and $\ell = 0$ have a definite parity, and determine the parity in each case.

B. [**20** points] Derive, from the Schrodinger equation, expression $(*)$ for the ground state energy E_1, and find the radial part of the ground state wave function $R_{10}(r)$ (unnormalized).

[1] The $\ell = 0$ and $\ell = 1$ spherical harmonics are $Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$, $Y_1^{+1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$, $Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$, and $Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$.

3.21 Hyperfine Splitting

The hyperfine structure in the ground state of the hydrogen atom is due to the magnetic interaction between the electron spin $\boldsymbol{S}$ and the proton spin $\boldsymbol{I}$. Both particles have spin 1/2. The spin Hamiltonian for this interaction may be written as $a\,\boldsymbol{S}\cdot\boldsymbol{I}$, with a a constant. (The 21 cm electromagnetic radiation which is of great importance in astrophysics originates in hydrogen hyperfine structure transitions.)

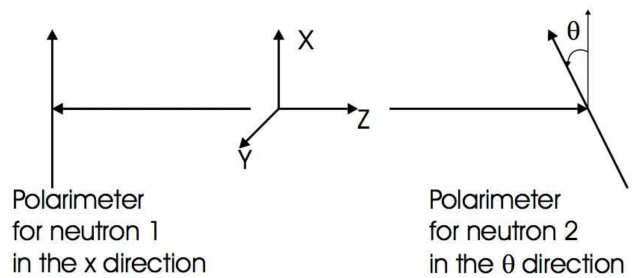
A. [**5** points] What are the possible values of the total angular momentum $\boldsymbol{F} \equiv \boldsymbol{S} + \boldsymbol{I}$?

B. [**10** points] Find the difference in energy between the eigenstates of the spin Hamiltonian. Express your answer in terms of the constant a.

3.22 Neutron Spins

A. [**8** points] The spin state of a spin-$\frac{1}{2}$ particle with magnetic moment $\boldsymbol{\mu} = \gamma\, \boldsymbol{S}$, in a constant external magnetic field $\boldsymbol{B} = \hat{e}_z\, B_0$, evolves according to $i\hbar\, \frac{\partial}{\partial t}\, |\Psi\rangle = -\gamma\, S_z B_0\, |\Psi\rangle$. Assume that the state at time $t = 0$ satisfies $S_x\, |\Psi\rangle = \frac{\hbar}{2} |\Psi\rangle$. What is the probability that at time $t > 0$ a measurement of the projection of the spin along the x-direction will yield $-\frac{\hbar}{2}$?

B. [**15** points] A system of two neutrons has zero total spin and zero orbital angular momentum. The two neutrons move apart with opposite momenta along the z-axis in an environment with zero magnetic field. After a certain time, the neutron moving to the left (neutron 1) enters a polarimeter that measures the neutron's spin projection along the x-axis. The measurement yields $+\hbar/2$. After this:



i. [**5** points] Give the spin wave function for the neutron moving to the right (neutron 2). This neutron enters a polarimeter that measures the neutron's spin projection along the y-axis. What is the probability of measuring $+\hbar/2$? Justify your answer.

ii. [**10** points] Suppose the polarimeter for neutron 2 is instead set at an angle θ with respect to the x-axis in a plane perpendicular to the z-axis. What is the probability of measuring $+\hbar/2$? Justify your answer.

C. [**12** points] Now, instead of neutrons, suppose the two particles are spin-1 nuclei, with the system initially in an $L = 0$, $S = 0$ state. The nucleus moving to the left enters a polarimeter that measures its spin projection along the x-axis. The measurement yields $+\hbar$. After this, the nucleus moving to the right enters a polarimeter that measures its spin along a direction at an angle θ with respect to the x-axis (in a plane perpendicular to the z-axis). What is the probability of measuring $\hbar$? Justify your answer.

3.23 Omega Baryons in a Harmonic Potential

Consider a particle of mass m moving in a one-dimensional harmonic potential of the form

$$V(\hat{x}) = \tfrac{1}{2} m\omega^2 \hat{x}^2 \,.$$

A. [**10** points] Show that the Hamiltonian for the system can be written in terms of the operator

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega}\, \hat{p} \right) ,$$

and its adjoint. Derive the commutation relation between $\hat{a}$ and $\hat{a}^\dagger$.

B. [**10** points] Use the fact that the operator $\hat{a}$ annihilates the ground-state, $\hat{a}|0\rangle = 0$, to show that the ground state wavefunction for the system is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \; e^{-\frac{m\omega}{2\hbar}x^2} \,.$$

[Simply showing that $\langle x|0\rangle$ satisfies the Schrödinger equation is not sufficient.]

The Ω^- baryon with spin and parity $J^\pi = \frac{3}{2}^+$ (a fermion) would be stable in the absence of weak interactions. For this problem, assume that Ω^- baryons are stable.

C. [**10** points] What is the spin and parity of the lowest energy eigenstate of a system comprised of one Ω^- moving in a three-dimensional harmonic potential with angular frequency ω? What is the energy and degeneracy of this state?

D. [**10** points] What are the spin and parity assignments of the lowest energy states of a system comprised of two non-interacting Ω^- baryons moving in the same isotropic three-dimensional harmonic potential?

E. [**15** points] A residual interaction between the two Ω^- baryons can be approximated by

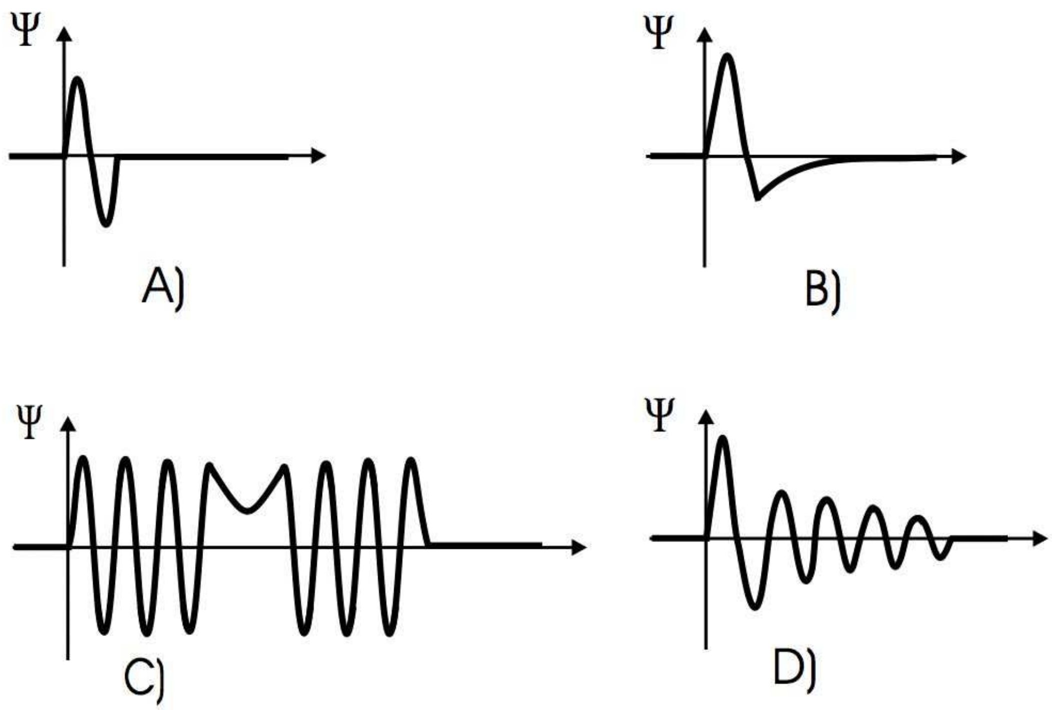
$$\delta\hat{H} = \eta \; \delta^3(\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2) \; \hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 \,.$$

What are the energies of the lowest-lying eigenstates of this system at leading order in the perturbation given by $\delta\hat{H}$? (The $\hat{\boldsymbol{s}}_i$ are spin-operators and the $\hat{\boldsymbol{r}}_i$ position operators of the i'th particle.)

F. [**15** points] What are the spin and parity assignments of the lowest energy states of a system comprised of three non-interacting Ω^- baryons moving in an isotropic three-dimensional harmonic potential?

3.24 One-dimensional Schrodinger Equation (1)

[**30** points] For each case shown in the figure below, make a sketch of a real (not complex) potential that would generate the stationary wave function shown. Explain your answer in each case. If you believe there is no such potential, explain why.



3.25 One-dimensional Schrodinger Equation (2)

A. [**5** points] Consider the one-dimensional time-independent Schrodinger equation with potential

$$V(x) = W(x) + V_0\, \delta(x - x_0)\,,$$

where $W(x)$ is any continuous function. Derive the discontinuity condition for $d\psi/dx$ at the singularity of the potential $V(x)$:

$$\lim_{\epsilon\to 0}\left[\left.\frac{d\psi}{dx}\right|_{x_0+\epsilon} - \left.\frac{d\psi}{dx}\right|_{x_0-\epsilon}\right] = \frac{2mV_0}{\hbar^2}\,\psi(x_0)\,.$$

B. [**8** points] Use this result to obtain an equation determining the energies of even parity energy eigenstates for a particle in the potential

$$V(x) = \begin{cases} \lambda\, \frac{\hbar}{2m}\, \delta(x) & \text{for } |x| < a\,; \\ \infty & \text{for } |x| > a\,, \end{cases}$$

where the real constant $\lambda \geq 0$. Sketch a graph which gives numerical solutions of the equation.

C. [**7** points] Draw the wavefunctions of the lowest parity even and parity odd energy eigenstates. Show the limiting forms of the wavefunctions when $\lambda \to 0$, and when $\lambda \to \infty$. Compare with the result of part B.

D. [**15** points] In the limit of large but finite λ, determine the period of oscillations if, at time $t = 0$, the particle is in a mixture of the lowest even and odd states, $\psi = (\psi_{\text{even}} + \psi_{\text{odd}})\,/\sqrt{2}$. (Determine the period explicitly, *i.e.*, in terms of m, a, and λ.)

3.26 One-dimensional Schrodinger Equation (3)

A. [**12** points] A particle of mass m moves in a one-dimensional potential given by

$$V(x) = G\,\delta(x)\,,$$

with $G \geq 0$. Derive the boundary conditions for the wave function, and show that the transmission coefficient through the delta function potential barrier for a given energy E is given by

$$T(E) = \left(1 + \frac{m\,G^2}{2E\hbar^2}\right)^{-1}\,.$$

(The transmission coefficient is the ratio of the fluxes of probabilities for the transmitted and incident waves.)

B. [**11** points] Now consider the potential

$$U(x) = \begin{cases} G\,\delta(x)\,, & -a \leq x < \infty\,; \\ +\infty\,, & x < -a\,, \end{cases}$$

with G again non-negative. In this potential, the wavefunction of an eigenstate with energy E has the form

$$\chi_k(x) = \begin{cases} \alpha(k)\,\sin k(x{+}a)\,, & -a \leq x < 0\,; \\ \beta(k)\,e^{ikx} + \gamma(k)\,e^{-ikx}\,, & x > 0\,, \end{cases}$$

where $k \equiv \sqrt{2mE}/\hbar$, and $\alpha(k)$, $\beta(k)$, and $\gamma(k)$ are functions of k. Consider the situation where the wavefunction at time $t = 0$ is given by

$$\Psi(x, t{=}0) = \begin{cases} \sqrt{\frac{2}{a}}\,\sin\frac{\pi x}{a}\,, & -a \leq x < 0\,, \\ 0\,, & x > 0\,. \end{cases}$$

[This situation resembles particle emission from unbound nuclear states, where a particle is trapped by a barrier and can slowly tunnel through it.] Write an expression (perhaps a complicated integral that you are not required to evaluate) for the wavefunction $\Psi(x, t)$ for $x < 0$ and $t > 0$.

C. [**7** points] For the potential $U(x)$, consider the quantity

$$\bar{E} = \int_0^\infty dx\,\Psi(x,t)^* \left(-\frac{1}{2M}\,\frac{d^2}{dx^2}\right)\Psi(x,t) \Bigg/ \int_0^\infty dx\,|\Psi(x,t)|^2\,,$$

which yields the mean energy 'outside the barrier'. Will this quantity increase, remain constant, or decrease with time? Give a qualitative explanation.

3.27 Oscillator Excitement

A. [**10** points] A quantum system with Hamiltonian H_0 is in energy eigenstate $|i\rangle$, with energy E_i, at time $t = -\infty$. Let $|f\rangle$ be another energy eigenstate with energy E_f. A perturbation, $V(t)$ is applied. Prove that, to first order in time-dependent perturbation theory, the amplitude corresponding to the system being in the state $|f\rangle$ at $t = +\infty$ is:

$$a_{fi} = \frac{1}{i\hbar} \int_{-\infty}^{\infty} \langle f|V(t')|i\rangle \, e^{i\omega t'} \, dt' \,,$$

where $\hbar\omega \equiv E_f - E_i$. Assume $E_f \neq E_i$, and that the amplitude a_{fi} is small, $|a_{fi}| \ll 1$.

B. [**25** points] A charged particle (with charge e) bound in an isotropic three-dimensional harmonic oscillator potential $V_{\text{osc}} = \frac{1}{2}k\,(x^2 + y^2 + z^2)$ is exposed to an electric field of the form $\boldsymbol{E} = \hat{e}_x \, E_0 \, e^{-|t|/\tau}$. Assume that the electric field is small enough to be treated perturbatively, and neglect the magnetic field.

- **i.** [**18** points] Calculate the matrix element $V_{\boldsymbol{n},\mathbf{0}} = \langle n_x, n_y, n_z|V_{\boldsymbol{E}}|0,0,0\rangle$, where $V_{\boldsymbol{E}}$ is the potential generated by the electric field, and $|n_x, n_y, n_z\rangle$ is the eigenstate of the Hamiltonian at zero electric field with n_x, n_y, n_z quanta of vibration in the x, y, z directions, respectively. [Hint: reexpress the Hamiltonian in terms of raising and lowering operators.]
- **ii.** [**7** points] At $t = -\infty$ the system is in the ground state. Calculate the probability of finding the system *not* in the ground state after a time much longer than τ.

3.28 Perturbation Theory

The $1s$ and the $2p_z$ wave functions of the hydrogen atom are:

$$\psi_{1s}(\boldsymbol{r}) = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}\,, \qquad \psi_{2p_z}(\boldsymbol{r}) = r\cos\theta\,\frac{e^{-r/2a_0}}{\sqrt{32\pi a_0^5}}\,,$$

where a_0 is the Bohr radius, $r = |\boldsymbol{r}|$ is the radial coordinate, and θ is the polar angle coordinate defined with respect to the z-axis.

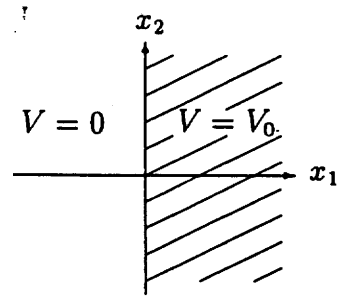
A. [**20** points] Find the shift in the energy of the $1s$ level due to the finite size of the proton at the lowest order in perturbation theory. Assume the proton is a sphere with radius $R \ll a_0$ with a uniform charge density distribution. Neglect relativistic corrections throughout this problem.

B. [**20** points] A constant electric field parallel to the z-axis is applied during a finite time interval Δt. Find the transition probability from the $1s$ to the $2p_z$ state.

