Compendium Master's Review Examinations 2018-current

Physics Department Physics University of Washington

Preface:

This is a compendium of problems from the Master's Review Examinations for physics graduate students at the University of Washington. Problems are grouped by year. The four exams are in Thermodynamics and Statistical Mechanics, Classical Mechanics, Quantum Mechanics, and Electromagnetism. This compendium covers the period after 2017. The MRE's for 2012-2017 are listed in a separate PDF file. UW physics Ph.D. students are strongly encouraged to study all the problems in these two compendia. Students should not be surprised to see a mix of new and old problems on future exams. Some bits of advice:

- Try to view your time spent studying for the Qual as an opportunity to integrate all the physics you have learned in that specific topic.
- Read problems in their entirety first, and try to predict qualitatively how things will work out before doing any calculations in detail. Use this as a means to improve your physical intuition and understanding.
- Some problems are easy. Some are harder. Try to identify the easiest way to do a problem, and don't work harder than you have to. Make yourself do the easy problems fast, so that you will have more time to devote to harder problems. Make sure you recognize when a problem is easy.
- Always include enough explanation so that a reader can understand your reasoning.
- At the end of every problem, or part of a problem, look at your result and ask yourself if there is any way to show quickly that it is wrong. Dimensional analysis, and consideration of simplifying limits with known behavior, are both enormously useful techniques for identifying errors. Make the use of these techniques an ingrained habit.
- Recognize that good techniques for studying Qual problems, such as those just mentioned, are also good techniques for real research.

1 Throttling (30 points total)

Consider a (non-ideal) gas that has a volume V, temperature T and pressure P in the initial state. It then undergoes a Joule-Thomson process (is throttled through a porous medium). As a result of this process the pressure changes by ΔP. Assume ΔP to be small.

A. [**5** points] Is there a thermodynamic potential that is conserved during the process? If so state what it is and briefly explain why it is conserved.

B. [**10** points] Find the entropy change ΔS.

C. [**15** points] Derive the following expression for the temperature change ΔT.

$$\Delta T = \frac{1}{C_P}\left[T\left(\frac{\partial V}{\partial T}\right)_P - V\right]\Delta P.$$

2 Thermal fluctuations [20 points]

Consider a container of volume V filled with $N \gg 1$ atoms of an ideal Boltzmann gas at temperature T. Determine the probability of finding an empty void of volume $V_0 \ll V$ in the center of the container. Explain your reasoning.

3 Ideal Bose gas (50 points total)

A monoatomic ideal Bose gas is placed in a thermally isolated container of volume V. The atoms are spinless and have a mass m. The number of atoms is N. In the initial state the temperature is T, and the gas is Bose condensed, with half its atoms in the condensate.

A. [**40** points] Determine the following quantities (you may express your answers in terms of dimensionless definite integrals, which you do not need to evaluate).

- **i.** [**10** points] The condensation temperature T_0.
- **ii.** [**15** points] The energy of the gas.
- **iii.** [**15** points] The pressure.

B. [**10** points] Now consider an adiabatic expansion of the gas in which its volume changes to $2V$. Find the number of particles in the condensate in the final state. Show your work.

This exam consists of three parts, Problem 1, Problem 2 (with sections A-D) and Problem 3 (with sections A-C). Write your solutions for each section in the indicated space.

Some useful formulas:

Newton's law:

$$\vec{F} = m\vec{A}$$

Euler-Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = \frac{\partial L}{\partial q_a}.$$

Rotating Frame:

$$\ddot{\vec{r}} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \dot{\vec{r}} - \vec{F}_{ext}/m$$

Moment of Inertia Tensor:

$$T_{ij} = \int d^3x\, \rho(x^2\delta_{ij} - x_i x_j).$$

Euler Equations (i, j, k cyclic, not summed over):

$$I_i\dot{\omega}_i - (I_j - I_k)\omega_j\omega_k = \tau_i.$$

Euler-angles:

$$R = R_3(\psi)R_1(\theta)R_3(\phi).$$

Hamilton's equations:

$$H(p_i, q_i) = p_i\dot{q}_i - L, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson brackets:

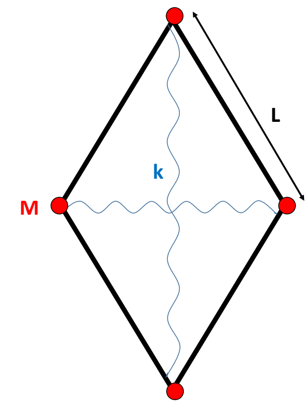
$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}, \qquad \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

Adiabatic Invariant / Action variable:

$$I = \frac{1}{2\pi}\oint p dq \qquad \text{with} \qquad p = \sqrt{2m(E - V)}.$$

1 Small oscillations (30 points)

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached at each joint. The opposite corners of the rhombus are joined by springs, each with a spring constant k. In the equilibrium (square) configuration the springs are unstretched. The motion is confined to a horizontal plane, and the particles only move along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about the equilibrium configuration.



2 Rigid Lamina (45 points total)

A rigid lamina is a flat rigid body with negligible thickness, that is its mass density has the form $\rho(x_1, x_2, x_3) = \delta(x_3)\mu(x_1, x_2)$. All angular velocities below are specified in the body fixed frame.

A. [**10** points] Show that for generic rigid lamina the principal moments of inertia obey $I_3 = I_1 + I_2$.

B. [**15** points] Show that for a lamina freely rotating in space the component of the angular velocity in the plane of the lamina (that is $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

C. [**10** points] Show that a $\vec{\omega}(t) = \omega_2 \hat{e}_2$ with constant ω_2 is a solution to the equations of motion. Study generic small perturbation around this configuration, that is consider both ω_1 and ω_3 components to be turned on with $\omega_1, \omega_3 \ll \omega_2$. Under what conditions is the solution with only ω_2 stable?

D. [**10** points] Define $\tan\alpha = \frac{\omega_2}{\omega_1}$ and derive an equation of motion for α alone.

1 Hamiltonian Dynamics and Adiabatic Invariants (25 points total)

A. [**15** points] Find the action variable I for a free particle moving in a one dimensional box of width $2a$. That is, it is subject to a potential $V(x)$=0 for $|x| \leq a$ and $V(x) = \infty$ for $|x| > a$. Derive the angular frequency $\omega = dE/dI$ of the corresponding angle variable and give a physical interpretation of your result.

B. [**5** points] Show that if the size a of the box changes adiabatically, the energy goes as $1/a^2$.

C. [**5** points] Find the time averaged force the particle applies to a single one of the walls.

Possibly useful equations, in the right context:

$$J^{\pm}\,|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1\rangle$$

$$r = \sqrt{x^2+y^2+z^2}, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

$$\left\{\frac{-\hbar^2}{2mr}\frac{d^2}{dr^2}r + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right\}\chi(r) = E_n\chi(r)$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{\ell -m} = (-1)^m Y_{\ell m}^*$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}, \quad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}, \quad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$i\hbar\dot{c_m} = \lambda\sum_n V_{mn}e^{i\omega_{mn}t}c_n, \quad \omega_{mn} \equiv \frac{E_m - E_n}{\hbar}, \quad V_{mn} \equiv \langle m|V|n\rangle$$

$$c_f^{(1)} = -\frac{i}{\hbar}\int_0^t \langle f|V|i\rangle e^{i\omega_{fi}t} \quad c_n = c_n^{(0)} + \lambda c_n^{(1)} + \cdots$$

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x + \frac{i}{(2m\hbar\omega)^{1/2}}p, \quad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x - \frac{i}{(2m\hbar\omega)^{1/2}}p, \quad [a,a^\dagger] = 1$$

1 Angular momentum, Clebsch-Gordon decomposition

Consider a system consisting of the spin states of 2 particles. Particle A has total spin quantum number $s_A = 2$, and particle B has total spin quantum number $s_B = 1$. Assume the particles are in a deep trap which allows us to neglect excitation of the spatial degrees of freedom. Possible observables for this system are the spin of particle A, known as $\vec{S}_A$, the spin of particle B, known as $\vec{S}_B$, and hermitian functions of $\vec{S}_A, \vec{S}_B$. You may assume as usual that $[S_{Ai}, S_{Bj}] = 0$ for any i, j where $i, j = 1, 2, 3$ label the three components, and that $[S_{Ai}, S_{Aj}] = i\epsilon_{ijk}S_{Ak}$, $[S_{Bi}, S_{Bj}] = i\epsilon_{ijk}S_{Bk}$ (repeated indices summed). The Hamiltonian for the system is

$$H = -\gamma \vec{S}_A \cdot \vec{S}_B$$

where γ is a numerical constant.