3.29 Potential Step

Non-relativistic particles (in three dimensions) impinge on the potential "step",

$$V(\boldsymbol{x}) = \begin{cases} V_0\,, & \text{if } x_1 > 0\,; \\ 0\,, & \text{otherwise,} \end{cases}$$



illustrated at right.

A. [**5** points] Explain why the complete spatial wavefunction which describes particles incident from the left with wavenumber $\boldsymbol{k}$ must have the form

$$\Psi(\boldsymbol{x}) = \begin{cases} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + R\, e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\,, & \text{if } x_1 < 0\,; \\ T\, e^{i\boldsymbol{k}''\cdot\boldsymbol{x}}\,, & \text{if } x_1 > 0\,, \end{cases}$$

for some coefficients R and T and wavevectors $\boldsymbol{k}'$ and $\boldsymbol{k}''$.

B. [**5** points] What are the wavevectors of the reflected and transmitted waves (in terms of $\boldsymbol{k}$)?

C. [**10** points] What are the reflection and transmission coefficients? Describe (or sketch) how they vary with $\boldsymbol{k}$.

D. [**5** points] Show that the angle of incidence (with respect to the x_1-axis) equals the angle of reflection.

E. [**5** points] Compute the ratio of the wavelengths of the incident and transmitted waves, and explain why the potential barrier may be said to have an (energy dependent) index of refraction n. What is n?

F. [**10** points] If $k_1^2 > 2mV_0$, show that the direction of the transmitted wave obeys Snell's law.

G. [**5** points] What happens if $k_1^2 < 2mV_0$?

H. [**10** points] How might these results for "matter waves" be tested experimentally?

3.30 Probability and Measurement

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two mutually orthogonal (and normalized) states of a physical system, so that $\langle\psi_1|\psi_2\rangle = 0$ and $\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$. Let α_n denote a nondegenerate eigenvalue of some physical observable A of the system, with $|\phi_n\rangle$ the corresponding eigenstate. Define the amplitudes $b_1(\alpha_n) \equiv \langle\phi_n|\psi_1\rangle$ and $b_2(\alpha_n) = \langle\phi_n|\psi_2\rangle$.

A. [**10** points] In the context of measurement theory, what are the physical interpretations of $|b_1(\alpha_n)|^2$ and $|b_2(\alpha_n)|^2$?

B. [**20** points] A particle is in the (unnormalized) state $3|\psi_1\rangle - 4i|\psi_2\rangle$. In terms of b_1 and b_2, what is the probability of obtaining a value α_n when the observable A is measured?

3.31 Scattering off Potential Wells

Consider a single particle whose dynamics are governed by the Schrödinger equation

$$H\psi = -\frac{\hbar^2}{2m}\psi'' + V(x)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

scattering from a 1d-potential well $V(x)$.

A. [**15** points] Assume the potential $V(x)$ allows for eigenfunctions of the hamiltonian H with energy $E = \frac{\hbar^2 k^2}{2m}$ that have the following asymptotic behavior:

$$\psi(x) \to \begin{cases} Ae^{ikx} & \text{for } x \to +\infty \\ Ce^{ikx} + De^{-ikx} & \text{for } x \to -\infty \end{cases}$$

Use this information to calculate the reflection and transmission probabilities, that is the ratios of transmitted and reflected probability fluxes to the incoming probability flux, for a plane wave incident from $x = -\infty$. Explain your reasoning.

B. [**20** points] As an example, calculate reflection and transmission probabilities for a wave with wavenumber k incident from $x = -\infty$ for a delta-function potential, that is of the form

$$V(x) = a\delta(x)$$

where a is a constant.

C. [**25** points] As a second example consider the potential

$$V(x) = -\frac{b}{\cosh^2(\alpha x)}$$

with $b > 0$.

i. [**15** points] Consider a trial function of the form $\psi_T = \frac{N}{\cosh(\alpha x)}$ where N is a normalization constant. Determine the normalization constant N so that ψ_T is properly normalized and then calculate the expectation value of the Hamiltonian in this trial state. What can you learn from this calculation about the existence of boundstates? You may find the following integrals useful:

$$\int_{-\infty}^{\infty}\frac{1}{\cosh x} = \pi, \quad \int_{-\infty}^{\infty}\frac{1}{\cosh^2 x} = 2, \quad \int_{-\infty}^{\infty}\frac{1}{\cosh^3 x} = \frac{\pi}{2}, \quad \int_{-\infty}^{\infty}\frac{1}{\cosh^4 x} = \frac{4}{3}.$$

ii. [**10** points] Show that for the special case $b = \frac{\alpha^2\hbar^2}{m}$

$$\psi(x) = e^{ikx}\frac{\hbar}{\sqrt{2m}}\left(-ik + \alpha\tanh(\alpha x)\right)$$

is an eigenfunction of the Hamiltonian for any real k. What does this imply for the transmission and reflection probabilities as a function of wavenumber k?

3.32 Spin Flip

[**40** points] A particle of spin 1/2 and magnetic moment μ is placed in a rotating magnetic field,

$$\boldsymbol{B}(t) = B_0\,\hat{e}_z + B_1\,(\hat{e}_x\,\cos\omega t - \hat{e}_y\,\sin\omega t)\,,$$

as is often used in magnetic resonance experiments. Assume that the particle has spin up along the z-axis ($m_z = +1/2$) at time $t = 0$, and derive the probability to find the particle with spin down ($m_z = -1/2$) at some later time $t > 0$. You may assume $t \ll 1/\omega$. [It will be convenient to express your answer in terms of the angular frequencies $\Omega_0 \equiv \mu B_0/\hbar$ and $\Omega_1 \equiv \mu B_1/\hbar$.]

3.33 Spin Measurements

A beam of spin $\frac{1}{2}$ particles has been prepared. You can measure a chosen spin component of each particle in the beam. When you measure the z-component of spin on a large sample of particles in the beam you observe that 50% of the particles have a spin projection $+\hbar/2$.

A. [**5** points] Let $|+\rangle_x$ and $|-\rangle_x$ denote eigenstates of S_x. Could each particle in the beam be in the state $|+\rangle_x$? Could each particle in the beam be in the state $|-\rangle_x$? Explain your answers.

B. [**5** points] Can you propose a measurement of spin components of the beam particles which would distinguish between the two cases? If you can, say what is measured and what the results might be. If not, why not?

C. [**5** points] In part A, it is assumed that all particles in the beam are in the same state. Is this assumption *required* in order to be consistent with your observations? Why or why not? Explain.

3.34 Spin-1/2 Systems

A. [**8** points] Construct the spin-$\frac{1}{2}$ rotation matrix $\mathcal{D}^{(1/2)}(\alpha\beta\gamma)$ from the definition of the rotation operator, for a rotation defined by the Euler angles α, β and γ.

B. [**7** points] For a spin-$\frac{1}{2}$ particle, what are the eigenstates and eigenvalues of the operator $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ where $\hat{S}_j$ are the spin-operators and $\boldsymbol{n}$ is the unit vector with Cartesian components $\boldsymbol{n} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$. Express the eigenvectors of $\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ in terms of eigenstates of $\hat{S}_z$.

Consider an ensemble of spin-$\frac{1}{2}$ particles. Half the ensemble is prepared in the $|+\rangle$ state of $\hat{S}_z$, and the other half is prepared in the state $|\psi\rangle = \cos\frac{\beta}{2}\,|+\rangle + \sin\frac{\beta}{2}\,|-\rangle$.

C. [**7** points] What is the density matrix ρ of the ensemble, and what is its trace? Is this a pure or mixed state?

D. [**7** points] What are $\langle\hat{S}_z\rangle$ and $\langle\hat{S}_x\rangle$?

This ensemble is immersed in a magnetic field oriented in the z-direction.

E. [**7** points] What is the time-dependent density matrix?

F. [**7** points] Show that a pure state cannot evolve into a mixed state under time evolution determined by the Schrodinger equation.

G. [**7** points] What are $\langle\hat{S}_z\rangle$ and $\langle\hat{S}_x\rangle$ as functions of time?

3.35 Spin-Orbit Interaction

An electron in a spherically symmetric potential $V(|\boldsymbol{r}|)$ is in an energy eigenstate with quantum numbers $S = \frac{1}{2}$, $L = 1$, and $J = \frac{1}{2}$. (As usual, S is the spin quantum number, L is the orbital angular momentum quantum number, and J is the total angular momentum quantum number.) The electron magnetic moment is

$$\boldsymbol{\mu} = -\frac{\hbar e}{2mc}\left(\boldsymbol{L} + g\boldsymbol{S}\right).$$

Here $-e$ and m are the charge and mass of the electron, respectively, c is the speed of light, and g is the gyromagnetic ratio of the electron. At time $t = 0$, a magnetic field $\boldsymbol{B} = B\,\hat{e}_z$ is switched on.

A. [**4** points] Write down an expression for the Hamiltonian, including the spin-orbit interaction.

In parts B, C, and D assume the magnetic field is arbitrarily weak.

B. [**8** points] Which combination of the quantum numbers J, L, S, J_z, L_z, and S_z are good quantum numbers for the energy eigenstates of this system?

C. [**17** points] Calculate the expectation value of the component of the magnetic moment which is along the $\boldsymbol{J}$ direction.

D. [**17** points] At time $t = 0$, assume that $J_x = +\frac{1}{2}$. Calculate the time dependence of the expectation value of J_x.

E. [**12** points] Will your answer to part B change if the magnetic field is made strong? If not, explain why not. If so, explain how strong the field has to be in order to change your conclusion.

3.36 Spin Projections and Spin Precession

A. [**15** points] In the spin 1/2 case, work out the eigenvalue problem for the operator $\mathbf{S}\cdot\hat{\boldsymbol{n}}$ where $\hat{\boldsymbol{n}}$ is a unit vector in the xz-plane making an angle $\beta < 90°$ with the $+z$ axis. Find the eigenvalues and corresponding eigenstates in terms of the $|\pm\rangle$ eigenstates of S_z.

B. [**15** points] One of the eigenstates in part A is $\left(\cos\frac{\beta}{2}\,|+\rangle + \sin\frac{\beta}{2}\,|-\rangle\right)$. Suppose a particle is in this state.

- **i.** Find the probability of observing S_x to have a value of -$\hbar/2$.
- **ii.** Find the expectation value of S_z.
- **iii.** Find the expectation value of S_x.
- **iv.** Check whether your answers are reasonable for the special cases of $\beta = 0°$ and $\beta = 90°$.

C. [**10** points] A spin 1/2 particle initially (at time $t = 0$) in the state shown in part B enters a magnetic field $\boldsymbol{B} = B_0\,\hat{\boldsymbol{e}}_z$. The particle has a magnetic moment $\boldsymbol{\mu} = \gamma\boldsymbol{S}$, where γ is a constant.

- **i.** Find the state of the particle at some later time t (expressed in terms of γ and B_0).
- **ii.** Use the time dependence of $\langle S_x\rangle$ to explain why precession of the spin around the field direction is expected.

3.37 Stark Shift

A hydrogen atom is in a uniform electric field $\boldsymbol{E}$ pointing in the z-direction. Express your answers to the following questions in terms of integrals over the radial functions $R_{n\ell}(r)$.

A. [**25** points] Find the lowest order non-zero correction to the energy for a hydrogen atom in the ground state $n = 1$. Comment on the sign of the result, and on the nature of the dependence of the energy on the magnitude of the electric field.

B. [**25** points] Work out the lowest order non-zero corrections to the energies of the $n = 2$ states. Discuss how and why these results differ from that of part A.

3.38 Superposition

[**25** points] Consider a one-dimensional harmonic oscillator with potential centered at $x=0$. Let $\psi_0(x)$ and $\psi_1(x)$ be real, normalized wavefunctions of the ground and first excited states, respectively. Consider the wavefunction $\psi(x) \equiv A\,\psi_0(x) + B\,\psi_1(x)$, with A and B real coefficients. Show that $\langle \hat{x} \rangle$ (the expectation value of x) for this state is in general different from zero, and determine the values of A and B that maximize and minimize this expectation value.

3.39 Symmetric Top

A symmetric top has moments of inertia $I_x = I_y$ and I_z in the body-fixed frame. The system is described by the Hamiltonian

$$H = \frac{1}{2I_x}(L_x^2 + L_y^2) + \frac{1}{2I_z}L_z^2\,,$$

where $\boldsymbol{L}$ is the total angular momentum.

A. [**7** points] Find the eigenvalues and eigenstates of the Hamiltonian.

B. [**10** points] What is the expectation value for a measurement of $L_x + L_y + L_z$ in an eigenstate of the Hamiltonian?

C. [**8** points] Suppose that at time $t = 0$, the top is in the state $|l{=}3, m{=}0\rangle$. What is the probability of obtaining the value $\hbar$ from a measurement of L_z at time $t = 4\pi I_z/\hbar$?

D. [**10** points] Suppose that at time $t = 0$, the top is in the state $|l{=}1, m{=}0\rangle$. What is the probability of obtaining the value $\hbar$ from a measurement of L_x at $t = 4\pi I_z/\hbar$?

3.40 Three Spins (1)

Consider a system of three collinear spin-1/2 particles. Nearest-neighbor spins interact, with the Hamiltonian given by

$$H = \beta \left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3 \right).$$

The interaction is ferromagnetic, $\beta > 0$.

A. [**10** points] Show that $\langle \uparrow_1 \downarrow_2 | \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 | \uparrow_1 \downarrow_2 \rangle = -1$.

B. [**10** points] Show that $\langle \uparrow_1 \downarrow_2 | \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 | \downarrow_1 \uparrow_2 \rangle = 2$.

C. [**30** points] The ground state of the Hamiltonian is

$$\frac{1}{\sqrt{6}} \big(|\uparrow_1 \uparrow_2 \downarrow_3\rangle - 2 |\uparrow_1 \downarrow_2 \uparrow_3\rangle + |\downarrow_1 \uparrow_2 \uparrow_3\rangle \big).$$

Show that this is an eigenstate and determine its energy.

D. [**5** points] How many independent states does the system have?

E. [**15** points] The Hamiltonian is invariant under rotations. Use this information to classify the eigenstates of the system.

F. [**20** points] Sketch a level diagram showing quantum numbers and degeneracy of states. Make sure the total number of states equals the answer to part D.

G. [**10** points] The Hamiltonian also is symmetric under the permutation of particles 1 and 3. Does this give rise to another quantum number? If so, define it and indicate its value on the level diagram.

3.41 Three Spins (2)

Consider a system composed of three spin-$\frac{1}{2}$ particles that have nearest neighbor interactions described by the Hamiltonian

$$\hat{H} = \beta\,(\,\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 + \hat{\boldsymbol{s}}_2 \cdot \hat{\boldsymbol{s}}_3\,) = \beta\,\hat{\boldsymbol{s}}_2 \cdot (\hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_3)\ ,$$

where $\hat{\boldsymbol{s}}_j$ are the spin-operators that act on the j'th particle.[1]

A. [**10** points] The ground state of this system is

$$|\psi_0\rangle = \frac{1}{\sqrt{6}}\left[\ |{\uparrow_1}{\uparrow_2}{\downarrow_3}\rangle - 2\,|{\uparrow_1}{\downarrow_2}{\uparrow_3}\rangle + |{\downarrow_1}{\uparrow_2}{\uparrow_3}\rangle\ \right].$$

Show that $|\psi_0\rangle$ is an eigenstate of $\hat{H}$, and find its energy eigenvalue.

B. [**5** points] How many independent eigenstates does this three-particle system have?

C. [**10** points] Determine the complete set of commuting operators that specify the eigenstates of the system, and show that they each commute with $\hat{H}$.

D. [**15** points] Find all the eigenstates of this system, along with their energy eigenvalues. Assuming that $\beta > 0$, sketch the spectrum of $\hat{H}$, indicating the total spin and degeneracy of each level.

[1]The following Clebsch-Gordan coefficients, $\langle j_1 m_1, j_2 m_2 | j_3 m_3\rangle$, may be helpful:

$$\left\langle \tfrac{1}{2}\ {-\tfrac{1}{2}},\, 1\ {+1}\middle|\tfrac{1}{2}\ {+\tfrac{1}{2}}\right\rangle = \sqrt{\tfrac{2}{3}}\,, \qquad \left\langle \tfrac{1}{2}\ {+\tfrac{1}{2}},\, 1\ 0\middle|\tfrac{1}{2}\ {+\tfrac{1}{2}}\right\rangle = -\sqrt{\tfrac{1}{3}}\,.$$

Matrix elements of the angular momentum operators may also be useful:

$$\langle j', m'|\hat{J}_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,\hbar\,\delta_{jj'}\,\delta_{mm'}\,.$$

3.42 Three Spins (3)

This QM question consists of a single (multi-part) problem worth 100 points.

Consider a system composed of three spin-$\frac{1}{2}$ particles that have nearest neighbor interactions described by the Hamiltonian

$$\hat{H} = \beta \left(\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 + \hat{\boldsymbol{s}}_2 \cdot \hat{\boldsymbol{s}}_3 \right) = \beta \, \hat{\boldsymbol{s}}_2 \cdot \left(\hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_3 \right) ,$$

where $\hat{\boldsymbol{s}}_j$ are the spin-operators that act on the j'th particle.[1]

A. [**10** points] First consider the subspace associated with particles 1 and 2. Show that the total spin operator in this subspace, $\hat{\boldsymbol{S}}_{12} = \hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_2$, satisfies the following commutation relations,

$$\left[\hat{\boldsymbol{S}}_{12}^2, \hat{\boldsymbol{s}}_1^2 \right] = \left[\hat{\boldsymbol{S}}_{12}^2, \hat{\boldsymbol{s}}_2^2 \right] = \left[\hat{\boldsymbol{S}}_{12}^2, (\hat{\boldsymbol{s}}_1 + \hat{\boldsymbol{s}}_2)_z \right] = 0.$$

Thus, depending on the dynamics, the eigenvalues of these operators can provide a useful labeling for the two-particle states in this subspace.

B. [**15** points] Still in the 1-2 subspace, find an expression for

$$\hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 | \uparrow_1 \downarrow_2 \rangle \quad ,$$

and show that

$$\langle \uparrow_1 \downarrow_2 | \hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 | \uparrow_1 \downarrow_2 \rangle \;=\; -\frac{1}{4} \quad , \quad \langle \uparrow_1 \downarrow_2 | \hat{\boldsymbol{s}}_1 \cdot \hat{\boldsymbol{s}}_2 | \downarrow_1 \uparrow_2 \rangle \;=\; +\frac{1}{2} \quad .$$

C. [**5** points] Now we return to the three-particle system and consider first its general properties. How many independent eigenstates does this three-particle system have?

D. [**20** points] Determine a complete set of commuting operators that also commute with $\hat{H}$ and that serve to specify the eigenstates of the system corresponding to the original three-spin Hamiltonian. Demonstrate explicitly that each of the chosen operators commutes with $\hat{H}$. Note in particular that the Hamiltonian is rotationally invariant.

[1] The following Clebsch-Gordan coefficients, $\langle j_1 m_1, j_2 m_2 | j_3 m_3 \rangle$, may be helpful:

$$\left\langle \tfrac{1}{2} \, -\tfrac{1}{2}, 1 \, +1 \middle| \tfrac{1}{2} \, +\tfrac{1}{2} \right\rangle = \sqrt{\tfrac{2}{3}} , \qquad \left\langle \tfrac{1}{2} \, +\tfrac{1}{2}, 1 \; 0 \middle| \tfrac{1}{2} \, +\tfrac{1}{2} \right\rangle = -\sqrt{\tfrac{1}{3}} .$$

Matrix elements of the angular momentum operators may also be useful:

$$\langle j', m' | \hat{J}_{\pm} | j, m \rangle = \sqrt{(j \mp m)(j \pm m + 1)} \, \hbar \, \delta_{jj'} \, \delta_{mm' \mp 1} .$$

Finally recall the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

E. [**15** points] For $\beta > 0$ the ground-state of the three-spin system is

$$|\psi_0\rangle \;=\; \frac{1}{\sqrt{6}}\left[\,|\uparrow_1\uparrow_2\downarrow_3\rangle \;-\; 2|\uparrow_1\downarrow_2\uparrow_3\rangle \;+\; |\downarrow_1\uparrow_2\uparrow_3\rangle\,\right].$$

Show that $|\psi_0\rangle$ is an eigenstate of $\hat{H}$, and find its energy eigenvalue.

F. [**20** points] Find all the energy eigenstates of the three-spin system, along with their energy-eigenvalues. Still assuming that $\beta > 0$, sketch the spectrum of $\hat{H}$, indicating the total spin and degeneracy of each level. Check that the total number of states matches the result in part C.

G. [**15** points] Assume that at some initial time ($t = 0$) the system is in the three-spin state $|\uparrow_1\downarrow_2\uparrow_3\rangle$. In terms of the quantities already defined in the previous parts of this problem, find the probability that the system will remain in this three-spin state as a function of the time t.

3.43 Time Evolution

Consider a particle of mass m moving in a one-dimensional potential

$$V(x) = -\alpha\,\delta(x)\,,$$

where α is a positive constant.

A. [**15** points] Find the energy level(s) and the normalized wave function(s) of the bound state(s).

B. [**25** points] At time zero, the wavefunction of the particle (which is not necessarily an eigenfunction) is:

$$\psi(t{=}0, x) = A\,e^{-\beta|x|}\,.$$

with β being an arbitrary positive parameter not related to α.

- **i.** [**5** points] Explain qualitatively what happens to the wave function in the limit $t \to \infty$.
- **ii.** [**10** points] Find the probability $W(x)\,dx$ of finding the particle in the interval $(x, x + dx)$ in the limit $t \to \infty$.
- **iii.** [**5** points] Evaluate the integral $\int_{-L}^{L} dx\,W(x)$.
- **iv.** [**5** points] Consider the $L \to \infty$ limit of the integral you evaluated in part (iii). What is the physical interpretation of this quantity? Compare with the analogous quantity at $t = 0$ and qualitatively explain the result.

C. [**10** points] Now put the system in a box of width $2L$. That is the potential is as above for $|x| < L$, but $V = \infty$ for $|x| \geq L$. Qualitatively describe the spectrum of normalizable eigenstates in this case. How does this change affect the answer to problem B(iv)? Explain.

3.44 Time-Reversal

The transformation

$$|\alpha\rangle \to |\tilde{\alpha}\rangle \equiv \hat{\Theta}|\alpha\rangle\,, \qquad |\beta\rangle \to |\tilde{\beta}\rangle \equiv \hat{\Theta}|\beta\rangle\,,$$

is said to be anti-unitary if

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^* \qquad \text{and} \qquad \hat{\Theta}\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^*\,\hat{\Theta}|\alpha\rangle + c_2^*\,\hat{\Theta}|\beta\rangle\,.$$

A. [**8** points] Show that $\langle\beta|\,\hat{\eta}\,|\alpha\rangle = \langle\tilde{\alpha}|\,\hat{\Theta}\,\hat{\eta}^\dagger\,\hat{\Theta}^{-1}\,|\tilde{\beta}\rangle$ for an arbitrary linear operator $\hat{\eta}$.

B. [**4** points] If $\hat{\Theta}_T$ is the time-reversal operator, or more precisely the motion reversal operator, and $|k\rangle$ is a momentum eigenstate, what is $\hat{\Theta}_T\,|k\,\rangle$?

C. [**8** points] Using the results of part (A) and part (B), show that for a Hamiltonian that is invariant under time-reversal, the matrix elements of the scattering operator $\hat{T}$ between eigenstates of momentum are related by

$$\langle k'|\,\hat{T}\,|k\rangle = \langle -k|\,\hat{T}\,|-k'\rangle\,.$$

D. [**10** points] One may choose a phase convention such that $\hat{\Theta}_T|x\rangle = +|x\rangle$. By considering the action of $\hat{\Theta}_T$ on an arbitrary state $|\alpha\rangle$, show that

$$\hat{\Theta}_T\,|l,m\rangle = (-1)^m\,|l,-m\rangle\,,$$

where the states $|l,m\rangle$ are eigenstates of $\hat{L}^2$ and $\hat{L}_z$.

E. [**10** points] Prove that

$$\langle\alpha,j,m|\,T^{(k)}_{\lambda=0}\,|\alpha,j,m\rangle = \pm(-1)^k\,\langle\alpha,j,m|\,T^{(k)}_{\lambda=0}\,|\alpha,j,m\rangle$$

where $T^{(k)}_{\lambda=0}$ is the λ=0 component of a rank-k spherical tensor that is either even or odd under time-reversal, so that $\hat{\Theta}_T\,T^{(k)}_{\lambda=0}\,\hat{\Theta}_T^{-1} = \pm T^{(k)}_{\lambda=0}$.

F. [**10** points] Explain what the result of part (E) implies for the electric dipole moment of the neutron or electron.

3.45 Toy Nuclei

[**16** points] Four neutrons and three protons are in a three-dimensional harmonic oscillator potential described by the Hamiltonian

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m} + \tfrac{1}{2}k\,\boldsymbol{r}_i^2\,,$$

where the sum on i is over the particles. There are no interactions between the particles, and the proton-neutron mass difference is neglected. Find the ground state energy of this system.

3.46 Transmission and Reflection (A)

A particle of mass m moves in a one dimensional potential $V(x)$. The potential vanishes exponentially fast as $x \to \pm\infty$, and $V(-x) = V(x)$. Let $\psi(x)$ be a solution of the time independent Schrodinger equation with a positive energy E,

$$H\,\psi(x) = E\,\psi(x)\,. \tag{$*$}$$

A. [**5** points] Show that the Schrodinger equation $(*)$ has two linearly independent solutions.

B. [**5** points] Show that these solutions can be chosen to be eigenstates $\psi_\pm$ of the parity operator P (defined by $P\psi(x) \equiv \psi(-x)$),

$$P\,\psi_+(x) = +\psi_+(x)\,,$$
$$P\,\psi_-(x) = -\psi_-(x)\,.$$

C. [**5** points] Show that $\psi_+(x)$ and $\psi_-(x)$ have the asymptotic behaviors

$$\psi_+(x) \sim \cos(kx + \delta_+(E))\,,$$
$$\psi_-(x) \sim \sin(kx + \delta_-(E))\,,$$

as $x \to +\infty$ (neglecting overall multiplicative factors), where $k \equiv \sqrt{2mE/\hbar^2}$, and the phase shifts $\delta_\pm(E)$ are real functions of E that vanish when $V(x) = 0$.

D. [**5** points] Consider an alternate set of two linearly independent solutions $\psi_1(x)$ and $\psi_2(x)$ satisfying the boundary conditions

$$\psi_1(x) \sim e^{ikx} \quad \text{as } x \to +\infty,$$
$$\psi_2(x) \sim e^{-ikx} \quad \text{as } x \to -\infty.$$

In the opposite limits,

$$\psi_1(x) \sim a(E)\,e^{ikx} + b(E)\,e^{-ikx} \quad \text{as } x \to -\infty,$$
$$\psi_2(x) \sim a'(E)\,e^{-ikx} + b'(E)\,e^{ikx} \quad \text{as } x \to +\infty.$$

Find $a'(E)$ and $b'(E)$ in terms of $a(E)$ and $b(E)$. Explain your reasoning.

E. [**25** points] A particle of energy E is incident on the potential from $x = -\infty$. Show that the probability R that the particle is reflected, and the probability T that the particle is transmitted, are given by

$$R = |r(E)|^2 \quad \text{and} \quad T = |t(E)|^2\,,$$

with the reflected and transmitted amplitudes determined by the phase shifts,

$$r(E) = \tfrac{1}{2}(e^{2i\delta_+} - e^{-2i\delta_-})\,,$$
$$t(E) = \tfrac{1}{2}(e^{2i\delta_+} + e^{-2i\delta_-})\,.$$

F. [**5** points] What is the reflection coefficient R for a particle of energy E incident from $x = +\infty$?

3.47 Transmission and Reflection (B)

Consider the potential $V(x) = -\alpha\,\delta(x)$, with $\alpha > 0$.

A. [**20** points] Find all bound state energies and wavefunctions, and give their parities.

B. [**20** points] Now consider positive energy scattering states. Show that $\tan\delta_+(E) = \sqrt{m\alpha^2/(2\hbar^2 E)}$, and find $\delta_-(E)$.

C. [**10** points] Find the reflection coefficient for a particle of energy E incident on the potential from $x = -\infty$, and show that the amplitude of the reflected wave $r(E)$, analytically continued to complex E, has a pole at $E = -m\alpha^2/(2\hbar^2)$. What is the physical significance of this value of E? [Hint: Note that $\frac{1}{2i}(e^{i\delta} - 1) = e^{i\delta}\sin\delta$.]

3.48 Two-dimensional Oscillator

Consider a two-dimensional harmonic oscillator with potential

$$V(x) = \tfrac{1}{2}\left(k_1\,x^2 + k_2\,y^2\right).$$

Let $E_{n_x n_y}$ denote the energy of the unpertuebed state $|n_x, n_y\rangle$. Add a perturbation $\Delta H \equiv \lambda\, xy$ and determine, at first order in λ, the shift in the eigen-energies E_{00}, E_{10}, and E_{01}. Do this:[1]

A. [**5** points] For the non-symmetric case, $k_1 \neq k_2$.

B. [**10** points] For the symmetric case, $k_1 = k_2$.

C. [**10** points] Describe the transition between cases A and B, *i.e.*, determine the eigenenergies for the "quasi-symmetric case".

D. [**5** points] What criterion determines the domain of applicability of the result in part A? What criterion determines the domain of applicability of the result in part B?

[1] For a one-dimensional harmonic oscillator, recall that $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$ with $a^\dagger|n_x\rangle = \sqrt{n_x+1}\,|n_x{+}1\rangle$ and $a|n_x\rangle = \sqrt{n_x}\,|n_x{-}1\rangle$.

3.49 Two Spins

Two spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \,,$$

where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are Pauli spin matrices acting on the indicated particle.

A. [**5** points] Rewrite the Hamiltonian in terms of

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

B. [**5** points] Rewrite the Hamiltonian as a 4×4 matrix using the eigenstates of σ_1^z and σ_2^z as a basis.

C. [**5** points] Find the energy levels and eigenstates of H using this 4×4 matrix.

D. [**5** points] Write the rotation operator $R(\alpha)$, which rotates the spins through an angle α about the x-axis, in terms of Pauli-spin operators σ^x, σ^y, and σ^z.