A. [**7** points] Find the energy eigenvalues, and the degeneracy of each eigenvalue.

B. [**8** points] Find two different complete sets of compatible observables.

C. [**6** points] Define $\vec{S}_T \equiv \vec{S}_A + \vec{S}_B$. Mark each of the the following statements True or False and explain your answers.

(c1) It is possible to find a basis in which all the basis states are simultaneously eigenstates of H and eigenstates of S_{Tx} .

(c2) It is possible to find a basis in which all the basis states are simultaneously eigenstates of H and eigenstates of S_{Ax}.

D. [**9** points] Choose one of (All, Some, None) to make each of the following statements true and explain.

(d1)(All, Some, None) of the eigenstates of $\vec{S}_A \cdot \vec{S}_T$ are eigenstates of H .

(d2)(All, Some, None) of the eigenstates of H are eigenstates of $\vec{S}_{Tz}$.

(d3) (All, Some, None) of the eigenstates of $\vec{S}_{Tz}$ are eigenstates of H .

2 Wigner Eckhart Theorem and Selection rules

A. [**15** points] A certain atom is in a state $|\psi\rangle = |\alpha j m\rangle$ with total angular momentum quantum number $j = 1/2$ and $z-$component m which is either $\frac{1}{2}$ or $-\frac{1}{2}$. α is used to represent all other quantum numbers. The position operator is $\vec{r}$, the momentum operator is $\vec{p}$, and the total angular momentum operator is $\vec{J}$. The radial component of position in spherical coordinates is r, and x, y, z are the Cartesian components of $\vec{r}$.

Which of the following matrix elements can be shown to vanish using rotational symmetry arguments? Explain why or why not.

(a1)

$$\langle \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} | r^2 | \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} \rangle$$

(a2)

$$\langle \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} | z^2 | \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} \rangle$$

(a3)

$$\langle \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} | p^2 - 3p_z^2 | \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} \rangle$$

(a4)

$$\langle \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} | J_x | \alpha \ \tfrac{1}{2} - \tfrac{1}{2} \rangle$$

(a5)

$$\langle \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} | J_x^2 - J_z^2 | \alpha \ \tfrac{1}{2} \ \tfrac{1}{2} \rangle$$

B. [**5** points] Given that

$$\langle \alpha' \tfrac{1}{2}\ \tfrac{1}{2} | Y_{10} | \alpha \tfrac{1}{2}\ \tfrac{1}{2} \rangle = A$$

where A is some number you are given, use the Wigner Eckhart theorem to find the following matrix element in terms of A:

$$\langle \alpha' \tfrac{1}{2} - \tfrac{1}{2} | Y_{1-1} | \alpha \tfrac{1}{2}\ \tfrac{1}{2} \rangle = ?$$

3 Stationary Perturbation Theory [30 points]

The Zeroth order Hamiltonian for two non identical particles of spin 1 in a magnetic field in the z direction is

$$H_0 = \mu_1 B S_{1z} + \mu_2 B S_{2z}$$

Treat the spin-spin interaction between the particles as a perturbation

$$H_1 = \gamma \vec{S}_1 \cdot \vec{S}_2$$

and find the energy levels of the system with Hamiltonian $H = H_0 + H_1$ to first order in the perturbation, assuming $|\mu_1| \neq |\mu_2|$. Would your answer change in the limit $\mu_1 = -\mu_2$? Explain why or why not.