E. [**5** points] Show that for $\alpha = \pi/2$ your rotation operator reduces to

$$R(\pi/2) = \tfrac{1}{2}(1 - i\sigma_1^x)(1 - i\sigma_2^x)\,.$$

F. [**5** points] Show by explicit evaluation that $[R(\pi/2), H] = 0$.

G. [**5** points] Rewrite H in terms of the total angular momentum $\boldsymbol{J} \equiv \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$.

H. [**5** points] Relate and identify the eigenstates of H in terms of those of definite angular momentum.

I. [**5** points] How do the eigenstates of H transform under the rotation $R(\pi/2)$?

J. [**5** points] What is the ground state expectation value of $\sigma_1^x\,\sigma_2^x$?

3.50 Unreal Helium

Consider two different single electron atomic wavefunctions $u_a(\boldsymbol{r}, s)$ and $u_b(\boldsymbol{r}, s)$. which are occupied by two electrons. Neglect the Coulomb repulsion between the electrons.

A. [**7** points] Show that a properly normalized two-particle wave function is

$$\Psi(r_1, r_2, s_1, s_2) = \frac{1}{\sqrt{2}}\big[u_a(1)u_b(2) - u_b(1)u_a(2)\big]\,,$$

where 1 and 2 denote both space and spin coordinates of the indicated electron.

B. [**7** points] Show that the expectation value of the mean square separation of the electrons is

$$\big\langle(\Delta r)^2\big\rangle = (r^2)_a + (r^2)_b - 2(\boldsymbol{r})_a \cdot (\boldsymbol{r})_b + 2|\boldsymbol{r}_{ab}|^2\,.$$

where

$$(\boldsymbol{r})_a \equiv \int d^3r\; u_a^*(\boldsymbol{r}, s)\,\boldsymbol{r}\,u_a(\boldsymbol{r}, s)\,,$$

$$(r^2)_a \equiv \int d^3r\; u_a^*(\boldsymbol{r}, s)\,r^2\,u_a(\boldsymbol{r}, s)\,,$$

$$\boldsymbol{r}_{ab} \equiv \int d^3r\; u_a^*(\boldsymbol{r}, s)\,\boldsymbol{r}\,u_b(\boldsymbol{r}, s)\,,$$

and summation over spins is implied.

C. [**7** points] Show that the expectation value of $\boldsymbol{r}$ vanishes if states a and b have the same parity, so that

$$\big\langle(\Delta r)^2\big\rangle = (r^2)_a + (r^2)_b + 2|\boldsymbol{r}_{ab}|^2\,.$$

D. [**7** points] If the two electrons also have different spin directions, show that $\boldsymbol{r}_{ab} = 0$ so that

$$\big\langle(\Delta r)^2\big\rangle = (r^2)_a + (r^2)_b\,.$$

E. [**7** points] Now show that electrons having the same spin directions are on average further apart than those with different spin directions. The contrast between parts D and E is an example of what fundamental physical effect?

Chapter 4

Thermodynamics and Statistical Mechanics

4.1 Atmospheric Physics

The thickness of the atmosphere is small compared to Earth's radius, so one may consider the acceleration of gravity g to be constant. Assume that the atmosphere is entirely nitrogen.[1]

A. [**15** points] The condition for hydrostatic equilibrium can be written as

$$\frac{dP(z)}{dz} = f\big(P(z), T(z)\big)\,,$$

where z is the height above the Earth's surface. Find the function f. Check that for $T =$ const. the result is consistent with Boltzmann's distribution.

B. [**10** points] The temperature profile $T(z)$ is often approximated by a linear function of height,

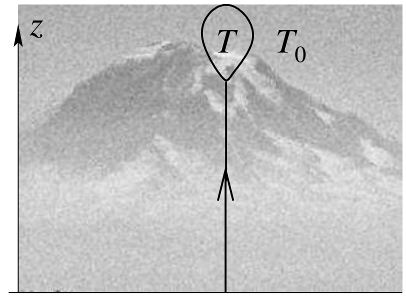
$$T(z) = T_0 - \Gamma\, z\,.$$

Show that the pressure $P(z)$ is proportional to a power of temperature,

$$\frac{P(z)}{P(0)} = \left(\frac{T(z)}{T(0)}\right)^{\beta}\,,$$

and find the exponent β.

[1] Possibly useful information: Boltzmann's constant $k_{\rm B} \simeq 8.6\times10^{-5}\,{\rm eV/K}$, proton mass $m_p \simeq 938\,{\rm MeV}/c^2$, electron mass $m_e \simeq 0.511\,{\rm MeV}/c^2$, deuteron mass $m_d \simeq 1875\,{\rm MeV}/c^2$.



Now consider an ideal hot air balloon, whose envelope is made of a very elastic material that can stretch and contract with almost no tension. In other words, the pressure inside the balloon differs negligibly from the pressure of the surrounding air. The envelope of the balloon is also an ideal thermal insulator, with negligible mass.

Initially the balloon is at sea level, $z = 0$, occupies a volume V_0, and has the same temperature T_0 as the surrounding air. The air inside the balloon is heated to some temperature $T_1 > T_0$. The balloon is then released and starts to rise.

C. [**20** points] Show that if β is smaller than some β_{crit}, or $\Gamma > \Gamma_{\text{crit}}$, then the balloon will go up forever (at least until one of our assumptions breaks down). Find the numerical value for Γ_{crit}.

D. [**15** points] Assume now that the temperature does not depend on altitude, or $\Gamma = 0$. By attaching a (nearly massless) string to the balloon, the ascent of the balloon may be used to do work. The balloon pulls the string during its very slow ascent to its maximum height. What fraction of the initial heat Q used to warm the air inside the balloon from T_0 to T_1 is converted to work? Give a numerical value of this fraction for $T_1 = 2T_0$.

4.2 Basic Kinetic Theory (this problem has 2 pages)

A. [**15** points] A homogenous, isotropic gas has a velocity ($\vec{v}$) distribution such that $\langle \vec{v} \rangle = 0$ and $\frac{m}{2}\langle v^2 \rangle = \frac{3}{2} k_\mathrm{B} T$, where k_B is the Boltzmann constant and T the temperature. The brackets indicate averaging over all N molecules. Assume that the gas molecules have zero size and experience no collisions among themselves. The gas is placed in a container of volume V whose side walls have a mass much larger than the molecule mass m. The wall is impenetrable to the molecules, they reflect back elastically. Derive the average momentum imparted on a single planar wall per unit time and unit area. From this show that the pressure of the gas is given by the ideal gas law

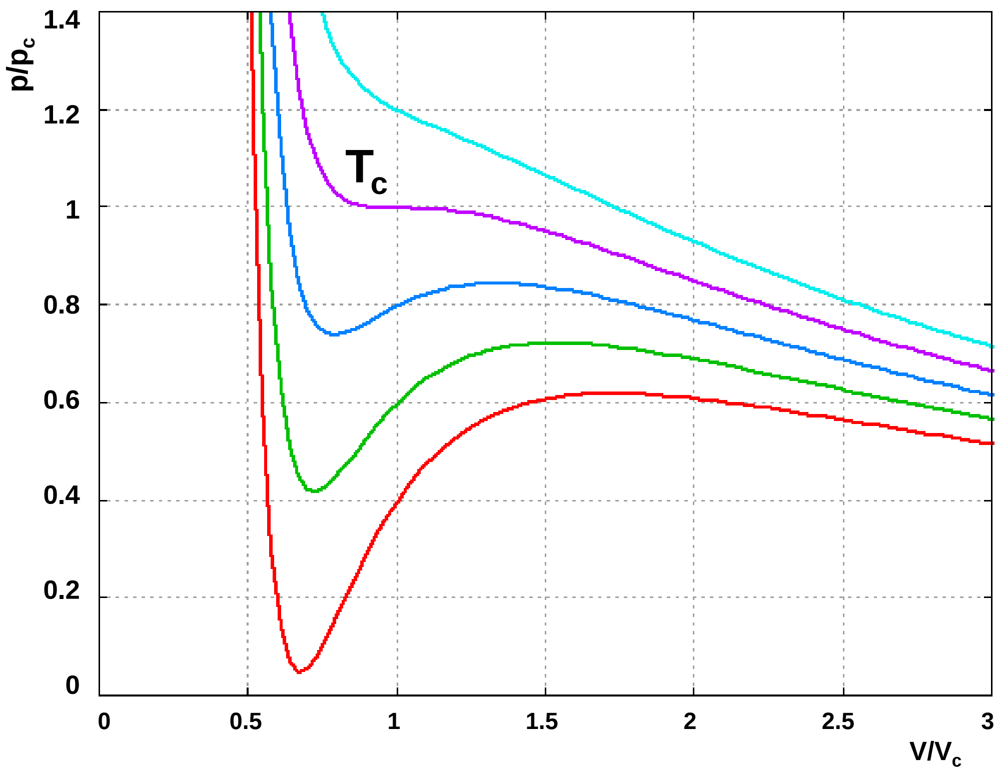
$$pV = N k_\mathrm{B} T.$$

Part B-D: For the remainder of the problem study the following generalization of the ideal gas. The effects of finite size molecules as well as 2-body short range interactions can be modeled by the van der Waals equation of state

$$\left(p + \frac{aN^2}{V^2} \right) \left(\frac{V}{N} - b \right) = k_\mathrm{B}\, T\,,$$

where a and b are material dependent constants.

B. [**15** points] The isothermal curves $p(V)$ of this van der Waals gas are displayed in the graph on the top of the next page. In particular, note the occurrence of two extrema for low temperatures that smoothly merge at a point with $p = p_c$, $V = V_c$ and $T = T_c$. Determine p_c, V_c and T_c in terms of a, b and N.



C. [**15** points] Determine the sign of the compressibility

$$\beta = -\frac{1}{V}\frac{\partial V}{\partial p}\bigg|_{T \text{ fixed}}$$

for T above and below T_c, and explain the consequences the sign of β has for the stability of the system. Use this to explain what happens to a physical system when its volume is reduced isothermally at $T < T_c$. In particular, explain whether it follows the isotherm and, if not, what it does instead.

D. [**5** points] Based on these considerations, sketch the phase diagram of the van der Waals gas in the $p-T$ plane. For any phase transition, indicate whether the transition is first or higher order.

4.3 Bose-Einstein Condensation

Consider a non-interacting non-relativistic Bose gas in a macroscopic three-dimensional box of volume V.

A. [**10** points] Write down the appropriate partition function and derive an equation that gives the occupation number as a function of energy at a given temperature and chemical potential.

B. [**10** points] Consider a gas composed of a finite number of particles. Make a sketch of the chemical potential as a function of temperature. Does it go to zero? If so, indicate wether this happens at $T \to 0$ or at some other temperature.

C. [**10** points] Compute the critical temperature, T_c, above which practically all the particles are in excited states, but below which a significant number is in the ground state.

D. [**10** points] Does the pressure at low temperatures ($T < T_\mathrm{c}$) depend on the particle density? If yes, how. If no, explain. What is the pressure of the ideal Bose gas at $T = 0$?

Consider now what happens for an infinite system, $V \to \infty$, in different dimensions, $1d$, $2d$, $3d$:

E. [**10** points] Can a non-relativistic ideal Bose gas of a given number density of particles undergo Bose condensation in $d = 1, 2, 3$ dimensions? Explain.

Useful Mathematical formulas

$$
\begin{aligned}
\int_0^\infty x^{n-1}\, e^{-x}\, dx &= \Gamma(n), \text{ with } n > 0 \\
\Gamma(n+1) &= n\,\Gamma(n); \quad \Gamma(1/2) = \sqrt{\pi} \\
\int_0^\infty e^{-ax^2}\, dx &= \tfrac{1}{2}\sqrt{\tfrac{\pi}{a}} \\
\int_0^\infty x^2 e^{-ax^2}\, dx &= \tfrac{1}{4}\sqrt{\tfrac{\pi}{a^3}} \quad \text{when } a > 0 \\
\int_0^\infty \tfrac{\sin^3(x)}{x^3}\, dx &= \tfrac{3\pi}{8} \\
\int_0^\infty \tfrac{x^{n-1}\,dx}{e^x - 1} &= \Gamma(n)\ \left(\tfrac{1}{1^n} + \tfrac{1}{2^n} + \tfrac{1}{3^n} + \cdots\cdot\right) \\
\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx &= \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\, dx = \tfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot 2n+1}\,\tfrac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx &= \int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx = \tfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}\,\tfrac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty \tfrac{x}{e^x - 1}\, dx &= \tfrac{\pi^2}{6} \\
\int_0^\infty \tfrac{x^3}{e^x - 1}\, dx &= \tfrac{\pi^4}{15} \\
\int_0^\infty \tfrac{\sin(x)}{x}\, dx &= \tfrac{\pi}{2} \\
\int_0^\infty e^{-ax^2} \cos bx\, dx &= \tfrac{1}{2}\sqrt{\tfrac{\pi}{a}} e^{-b^2/4a} \\
\int_0^\infty \ln \tfrac{e^x + 1}{e^x - 1}\, dx &= \tfrac{\pi^2}{4}
\end{aligned}
$$

4.4 Classical and Quantum Statistics

Consider a particle (with further characteristics to be specified below) having two energy levels, 0 and ε. You are given a system consisting of two such noninteracting particles in contact with a thermal reservoir at temperature T.

A. [**16** points] What are the energy levels and degeneracies of the *system* if:

- **i.** the particles obey classical statistics and are distinguishable?
- **ii.** the particles obey Bose-Einstein statistics with spin-0?
- **iii.** the particles obey Fermi-Dirac statistics with spin-1/2? In addition to the degeneracy also specify the total spin in each energy level.
- **iv.** the particles obey Bose-Einstein statistics with spin-1? In addition to the degeneracy also specify the total spin in each energy level.

B. [**8** points] In statistical mechanics, one often finds that the canonical partition function for a system of N indistinguishable, noninteracting particles is $Z = Z_1^N/N!$ where Z_1 is the partition function for a single particle. Briefly explain the range of validity of this result, and in particular explain why it does not hold for case A-ii above.

C. [**26** points] Suppose you are told that the system above is definitely composed of either spin-1/2 fermions or spin-1 bosons.

- **i.** [**5** points] What is a thermodynamic observable which can be used to distinguish the two cases? Briefly explain why, on general grounds, you expect that your observable can distinguish between the two cases.
- **ii.** [**16** points] For both cases (fermion or boson) calculate the temperature dependence of your chosen observable from part i.
- **iii.** [**5** points] Give a qualitative description of how one could actually measure your observable from part i.

4.5 Clausius-Clapeyron Relation

Consider a fixed amount (say, N molecules) of a substance than can be in two phases (*e.g.*, liquid and gas, or liquid and solid). Consider the (T, P) phase diagram. Let the phase-transition line be given by $P = P_{\mathrm{p.t.}}(T)$.

A. [**10** points] Write down the condition of equilibrium on the phase-transition line.

B. [**15** points] Show that the slope of this line is

$$\frac{dP_{\mathrm{p.t.}}}{dT} = \frac{Q}{T\,(V_2 - V_1)}\,,$$

where Q is the heat of transition, and V_1 and V_2 are the volumes the two phases occupy on the two sides of the phase-transition line.

C. [**5** points] Does the temperature of transition from ice to water increase or decrease with increasing pressure?

D. [**20** points] Consider a liquid-gas phase transition. Suppose the volume in the linquid phase is much smaller than the volume in the gas phase, $V_1 \ll V_2$. Assume the gas is ideal, and the the heat of transition is constant. Show that the logarithm of the phase transition pressure has the functional form

$$\ln P_{\mathrm{p.t.}}(T) = A - B/T\,.$$

What is the constant B?

4.6 Cold Copper

A. [**20** points] Why is the specific heat of copper at low temperatures, around 1 K or less, proportional to temperature? What is the main reason for the deviation of the specific heat from linearity at slightly higher temperatures? *A few sentences outlining the essential physics is sufficient.*

B. [**25** points] The spin of a copper nucleus is 3/2, so the energy levels of a copper nucleus in a magnetic field B are of the form $-mB\mu$, where m has the values 3/2, 1/2, $-1/2$, $-3/2$. Derive expressions for the magnetization M and magnetic energy U_{M} of noninteracting copper nuclei in a magnetic field B at temperature T, in the temperature region

$$k_{\mathrm{B}}T \gg \mu B\,,$$

where k_{B} is Boltzmann's constant.

C. [**25** points] Show that the contribution of the nuclear spins to the entropy per atom can be written in the form

$$S_{\mathrm{M}} = k_{\mathrm{B}} \ln 4 - \frac{5\mu^5 B^2}{8k_{\mathrm{B}}T^2}\,,$$

in this temperature regime. *For this part, you may find it helpful to use the thermodynamic identity $\partial U_{\mathrm{M}}/\partial T = T\partial S_{\mathrm{M}}/\partial T$. Both sides are expressions for the heat capacity.*

D. [**30** points] If the nuclei in copper metal are magnetized at a temperature of 0.20 K in a field of 0.70 T, and then the magnetic field is switched off reversibly and adiabatically, what will be the final temperature of the metal when the field B is zero?

[The specific heat of copper at low temperature T, in the absence of a strong magnetic field, is $\gamma k_{\mathrm{B}}T$ per atom, where $\gamma = 8.36 \times 10^{-5}$ K^{-1}. For a copper nucleus, $\mu = 7.47 \times 10^{-27}$ J/T. The Boltzmann constant $k_{\mathrm{B}} = 1.38 \times 10^{-23}$ J/K.]

4.7 Free Expansion

Consider a process in which a gas of initial temperature T occupying an initial volume V expands its volume from V to $V + \Delta V$ in such a way that the total internal energy E of the gas remains constant. The goal of this problem is to work out the associated change in temperature.

A. [**10** points] Describe how such an expansion from V to $V + \Delta V$ at constant E could be realized experimentally.

B. [**10** points] Show that for ideal gases the temperature drop due to the expansion has to be zero. A few sentences are sufficient.

C. Now consider a non-ideal gas whose pressure is no longer governed by the ideal gas expression but instead is taken to be a general function of the temperature and the volume, $P = P(T, V)$. The goal in this subproblem is to determine the temperature drop due to the expansion of this non-ideal gas. The answer will be expressed in terms of $P(T, V)$ and its derivatives.

i. [**8** points] Suppose the change in volume δV is small. We want to determine the differential change in temperature, $(\partial T/\partial V)_E$. Show that this quantity obeys the following relation:

$$\left(\frac{\partial T}{\partial V}\right)_E = -\frac{(\partial E/\partial V)_T}{(\partial E/\partial T)_V}.$$

ii. [**15** points] Express $(\partial T/\partial V)_E$ in terms of $P = P(T, V)$ (and its derivatives), and the specific heat $C_V = (\partial E/\partial T)_V$.

D. [**7** points] Determine the change in entropy per unit volume associated with this expansion of the gas and show that it is always positive.

4.8 Einstein solids

Throughout this problem you may need to make use of the relationship between entropy, S, and the number of quantum states or multiplicity of the system, Ω

$$S = k \ln \Omega$$

and of the definition of temperature, $1/T = (\partial S/\partial U)_V$.

A. [**12** points] Consider an Einstein solid, i.e., an idealized model for a crystal with N atoms, as N independent oscillators, each capable of oscillating in the 3 dimensions of space with energy given by $E = (n + 3N/2)\ \hbar\omega$. Show that the number of quantum states of the system for n quanta of energy distributed in N 3d oscillators is given by:

$$\Omega = \frac{(n + 3N - 1)!}{n!(3N - 1)!}$$

Explain clearly how you obtain the number of quantum states. Calculate the entropy.

(Hint: It may help you to first calculate the number of ways of distributing one quantum of energy, $n = 1$, in one 3d oscillator and to repeat for three quanta of energy, $n = 3$, in one 3d oscillator...)

B. [**10** points] Use the definition of temperature to calculate the temperature dependence of the internal energy, $U = n\hbar\omega$, of an Einstein solid at high temperatures. Calculate the specific heat at high temperatures.

C. [**14** points] Two Einstein solids, A and B, with number of atoms $N_A \gg 1$ and $N_B \gg 1$ are placed in thermal contact. Assume further that there are many energy quanta per oscillator so that $n \gg N_A,\ N_B$. Calculate what fraction of the total energy is in solid A at equilibrium. Calculate the entropy of each of the solids and of the system at equilibrium.

D. [**14** points] Imagine that aliens deliver into your hands two objects, C and D, made of substances whose multiplicities increase linearly with internal energy and number of 'atoms', like $\Omega = a\ N\ U/\epsilon$, where ϵ is some quantum of energy and a is a constant. The objects have initial internal energies $U_D^i = 2\ U_C^i$ but D has four times the number of 'atoms' as C, $N_D = 4\ N_C$. When placed in thermal contact, will energy spontaneously flow from C to D or from D to C? What will be the ratio of the final internal energies, U_C^f/U_D^f, long after they have been placed in thermal contact?

4.9 Entropy Change

A. [**5** points] Consider a monoatomic dilute gas in a container of volume V. Let $P(\boldsymbol{v})$ be the velocity probability distribution of each molecule. Which of the following formulas represents the entropy associated with this distribution function?

$$S = k_{\rm B} \int d^3v \, \log P(\boldsymbol{v}) \,, \tag{1}$$

$$S = -k_{\rm B} \int d^3v \, P(\boldsymbol{v}) \log P(\boldsymbol{v}) \,, \tag{2}$$

$$S = k_{\rm B} \int d^3v \, e^{-P(\boldsymbol{v}) \log P(\boldsymbol{v})} \,. \tag{3}$$

B. [**10** points] The total entropy S of a monoatomic ideal gas is the sum of two terms: its positional entropy S_x and its momentum entropy S_v. Positional entropy S_x is of no interest to us; it remains constant throughout this question because the volume does not change. At time $t = 0$ the system is prepared in a configuration where each particle has velocity $\boldsymbol{v}$ with probability

$$P(\boldsymbol{v}) = \begin{cases} (\frac{4}{3}\pi v_0^3)^{-1} & \text{if } |\boldsymbol{v}| \le v_0 \,; \\ 0 & \text{if } |\boldsymbol{v}| > v_0 \,. \end{cases}$$

Calculate the ideal gas entropy S_v for this initial condition.

C. [**10** points] Calculate the internal energy U_0 of the ideal gas in this initial configuration.

D. [**10** points] The above gas is allowed to reach thermal equilibrium while in thermal contact with the surrounding air, which is at temperature T_R. The initial condition in part B was chosen such that $U_0 = \frac{3}{2} N k_{\rm B} T_R$.

Derive the formula for the entropy of this gas after it reaches thermal equilibrium. There are many ways of doing this: *e.g.*, as a derivative of the canonical partition function, or by choosing the correct formula in part A.

E. [**5** points] How much heat has been exchanged between the gas and the surrounding air during the equilibration process?

F. [**10** points] Did the gas gain or lose entropy? Did the surrounding air gain or lose entropy? Base your answers on the second and first laws of thermodynamics, and reconcile this with your results in the above parts.

4.10 Equipartition

A. [**10** points] Formulate in words the content of the equipartition theorem in classical statistical mechanics.

B. [**10** points] An ideal gas is localized in space by a radial potential,

$$U(r) = \kappa\, r^3\,.$$

What is the contribution of the potential to the average energy per particle when the gas is at a temperature T?

C. [**10** points] Discuss your result in part B in the context of your answer in part A.

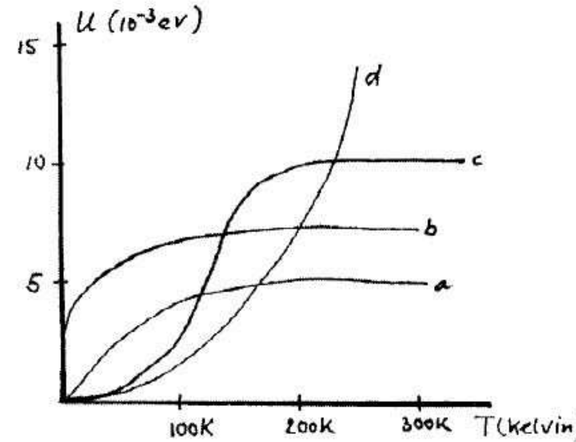
4.11 Excitable Particles

Consider a set of non-interacting, stationary particles, each having two internal energy levels. The excited state lies ϵ above the ground state in energy and is n-fold degenerate.

A. [**5** points] Calculate the Helmholtz free energy $f(T)$ per particle.

B. [**10** points] The figure on the right shows a set of internal energy per particle curves $u(T)$. One of them corresponds to this two level system — which one?

C. [**10** points] Estimate the energy gap ϵ and the degeneracy n of the excited state from the $u(T)$ curve in the figure. (Recall that $k_{\mathrm{B}}T \simeq 1/40$ eV at room temperature.)



4.12 Gambling

Gambling may be considered as a one-dimensional random walk. The probability q of moving to the left and the probability $p = 1-q$ of moving to the right in the random walk represent the probability that the gambler loses or wins on a specific bet. For each step n, the step size s_n to the left (losing) is equal to the amount bet b_n. The step size to the right (winning) is D times larger, Db_n. In other words, the gambler receives D times the amount of the bet when he wins. The parameters p, q and D are set by the gambling establishment. The gambler has the freedom to choose the total number of bets N and the amount b_n of each bet, for a given total gambling budget B, where $B = \sum_{n=1}^{N} b_n$. Assume for this problem that the gambler keeps any proceeds of each individual bet b_n separate from his original budget B and does not reinvest them.

A. [**15** points] The "first law of gambling" states that there is no way of varying the number and amount of one's bet within a given budget B that will enhance ones "expected" (average) winnings, $\bar{w}$, defined by:

$$\bar{w} = \sum_{n=1}^{N} \bar{s}_n \,,$$

where $s_n \in \{-b_n, Db_n\}$ is the result of a given bet b_n, and $\bar{s}_n$ is the average result if it could be repeated many times. Calculate $\bar{w}$. Show that it is consistent with the first law of gambling stated above.

B. [**9** points] The mean square fluctuation in the expected winnings is given by:

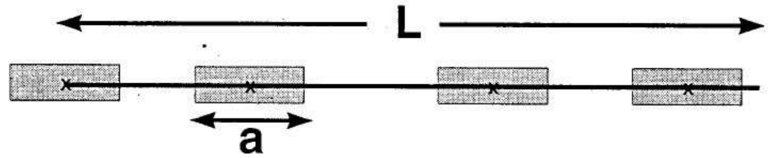
$$\overline{\Delta w^2} = \sum_{n=1}^{N} \overline{\Delta s_n^2} = \left[pD^2 + q - (pD - q)^2\right] \sum_{n=1}^{N} b_n^2 \,.$$

How do these fluctuations scale with the number of bets N in the special case where all bets are for the same amount, $b_n = b$? How do they scale with N for equal sized bets within a given budget B?

C. [**8** points] Describe (without extensive calculations) how the gambler can maximize or minimize these fluctuations by using his freedom to choose both N and the distribution of bets $\{b_n\}$ for a given fixed budget B.

D. [**8** points] Gambling establishments set the odds against you — they select p, q and D so that $\bar{w} < B$. Describe how the gambler can use the fluctuations to maximize his changes of ending up with more money than he started with.

4.13 Hard Rods



Consider a one-dimensional gas of N identical hard "rods" of length a and mass m, whose centers are confined to a region of length L (see figure). The Hamiltonian for this system can be taken to be

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{1 \le i < j \le N} \phi(|x_j - x_i|) \,, \tag{1}$$

where x_i denotes the location of the center of the i'th rod, and $\phi(\Delta x)$ is an interaction potential,

$$\phi(\Delta x) = \begin{cases} +\infty & \text{for } |\Delta x| < a\,; \\ 0 & \text{for } |\Delta x| > a\,. \end{cases} \tag{2}$$

Note that this form for the potential implies that the centers of neighboring rods can't get closer than a distance a apart, and that otherwise they don't interact.

A. [**14** points] By integrating over momenta, the partition function can be written in the form

$$Z(N,L,T) = \frac{A^N}{N!} \int_0^L dx_1 \int_0^L dx_2 \ldots \int_0^L dx_N \, e^{-\beta \sum_{1 \le i < j \le N} \phi(|x_j - x_i|)} \,, \tag{3}$$

where $\beta \equiv 1/(k_{\mathrm{B}}T)$. What is the coefficient A? Your answer should be given in terms of m and T.

B. [**14** points] Show that

$$Z = \left(A\,[L-(N-1)\,a]\right)^N / N! \,, \tag{4}$$

by performing the x_i integrations, noting that the exponential in Eq. (3) is either 0 or 1 because of the form assumed for the potential ϕ. One way to do this is to restrict the integration to the case $0 \le x_1 < x_3 \ldots x_N \le L$, and then multiply the answer by $N!$ to account for all permutations of this configuration.

C. [**14** points] Compare the result for Z in Eq. (4) for a gas of rods to the partition function for N non-interacting particles confined to a line of length L (an ideal gas) and explain the difference between the two.

D. [**14** points] In the limit that $N \to \infty$ and $L \to \infty$, with the density $\rho \equiv N/L$ held fixed, compute the free energy per particle, $f = -\ln Z/(N\beta)$, from the partition function in Eq. (4). Use Stirling's formula, $\ln N! \simeq N \ln N - N$.

E. [**14** points] In the same limit as in part D, compute the pressure of the gas as a function of the density ρ (*i.e.*, compute the equation of state).

4.14 Harmonic Oscillator

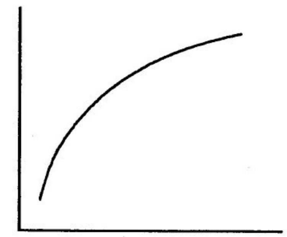
Consider a harmonic oscillator which has energy levels $E_n = (n + \frac{1}{2})\,\hbar\omega$.

A. [**8** points] Give an estimate of the temperature T, below which significant deviations from classical statistical behavior occur.

B. [**8** points] Calculate the entropy of the harmonic oscillator as a function of temperature.

C. [**9** points] Take the classical limit for the entropy and give a physical explanation for its value.

4.15 Helmholtz Free Energy



Someone measured the Helmholtz free energy F as function of temperature T for a particular system at constant volume V. The figure shows a schematic plot of the results. The axes are not labeled. It might be a plot of $F(T)$ or $T(F)$. The following line of arguments will determine which one it is:

A. [**7** points] Transform the conventional form of the second law of thermodynamics, based on $dQ = dU + p\,dV$ and the relation between entropy, S, and heat, into an equivalent form involving $F(T, V)$ by means of a Legendre transformation, and show that

$$\left(\frac{\partial F}{\partial T}\right)_V = -S\,.$$

B. [**7** points] Explain why the isochoric heat capacity, C_V, must be positive.

C. [**8** points] Express C_V/T as a second derivative of F.

D. [**8** points] Is $F(T)$ convex or concave? Which of the two variables is plotted along the vertical axis?

4.16 Ideal Fermions in any Dimension

Consider an ideal gas of spin-$\frac{1}{2}$ fermions with energy spectrum $\omega_p \propto p^\alpha$, confined in a box of volume V in n spatial dimensions.

A. [**10** points]

i. From the definition of the partition function of the grand canonical ensemble,

$$Z = \sum_n e^{-\beta(E_n - \mu N_n)} \, ,$$

show that for our system of fermions the partition function can be written as

$$Z = \prod_{p,s} \left[1 + e^{-\beta(\omega_p - \mu)}\right] ,$$

where the product is taken over all values of momentum and spin of a single electron.

ii. Define the thermodynamic potential $\Omega(V, T, \mu)$ so that $Z = e^{-\beta\Omega}$. This thermodynamic potential is related to the pressure through $\Omega = -PV$. Show that the pressure for our system is

$$P = 2T \int \frac{d^n p}{(2\pi\hbar)^n} \, \ln\left[1 + e^{-\beta(\omega_p - \mu)}\right] \, .$$

B. [**10** points] Show that the internal energy E and pressure P satisfy $E = (n/\alpha)PV$.

C. [**10** points] Show that the pressure is an homogeneous function of T and μ of order $(n/\alpha) + 1$,

$$P = T^{(n/\alpha)+1} f\left(\frac{\mu}{T}\right) ,$$

where f is some univariate function. In other words, if one increases both T and μ by a factor of λ, then P increases by a factor of $\lambda^{(n/\alpha)+1}$.

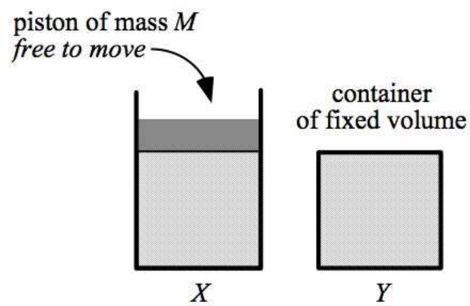
D. [**15** points] By using the result derived in part C, and thermodynamic relations, show that in adiabatic processes, the ratio μ/T remains constant.

E. [**5** points] Show that the equation for an adiabat is $PV^{1+(\alpha/n)} = \text{const}$.

4.17 Ideal Gas Entropy

Throughout this question, clearly state and justify any assumptions or approximations you make. Note that parts A and B are independent.

A. [**15** points] Two samples of an ideal gas (X and Y) are each confined in upright cylinders. Both are initially in equilibrium and are at the same pressure P_0 and temperature $T_0 = 300$K. Each occupies a volume V_0. The cylinder containing sample X is sealed at the top by a piston of mass M. The piston can move without friction, but no gas can enter or leave the cylinder. The cylinder containing sample Y has a fixed volume.



Initial conditions

Each cylinder is placed in thermal contact with a reservoir at $T_{\text{res}} = 400$K. The two samples are allowed to come to equilibrium with the reservoir. (The system of cylinders and reservoir is isolated from the rest of the universe.)

i. Can the process of heating sample X be considered reversible or must it be considered irreversible? Explain.

ii. Is the entropy change of sample X greater than, less than, or equal to the entropy change of sample Y? Explain. (If either sample undergoes no entropy change, state so explicitly.)

B. [**20** points] Two fixed partitions divide an insulating box into three compartments of equal volume. The left and right compartments each contain n moles of an ideal gas at standard temperature and pressure; the middle compartment is evacuated. Two processes, each starting from the same initial conditions, are described below.

In process 1, the two partitions are removed simultaneously. The system is allowed to come to equilibrium. In process 2, the right-most partition is removed first. The system is allowed to come to equilibrium. Then the left-most partition is removed and the system is allowed to re-equilibrate.

i. Is the absolute value of the change in entropy of the system composed of the entire box and its contents in process 2 greater then, less than, or equal to that for process 1? Explain.

ii. Calculate the change in entropy of the system composed of the entire box and its contents for process 2.

4.18 Ideal Gas Zap

A. [**15** points] For an ideal gas, show that the molecular specific heats at constant volume (c_v) and pressure (c_p) are related by $c_p = c_v + k_B$, where k_B is Boltzmann's constant.

B. [**15** points] Consider an ideal monoatomic gas at room temperature and pressure, $T_0 = 300$K, $p_0 = 10^5$ Pa. A pulsed laser beam deposits an energy E in a small volume V in 10^{-9} seconds. If no volume increase occurs during this short interval, show that there is a pressure rise $\Delta p = (\gamma - 1)\, E/V$ and a temperature rise $\Delta T = (T_0/p_0)(\gamma - 1)(E/V)$, where $\gamma = c_p/c_v$.

C. [**15** points] Assume that this volume expands before there is any heat transfer to the surrounding gas. What is the final density of the gas? What is its final temperature? You may express your answers in terms of the initial pressure and temperature p_0 and T_0 of the room temperature gas and the Δp you found in part B.

D. [**7** points] It is straightforward to make nanosecond laser pulses with microJoules of energy that are focused to a diffraction limited spot. Assume the laser pulse deposits 10^{-6}J of energy uniformly in sphere of radius 2×10^{-6} m. Return to your formulas in parts B and C and use these values to estimate: (i) the local pressure rise Δp immediately following absorption of the laser pulse, (ii) the temperature rise ΔT, and (iii) the final temperature after expansion.

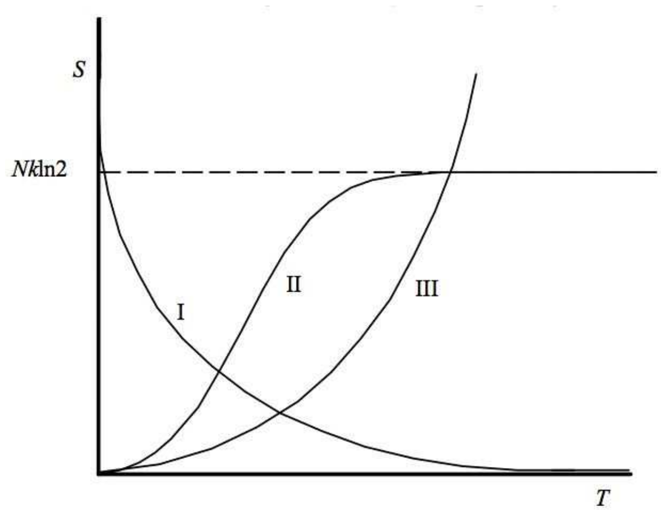
E. [**8** points] The answers you got in part D should be quite far above room temperature and pressure. Discuss the validity of the assumptions (energy deposition before expansion, and then expansion before heat loss) you made in obtaining these values.

4.19 Ideal Paramagnet

A system consists of N localized non-interacting spin 1/2 particles in a magnetic field $\boldsymbol{B} = B\,\hat{\boldsymbol{e}}_z$. The energy associated with each magnetic moment $\boldsymbol{\mu}$ is $\varepsilon = -\boldsymbol{\mu}\cdot\boldsymbol{B}$. The system is quantum-mechanical: the z-component of each magnetic moment is $\mu_z = s_z\, g\mu_{\mathrm{B}}$, where g is the g-factor, μ_{B} is the Bohr magneton, and s_z can take the values $\pm 1/2$.

A. [**10** points] To what value will the total energy of the system tend (i) as $T \to 0\,\mathrm{K}$? (ii) as $T \to \infty$? Explain. To what value will the entropy tend in the same limits? Explain.

B. [**10** points] Explain which of the following proposed graphs of entropy as a function of temperature (for some arbitrary field strength B) could possibly be correct. Qualitative reasoning is sufficient.



Explain how the graph would change, if at all, if (i) the field had a greater magnitude and (ii) the field were zero.

C. [**15** points] Derive an expression for the magnetization of the system as a function of temperature and applied field.

D. [**15** points] Show that the entropy of the system is given by:

$$S = Nk_B\left\{\ln\left[2\cosh\left(\frac{\beta g\mu_B B}{2}\right)\right] - \frac{\beta g\mu_B B}{2}\tanh\left(\frac{\beta g\mu_B B}{2}\right)\right\},$$

where $\beta = 1/(k_B T)$ with k_B the Boltzmann constant.

4.20 Joule Free Expansion

"Joule free expansion" is a type of process where a gas expands while in thermal and mechanical isolation. It can not exchange heat or mechanical energy with the environment, or any other object. The temperature of a classical dilute gas (an ideal gas) does not change during such a process. This is not true for a quantum gas. Consider a gas of fermions. For simplicity let it be one dimensional: N electrons moving freely along a wire of length L. (Ignore Coulomb interactions.) Assume that the energy eigenstates of the fermions are of the familiar form

$$E_n = \frac{\hbar^2 k^2}{2m}\,, \qquad k = \frac{2\pi n}{L}\,, \qquad \text{with } n = 0, \pm 1, \pm 2, \cdots .$$

A. [**15** points] Show that at zero temperature the Fermi momentum $k_{\rm F}$ and the Fermi energy $E_{\rm F}$ are functions of only the density $\rho \equiv N/L$, and have the form

$$k_{\rm F} \sim \rho^a\,, \qquad \text{and } E_{\rm F} \sim \rho^b\,.$$

Determine the exponents a and b.

B. [**10** points] Show that the total zero temperature energy per particle has the form

$$u(\rho) \sim \rho^c\,.$$

Determine the exponent c.

C. [**15** points] Assume that at room temperature the Fermi energy is much larger than $k_{\rm B}T$ such that the specific heat can be estimated in the same manner as for electrons in a metal. Only the fermions close to the Fermi level are thermodynamically active, and they can be treated as behaving classically, obeying the equipartition theorem. Show that the specific heat per particle has the form

$$c \sim \rho^d\, T^e\,.$$

Determine the exponents d and e.

D. [**15** points] The above results combine into the following low temperature approximation for the energy per particle

$$u(\rho, T) \simeq a_0\, \rho^c + a_1\, \rho^d\, T^{e+1}\,,$$

with a_0 and a_1 constants. Make some reasonable guesses for c, d, and e if you were unable to determine their values in parts A–C.

Suddenly the length of the wire changes and the fermions undergo a Joule free expansion process. (This might be realizable in the context of nanotubes.) Determine from the above formula whether the Fermi gas temperature increases or decreases during this free expansion process.

E. [**15** points] Does your answer agree with your physical intuition? Discuss this briefly.

4.21 Free Expansion

A gas of initial volume V and initial temperature T is allowed to expand into vacuum (which has zero pressure), thereby increasing its volume to $V + \delta V$. The task is to compute the associated temperature change of the gas, δT, as a result of the expansion.

A. [**10** points] Show that for ideal gases the temperature drop has to be zero. A few sentences are sufficient.

B. [**10** points] Suppose the change in volume δV is small. We want to determine the differential change in temperature, $(\partial T/\partial V)_E$. Show that this quantity obeys the following relation:

$$\left(\frac{\partial T}{\partial V}\right)_E = -\frac{(\partial E/\partial V)_T}{(\partial E/\partial_T)_V}.$$

C. [**20** points] Express $(\partial T/\partial V)_E$ in terms of the equation of state $P = P(T, V)$, and the specific heat $C_V = (\partial E/\partial T)_V$.

D. [**10** points] Determine the change in entropy per unit volume associated with this expansion of the gas and show that it is always positive.

4.22 Landau Diamagnetism

Solving Schrödinger's equation for an electron in a uniform magnetic field $\boldsymbol{B} = B\,\hat{\boldsymbol{e}}_z$ yields a set of energy levels (known as Landau levels) given by

$$\epsilon(p_z, n) = \frac{p_z^2}{2m} + \frac{\hbar e|B|}{mc}\left(n + \tfrac{1}{2}\right). \qquad (*)$$

Here, e is the electron charge, $p_z \in [-\infty, \infty]$ is the electron momentum in the z-direction, and the quantum number $n = 0, 1, 2, \cdots, \infty$. The degeneracy of each state is $g = 2L^2 e|B|/(2\pi\hbar c)$ when states are confined to lie within a cube of linear size L (with L much larger than any microscopic scale). The electron magnetic moment is neglected in the result $(*)$, and throughout the rest of this problem.

A. [**10** points] Write down the logarithm of the grand canonical partition function for this gas of (non-interacting) electrons in a magnetic field. Do not evaluate the sum or integral.

B. [**20** points] Calculate the fugacity $z = e^{\beta\mu}$ in the high-temperature limit. Express the result in terms of the particle density N/V.

C. [**20** points] In the same high-temperature limit, show that the zero-field magnetic susceptibility is negative (indicating diamagnetic response) and varies as $-1/T$. Compute the numerical coefficient multiplying $-1/T$.

4.23 Magnetic Cooling

A. [**10** points] Consider a system of N identical non-interacting spins with associated magnetic moment μ, at a finite temperature T, and in an external magnetic field of magnitude B. (This is a very crude model of paramagnetic salts.) Consider an adiabat on the (T, B) plane. Is dB/dT positive or negative on the adiabat? You only need to give a qualitative argument.

Now consider a piece of paramagnetic salt in an external magnetic field B. The energy of the system satisfies the equation

$$dE = T\,dS + B\,dM\,,$$

where M is the magnetization. The paramagnetic salt satisfies the Curie law: the magnetization, as a function of external field and temperature is

$$M = a\,\frac{VB}{T}\,,$$

where a is some constant and V is the volume, considered to be constant. It also follows the Schottky law: the specific heat at zero magnetic field is

$$C_B(T, B{=}0) = \left(\frac{\partial E}{\partial T}\right)_{B=0} = \frac{b\,V}{T^2}\,,$$

where b is another constant.

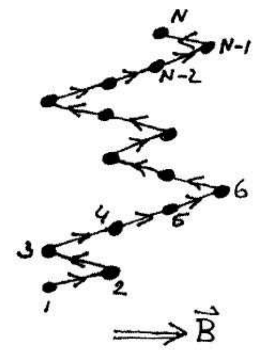
B. [**20** points] Show that the specific heat at constant magnetic field $c_B(T, B)$ is

$$c_B(T, B) = \frac{V}{T^2}\left(b + aB^2\right).$$

C. [**20** points] Find the equation for the adiabats on the (T, B) plane.

4.24 Magnetic One-dimensional Chain

A chain-like molecule is composed of N rigid straight line segments connected by hinges that can only be in two states, namely at an angle $\theta = 0$ or π. The figure on the right shows a typical configuration (but with the $\theta = \pi$ angles slightly smaller than 180° for graphical clarity). Each rigid line segment carries a magnetic moment μ aligned with the segment, and the magnetic field points in the x-direction, $\boldsymbol{B} = B\,\hat{\boldsymbol{e}}_x$. Since all line segments point in the x-direction, all magnetic moments point either along or opposite to the magnetic field.



A. [**15** points] Calculate the entropy $S(N,T)$ of the chain.

B. [**15** points] Calculate the average distance between the end points of the chain,

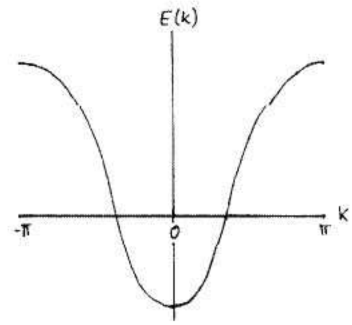
$$D(N,T) = \langle x(N) - x(0) \rangle \, .$$

C. [**15** points] Calculate the root-mean-square fluctuations in the endpoint separation,

$$\Delta D(N,T) = \sqrt{\langle [x(N) - x(0)]^2 \rangle - \langle x(N) - x(0) \rangle^2} \, .$$

D. [**15** points] Discuss your result in part C in the limit of very high temperatures from the perspective of random walks. In case you did not succeed in solving part C, you should still be able to predict how $\Delta D(N)$ depends on N using random walk arguments.

4.25 One-dimensional Conductors



A. [**10** points] The figure on the right shows the band structure of a one dimensional conductor, *i.e.*, the energy $E(k)$ of an electron as a function of momentum k (in units of inverse lattice spacing). The band width $\Delta \equiv E(\pi) - E(0) = 10$ eV. At $T = 300$ K, the electron occupation number of the highest energy state in the band is equal to $n(\pi) = 0.001$. What is the location of the chemical potential relative to the bottom of the band?

B. [**15** points] Let the same $E(k)$ represent the energy levels of bosons instead of electrons. Discuss (in a few sentences) under what circumstances the chemical potential is positive, zero, and negative compared to the bottom of the band.

4.26 Maxwell-Boltzmann Distribution

The Maxwell-Boltzmann distribution for a collection of simple molecules of mass m is

$$\Phi(x, y, z, p_x, p_y, p_z)\, d\tau = \frac{e^{-\beta\epsilon} d\tau}{\int_{-\infty}^{\infty} e^{-\beta\epsilon}\, d\tau}\,, \tag{1}$$

where $d\tau \equiv dx\, dy\, dz\, dp_x\, dp_y\, dp_z$, $\beta = 1/k_{\mathrm{B}}T$, and the total energy $\epsilon \equiv (p_x^2 + p_y^2 + p_z^2)/2m + U(x, y, z)$, with $U(x, y, z)$ an external potential. Momentum is non-relativistic, $\boldsymbol{p} = m\boldsymbol{v}$.

A. [**20** points] Integrate over the coordinates x, y, and z, and show that Eq. (1) reduces to the product of three factors of the form:

$$\Psi(v_x)\, dv_x = \left(\frac{\beta m}{2\pi}\right)^{1/2} e^{-\beta m v_x^2/2}\, dv_x\,, \tag{2}$$

with similar expressions for the velocity components v_y and v_z. Sketch the distribution function $\Psi(v_x)$ as a function of v_x.

B. [**10** points] Show that the average kinetic energy associated with each velocity component is $\frac{1}{2}k_{\mathrm{B}}T$, in agreement with equipartition of energy.

C. [**20** points] Derive the speed distribution function $f(v)\, dv$ and sketch $f(v)$ as a function of v.

D. [**20** points] Suppose that some of the N molecules within a container of volume V can escape in the x-direction over a potential barrier of height E_0. Derive an expression for the number which escape per second per unit area. Begin by considering a small area dS perpendicular to the x-axis, and evaluate the number of molecules/sec which approach dS using Eq. (2), then integrate to include only those molecules which have energy greater than E_0.

E. [**10** points] Free electrons in a metal obey Fermi-Dirac rather than Maxwell-Boltzmann statistics. Sketch the Fermi-Dirac distribution in energy for a free electron gas and compare it with a sketch of the classical Maxwell-Boltzmann distribution at, say, room temperature. Explain qualitatively how the two distributions change with temperature.

F. [**10** points] Give an order of magnitude estimate for the average electron speed at room temperature according to the classical Maxwell-Boltzmann distribution. Compare with the electron speed at the Fermi surface in a metal according to the Fermi-Dirac distribution.

G. [**10** points] Photoelectron emission experiments on a clean metallic surface in high vacuum show that electrons in metals obey Fermi-Dirac rather than Maxwell-Boltzmann statistics. Explain qualitatively how such an experiment might be performed. Recall Einstein's theory of the photoelectric effect, $h\nu = \frac{1}{2}mv^2 + W_b$, where W_b is the potential barrier height at the surface of the metal. Describe how the wavelength or frequency dependence of the photoemitted electrons would differ for the Maxwell-Boltzmann and Fermi-Dirac distributions. Assume that ultraviolet photons up to say 10 eV can be used, and that $W_b = 2.0$ eV.

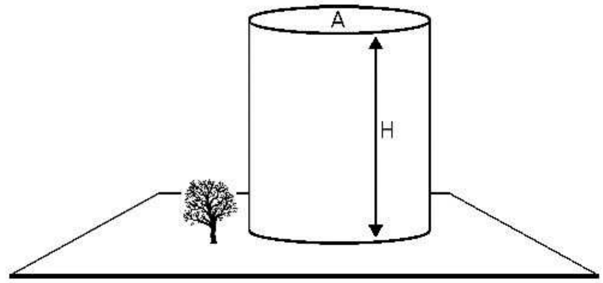
4.27 Mixture of Isotopes

Consider a solid consisting of a mixture of two different isotopes. Treat the isotopes as classical objects, which are completely identical except for an infinitesimal difference in mass that allows us to distinguish the two types.

A. [**10** points] If the number of atoms of isotope 1 is N_1 and that of isotope 2 is N_2, so that the concentration of isotope 1 is $c_1 = N_1/N$, where $N = N_1 + N_2$, what is the increase in the entropy of the solid from the value it would have if all N atoms were of isotope 1? (In other words, what is the entropy of mixing?)

B. [**15** points] Consider a solid and a liquid which are in coexistence with each other at the melting temperature T_a. Each is a mixture of the above two types of isotopes. The solid has a concentration $c_1^{(s)}$ of isotope 1 and the liquid a concentration of $c_1^{(\ell)}$. Assume that $c_1^{(\ell)} < c_1^{(s)} \ll 1$. Is the temperature T_b where the pure solid melts (the one with $c_1^{(s)} = 0$), less or greater than T_a? Explain your answer. A qualitative physics argument is sufficient (although a quantitative result is also acceptable).

4.28 Natural Gas Silo



Natural gas is often stored in large cylindrical silos like the one in the figure on the right. It has volume $V = HA$, with height H and cross sectional area A. The number of monoatomic gas molecules N inside the silo is small enough for it to be treated as a classical ideal gas. The gravitational field $\boldsymbol{g}$ points vertically down.

In the following, assume that the temperature of the gas is low enough for the variation in potential energy of the gas molecules with altitude to be important. You need to include it in the following calculations.

A. [**10** points] Evaluate the canonical partition function and show that in the thermodynamic limit the Helmholtz free energy has the form

$$F = -k_{\mathrm{B}} T N \left[c + \log \frac{A}{N} + \frac{5}{2} \log T + \log \left(1 - e^{-mgH/k_{\mathrm{B}}T} \right) \right],$$

with c a constant.

B. [**5** points] Evaluate the derivative of the free energy with respect to A,

$$\left(\frac{\partial F}{\partial A} \right)_{H,T,N}.$$

What physical quantity does this derivative represent? What type of physical process does it relate to? What type of term in the first law of thermodynamics does it represent?

C. [**10** points] Evaluate the derivative of the free energy with respect to H.

$$\left(\frac{\partial F}{\partial H} \right)_{A,T,N}.$$

What physical quantity does this derivative represent? What type of physical process does it relate to? What type of term in the first law of thermodynamics does it represent?

D. [**15** points] Derive the formula for the isochoric (constant volume) heat capacity C_V.

E. [**5** points] For all silos on Earth, one can safely ignore the variation of the gravitational energy of the gas molecules with height inside the silo. Estimate (in the form of a formula) the characteristic temperature T_g above which this is a good approximation.

F. [**5** points] Show that your formula for the heat capacity C_V in part D reduces far above T_g to the conventional ideal gas formula for an ideal gas in which the gravitational potential is completely ignored.

4.29 Number Fluctuations

A. [**10** points] Derive the relation between particle number fluctuations and particle number susceptibility of an arbitrarily interacting gas (in contact with a reservoir with which both heat and particles may be exchanged),

$$\langle (\Delta N)^2 \rangle \equiv \langle N^2 \rangle - \langle N \rangle^2 = T \left. \frac{\partial \langle N \rangle}{\partial \mu} \right|_T .$$

B. [**15** points] Using this relation (or any other valid approach), show that the number of particles N and the occupation number $n_{\boldsymbol{k}}$ for a specific momentum $\boldsymbol{k}$ satisfy the conditions:

i. $\langle n_{\boldsymbol{k}}^2 \rangle - \langle n_{\boldsymbol{k}} \rangle^2 = \langle n_{\boldsymbol{k}} \rangle (1 \pm \langle n_{\boldsymbol{k}} \rangle)$ for quantum statistics. Indicate which sign corresponds to bosons, and which to fermions.

ii. $\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$ for classical statistics.

C. [**15** points] For a spin-$\frac{1}{2}$ ideal Fermi gas at a temperature T small compared to the Fermi energy ($T \ll \epsilon_F$), evaluate $\langle (\Delta N)^2 \rangle / \langle N \rangle$, and express the result in terms of T/ϵ_F.

4.30 Particles in Harmonic Potential

Consider two non-interacting particles with the same mass in a one-dimensional harmonic potential with frequency ω. Find the free energy of the system in the following cases:

A. [**9** points] The two particles are distinguishable.

B. [**9** points] The particles are identical bosons.

C. [**9** points] The particles are identical spinless fermions.

D. [**13** points] The particles are spin-1/2 fermions.

4.31 Pendulum Oscillations

A pendulum hangs in a room, in equilibrium with the air in that room, at temperature $T = 300\,\mathrm{K}$ and 1 atm pressure. The pendulum consists of a mass, $m = 1.0$ kg, at the end of a massless string of length $\ell = 10$ cm. Its natural frequency, for small amplitudes, is $\nu_0 = (10\ \mathrm{Hz})/2\pi = \frac{\pi}{2}\sqrt{g/\ell}$, with $g \simeq 10\,\mathrm{m/s}^2$ the acceleration of gravity. The Boltzmann constant $k_\mathrm{B} \simeq 1.4 \times 10^{-23}\ \mathrm{JK}^{-1}$.

A. [**8** points] What is the average potential energy of the pendulum?

B. [**12** points] What is its root mean squared amplitude of the oscillation?

4.32 Photon Gas

Consider a photon gas in a conducting rectangular cavity.

A. [**15** points] For a given mode frequency ω, derive (stating assumptions) the average number of photons $\langle n \rangle$ in this mode.

B. [**10** points] Derive the root-mean-square fluctuations in n. Compare it with part A: are the photon number fluctuations always greater or less than the mean?

4.33 Photons and Radiation Pressure

A. [**20** points] Consider a 3-dimensional photon gas with energy spectrum $E = \hbar c q$ where $q = |\boldsymbol{q}|$ is the magnitude of the wave vector. *Solve the questions below using quantum statistical mechanics.*

i. [**5** points] Discuss why the chemical potential of this photon gas is zero.

ii. [**8** points] Show that the radiation pressure of the photon gas is

$$p = \frac{4\sigma}{3c}\, T^4 \,,$$

where $\sigma = \pi^2 k^4/(60\hbar^3 c^2)$.

iii. [**7** points] Show that the energy density u of the photon gas can be expressed as $u = 3p$.

B. [**20** points] You are given an evacuated container of volume V whose walls are perfectly reflective for electromagnetic radiation.

i. [**6** points] By considering the density of states of the 3-dimensional photon gas in the container, show that the energy U of the gas is a linear function of V.

ii. [**7** points] Starting from the second law of thermodynamics and using Maxwell's relations, derive the general relation

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P\,.$$

iii. [**7** points] Using the results of parts B.i. and B.ii., prove that the radiation pressure is given by $p = a\,T^4$ where a is an undetermined constant. The result $u = 3p$, which was derived statistically mechanically in part A.iii., may be used here.

4.34 Quantum Particles in Harmonic Oscillator

[**50** points] Consider quantum particles of mass m in a potential $V(x) = \frac{m\,\omega^2}{2}x^2$ in one dimension at temperature T.

A. [**10** points] Calculate the canonical-ensemble partition function for each case listed below. Express your results in terms of the dimensionless parameter $\alpha = \frac{\hbar\omega}{kT}$.

i. [**5** points] A single particle in the oscillator potential.

ii. [**5** points] N *spinless and distinguishable* particles in the oscillator.

B. [**10** points] For the two cases above, obtain expressions for and make sketches of the average energy, E, and specific heat, C_V, as a function of the temperature.

C. [**20** points] Now consider particles in the oscillator.

i. [**15** points] Calculate the canonical-ensemble partition function for each case listed below and express it as an expansion in $\xi = e^{-\alpha}$. Keep terms up to order ξ^4.

- Two *indistinguishable* spinless particles.
- Two spin-1/2 particles, one with spin up and the other with spin down.
- Two *indistinguishable* spin-1/2 particles, both with spin up.

ii. [**5** points] Rank the entropy at a given temperature for the first and last of the three cases listed above.

D. [**10** points] Consider now *spinless and indistinguishable* particles (bosons) in a 1-dimensional harmonic oscillator with chemical potential, μ, being held by external conditions, but allowed to vary. Indicate whether this system will, under certain conditions, yield a large number of particles in the ground state. If yes, indicate under what conditions. If not, explain why.

4.35 Quantum Statistics

A container of fixed total volume V is divided into two compartments by a wall which is heat conducting, but impermeable to gas particles. One compartment contains N particles of an ideal Fermi gas A, while the other compartment contains N particles of a different ideal Fermi gas B. The masses of the particles of the two gases are equal, but the particles of gas A have spin $\frac{1}{2}$ and the particles of gas B have spin $\frac{3}{2}$. The two compartments have the same temperature.

A. [**15** points] Assuming that the two compartments have the same pressure, find the ratio V_A/V_B at very high temperatures. (Both gases are non-relativistic.)

B. [**15** points] Assume that the chemical potentials in the two compartments are equal. What is V_A/V_B? What is the ratio of pressures P_A/P_B?

C. [**20** points] Let the temperature be zero, find the ratio V_A/V_B so that the two compartments have the same pressure.

4.36 Refrigerating Ideal Gases

Consider a block of refrigerating material and three samples that need to be cooled. In the temperature range of interest, $0.01 \leq T \leq 0.1$K, the heat capacities of the three samples are approximately given by

$$\text{Sample F} \qquad C_F = (\pi^2/2) N_F \, k_\text{B} \, (T/T_F) \, , \tag{1}$$

$$\text{Sample B} \qquad C_B = 1.9 \, N_B \, k_\text{B} \, (T/T_B)^{3/2} \, , \tag{2}$$

$$\text{Sample C} \qquad C_C = (3/2) N_C \, k_\text{B} \, . \tag{3}$$

Here $k_\text{B} = 1.38 \times 10^{-23}$ Joule/Kelvin is Boltzmann's constant, and N_F, N_B, and N_C are the numbers of particles in each sample. Sample F is an ideal Fermi gas with $T_F = 2$K. Sample B is an ideal Bose gas with $T_B = 2$K. Sample C is a very dilute classical mono-atomic ideal gas.

The refrigerating block consists of a paramagnetic type solid. At zero external magnetic field the lowest two levels of the paramagnetic ions are separated by an energy $T_R = \Delta\epsilon/k_\text{B} = 0.003\,$K, with all other excitation energies (divided by k_B) above $10\,$K. In the temperature range of interest the heat capacity of the refrigerating block has the form:

$$\text{Refrigerator R} \qquad C_R = (1/4) N_R \, k_\text{B} \, (T_R/T)^2 \, , \tag{4}$$

with N_R the number of particles.

A. [**15** points] Derive the canonical partition function of a classical mono-atomic ideal gas, $Z = (1/N!)(V/h^3)^N (2\pi m k_\text{B} T)^{3N/2}$, with V the volume, m the mass of the particles, and h the Planck's constant. Derive the heat capacity (3) from Z.

B. [**25** points] Calculate the partition function for the paramagnetic substance R, for temperatures $T \leq 1$K, and then use your answer to derive the heat capacity. Show all your work. Justify the approximations needed to arrive at the result (4). Neglect vibrations of the crystal lattice, *etc.*

C. [**20** points] Explain intuitively, using Fermi surface and quasi-classical ideal gas properties, why the heat capacity of an ideal Fermi gas is proportional to temperature.

D. [**20** points] All three samples are at an initial temperature $T_1 = 0.1\,$K, and the refrigerating block R is at $T_2 = 0.01\,$K. Calculate N_F/N_R, N_B/N_R, and N_C/N_R, assuming that these ratios are fine-tuned in such a way that when any of the samples is brought in thermal contact with the refrigerator, the final equilibrium temperature is $T_f = 0.05\,$K for all three processes.

E. [**20** points] For a given value N_R, it is possible to cool down many more liquid ^{4}He particles than liquid ^{3}He particles. Why is that so? Describe the relevant physics going on in samples F and B as they are being cooled down.

4.37 Rotating Cylinder

Consider an ideal gas which is confined in a cylinder rotating around its axis with a frequency Ω. The radius R of the cylinder, the temperature T, the number of particles N, and their mass m, are given.

A. [**25** points] Find the frequency-dependent part of the free energy, $F(\Omega)$, in the rotating system of coordinates.

B. [**25** points] Find the angular momentum of the system in the lab frame. (Use the fact that the rotation is equivalent to the existence of a potential $u(r)$, where r is the perpendicular distance from the axis of the cylinder.)

4.38 Rubber Band versus Gas

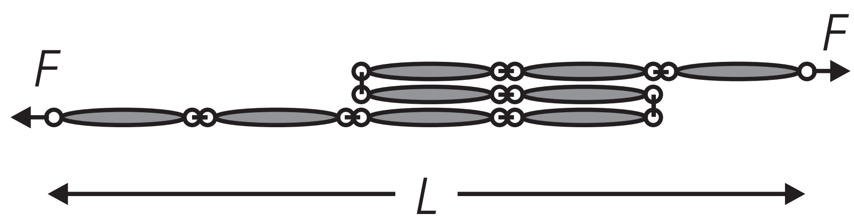
Useful approximation: $\ln N! \approx N \ln N - N$ for $N \to \infty$ (Stirling's approximation).

A. [**20** points] Consider a classical ideal gas of identical non-interacting molecules.

i. [**15** points] Derive an expression for the entropy and express it in terms of its temperature, T, its volume, V, and the number of molecules, N, for large N. Show your work.

ii. [**5** points] Note that the conservation of energy, $dE = TdS - PdV$, implies that there are two possible contributions to the pressure: an 'entropic' one, $T\ \partial S/\partial V$, and an energetic one, $-\partial E/\partial V$. Derive expressions for these two contributions to the pressure.

B. [**30** points] Consider a simplistic model for a polymer chain of the kind of which rubber bands are made. We will consider the polymer as made of N molecules of length d linked together end-to-end. The angle between successive links can be either 0 or 180 degrees, as shown in the figure below, with the same potential energy. Assume that N is even and that the system is at temperature T.



i. [**10** points] Show that the number of arrangements that yields length $L = 2\,m\,d$ is

$$g(N,m) = \frac{N!}{(\frac{N}{2}+m)!(\frac{N}{2}-m)!}\,,$$

where m is a positive integer. Clearly indicate the reasoning needed to derive this result.

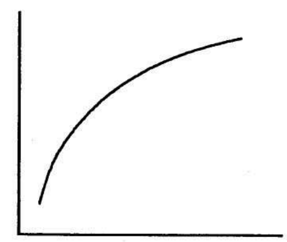
ii. [**8** points] Calculate the force needed to keep the chain at a given length, L. Comment on differences or similarities with respect to part A.ii regarding the relative contributions of the entropic versus energetic contributions to the force.

iii. [**7** points] By what amount does the energy of a system of polymers at temperature T change when a link flips direction so as to increase the overall length L by $2d$? What is the ratio of the probabilities for these two configurations? Briefly explain your answer to these questions.

iv. [**5** points] Consider having a polymer chain extended by applying a force to it. Derive an expression for its average length. Would the average length increase, stay the same, or decrease if the temperature is increased? Explain.

4.39 Stability and Convexity

Someone has calculated the entropy S as a function of energy U for a particular system. The figure on the right is a schematic plot of the results. The axes are not labeled. It might be a plot of $S(U)$ or $U(S)$; *i.e.*, the vertical axis might represent S or U. Show that thermodynamic stability requirements determine which one it is. Specifically:



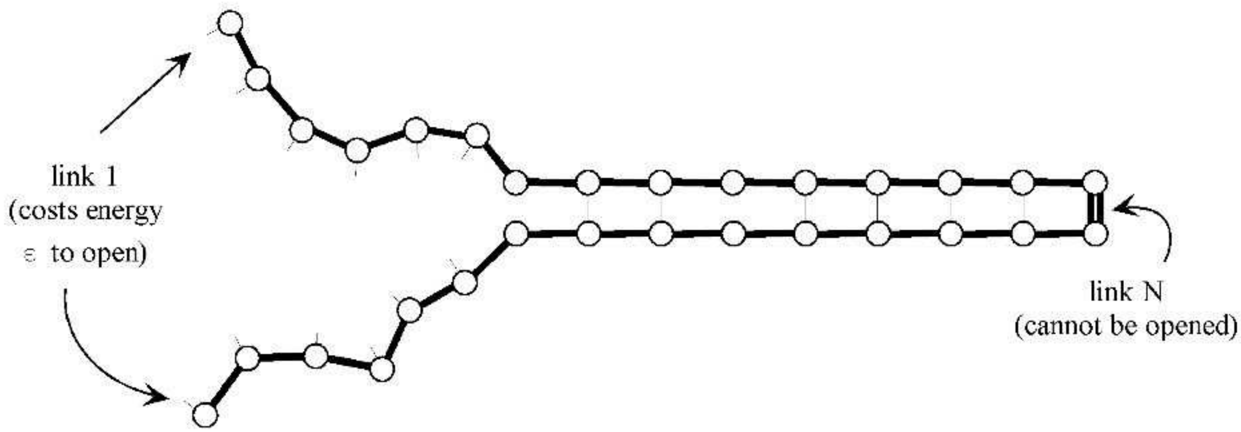
A. [**5** points] Write down the second law of thermodynamics.

B. [**5** points] Write down the calorimetric definition of heat capacity.

C. [**10** points] Derive from parts A and B the relation between the heat capacity and the second derivative of the energy with respect to entropy.

D. [**10** points] Determine from your answer to part C whether $U(S)$ is convex or concave. Which of the two variables is plotted along the vertical axis?

4.40 Thermal Zipper



Consider a "zipper" of N parallel links that can be opened only from one end. The zipper is in equilibrium with its environment at temperature T. The energy required to open link j is ε if link $j-1$ is open, but is infinite if link $j-1$ is still closed. Similarly, link j can be closed with a decrease in energy $-\varepsilon$ only if link $j+1$ is already closed. Link N cannot be opened under any circumstances. Assume that the open state of a link is g-fold degenerate to account for the large number of rotational conformations of the open structure. This model has been used as a starting point for understanding the thermally-driven unbinding of the DNA double-helix.

A. [**10** points] Find the canonical partition function of the system. Express it in terms of $x \equiv g\,\exp(-\varepsilon/kT)$.

B. [**15** points] Show that the average number of open links $\langle s \rangle$ is

$$\langle s \rangle = \frac{x}{1-x} - \frac{Nx^N}{1-x^N}\,.$$

C. [**20** points] Suppose $g > 1$. The previous result implies that for large N the number of opened links becomes large as $x \to 1$. Why does this happen, given that opening up a link costs an energy ε? Explain this fact using the free energy $F = E - TS$.

D. [**20** points] A Taylor series expansion of $\ln Z$ about $x = 1$ reads

$$\ln Z = \ln N - \tfrac{1}{2}(N-1)\,\eta + \tfrac{1}{24}(N-1)(N-5)\,\eta^2 + O(\eta^3)\,,$$

where $\eta \equiv 1 - x$. Use this result to evaluate the heat capacity at $x = 1$. You may assume $N \gg 1$ if it simplifies the calculation.

4.41 Thermodynamic Relations

For each of the following thermodynamic conditions, describe a system or class of systems (the components or range of components, temperatures, *etc.*) which satisfy the specified condition. Confine yourself to classical, chemical systems of constant mass.

A. [**5** points] $\left(\frac{\partial U}{\partial V}\right)_T = 0.$

B. [**10** points] $\left(\frac{\partial S}{\partial V}\right)_P < 0.$

C. [**5** points] $\left(\frac{\partial T}{\partial S}\right)_P = 0.$

4.42 Two-dimensional Bosons

Consider an ideal gas of spin-0 bosons of mass m in two dimensions.

A. [**20** points] Starting from the grand canonical ensemble, show that the average occupancy of a state with energy ε is

$$\bar{n}(\varepsilon, T) = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1} .$$

This is, of course, a general result independent of the number of spatial dimensions.

B. [**15** points] Consider a sub-area of the two-dimensional space with size $L_x \times L_y$. Write down, without proof, the (non-relativistic) energy-momentum relationship $\varepsilon(k_x, k_y)$ for the bosons. Then use your energy-momentum relationship together with the result of part A to show that the total number of particles can be expressed as:

$$N = L_x L_y \frac{mkT}{2\pi\hbar^2} \sum_{j=1}^{\infty} \frac{e^{j\mu/kT}}{j} .$$

Feel free to work in the limit where L_x and L_y are arbitrarily large,

C. [**15** points] Based on the result of part B, is Bose-Einstein condensation expected to occur in two dimensions? Explain your answer.

4.43 Two Levels

A particular quantum system has two different energy levels, the ground energy $E_0 = 0$, and the excited energy E_1. The ground state is non-degenerate, and the excited state has a degeneracy of n, *i.e.*, there are n distinguishable states of energy E_1. At a temperature of T:

A. [**6** points] Give the free energy of the system.

B. [**6** points] Give the probability that the excited state will be occupied.

C. [**6** points] Give the average energy of the system.

D. [**12** points] Give the entropy of the system. Calculate or state the values as $T \to 0$ and $T \to \infty$. Explain why these results are reasonable.

4.44 Two Levels and Beyond

A. [**20** points] Consider a two-level system with energy states 0 and ε.

i. [**10** points] In the canonical ensemble, derive an expression for the heat capacity of the two-level system.

ii. [**10** points] In the 1970's, it was discovered that the low-temperature heat capacity of insulating glasses is linear in T. Approximate the internal degrees of freedom for a glass as a superposition of independent two-level systems with a broad distribution of energy differences $g(\varepsilon)d\varepsilon$. Show that the heat capacity of a glass is given by $C_V(T) \sim A k_{\rm B}^2 T\, g(0)$, where A is a constant of order unity.

B. [**20** points] A myoglobin molecule in solution can either have exactly one adsorbed O_2 molecule, or else zero adsorbed O_2 molecules. Let ε denote the energy of an adsorbed molecule of O_2 relative to an O_2 in solution at infinite distance from the myoglobin.

i. [**12** points] Approximate the O_2 molecules in solution as an ideal gas (exclude rotational and vibrational degrees of freedom). Prove that the chemical potential of an ideal gas is $\mu = k_{\rm B}T\log(n/n_Q)$, where n is the particle density and $n_Q = (MkT/2\pi\hbar^2)^{3/2}$.

ii. [**8** points] Prove that the fraction of myoglobin molecules with an adsorbed O_2 molecule is given by

$$f = \frac{n}{n_Q \exp(\varepsilon/k_{\rm B}T) + n}\,,$$

where n is the concentration of O_2 molecules in the surrounding solution.

C. [**20** points] Consider a gas of free electrons at $T = 0$. An electron in a magnetic field has an energy of $\pm\mu_B H$ according to whether the spin is parallel or antiparallel to the field $\boldsymbol{H}$.

i. [**5** points] Sketch the spin-dependent density of states and indicate a typical occupancy.

ii. [**15** points] Show that the spin paramagnetic susceptibility is $\frac{3}{2}n\mu_B^2/\mu_0$, where n is the electron density and μ_0 is the chemical potential at $T = 0$.

4.45 Van der Waals Gas

A. [**10** points] Using a Maxwell relation, show that:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P\,.$$

(U is the internal energy, V is the volume, T the temperature, and P the pressure).

B. [**20** points] A van der Waals gas has the following equation of state

$$\left(P + \frac{aN^2}{V^2}\right)\left(\frac{V}{N} - b\right) = k_{\mathrm{B}}\,T\,,$$

where N is the number of particles, k_{B} is Boltzmann's constant, and a and b are two positive constants. Show that, at a given temperature, the specific heat at constant volume C_V of a van der Waals gas with a fixed number of particles N is independent of its volume. Use an appropriate Maxwell relation.

C. [**20** points] This van der Waals gas, initially occupying a volume V at temperature T_i, undergoes a free expansion (*i.e.*, an expansion in which U remains constant) up to a slightly larger final volume $V + \Delta V$. Does the temperature of the gas rise or drop? If needed, use the relation derived in part A. Is the interaction between the molecules of the van der Waals gas attractive or repulsive? Explain.

4.46 White Dwarfs

Consider a degenerate Fermi gas of N identical spin-$\frac{1}{2}$ nonrelativistic fermions of mass m confined in a volume V. Neglect the interaction between fermions.

A. [**8** points] Find the ground state energy.

B. [**12** points] Find the heat capacity at constant volume C_V.

The final evolutionary state of a star whose mass is not too high is a white dwarf. In white dwarfs, matter is highly compressed, so the electrons are not pinned to nuclei, but move freely. White dwarfs are very cold so electrons form a degenerate Fermi gas. The pressure of this gas is balanced by the gravitation force.

In the rest of this problem, consider a simplified model in which the white dwarf is a sphere with uniform density. Assume the white dwarf is made out of electrons and ^{12}C nuclei. The ^{12}C nucleus has equal numbers of protons and neutrons. The kinetic energy of the ^{12}C nuclei can be neglected, but gravitational energy cannot be neglected.

Order of magnitude estimates are sufficient — you may ignore $O(1)$ pure numerical coefficients.

C. [**12** points] By minimizing the total energy of the star with respect to the radius, derive the relationship between the mass and radius of a cold white dwarf. Assume the electrons in the star are nonrelativistic.

D. [**8** points] Can a white dwarf be in equilibrium if the electrons are ultrarelativistic?

E. [**10** points] Estimate the maximal mass of a white dwarf and compare it to the mass of the Sun. Some physical constants which might be useful are:

$$m_p\,c^2 = 0.94 \text{ GeV}, \quad M_{\text{Pl}} \equiv (\hbar c/G)^{1/2} = 1.2 \times 10^{19} \text{ GeV}/c^2,$$
$$m_e\,c^2 = 0.51 \text{ MeV}, \quad M_{\text{sun}} = 1.2 \times 10^{57}\,m_p.$$