4 Time Dependent Perturbation Theory [**20** points]

A one dimensional harmonic oscillator is in its ground state at time $t = 0$. A weak field is turned on at time $t = 0$ and turned off at time $t = T$. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 m x^2 + V(t)$$

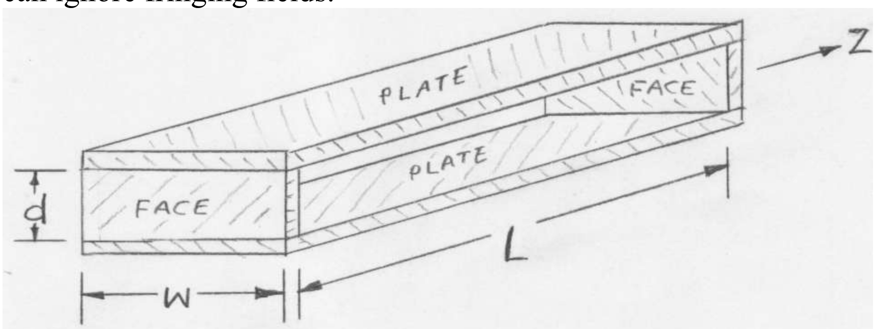
with

$$V(t) = 0, t < 0 \text{ or } t > T$$

$$V(t) = -\epsilon x^2,\ 0 < t < T\ .$$

In the weak field limit, use first order time dependent perturbation theory to find the probability that the oscillator is in the n^{th} excited state at time $t > T$, for all n for which this probability is non zero.

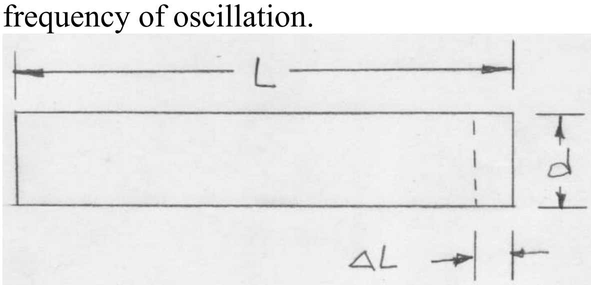
I. (35 points total) Open-sided electromagnetic resonator.
Consider an open-sided electromagnetic resonator consisting of two parallel conducting plates of width w separated by a gap d, and two parallel conducting end-faces a distance L apart. Consider only the lowest TEM mode of the resonator. Note $L >> w >> d$ so you can ignore fringing fields.



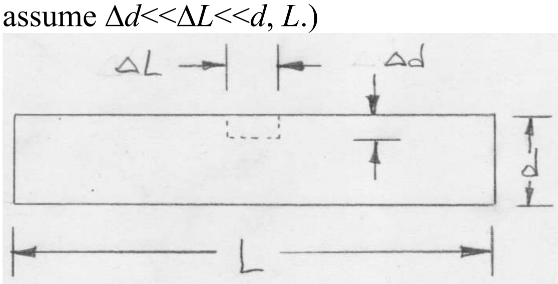
a. (5 points) Resonant frequency. What is the angular frequency of oscillation?

b. (10 points) Fields. What are the **E** and **H** fields within the resonator?

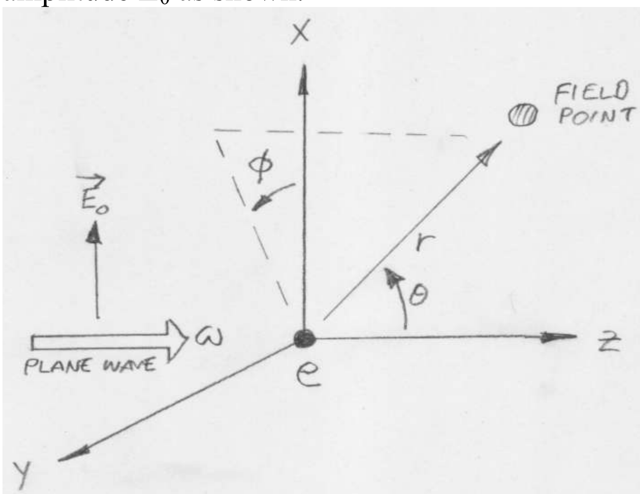
c. (10 points) Length perturbation. The resonator is perturbed in shape by moving one face slightly inwards by ΔL as shown. Find the corresponding change in the angular frequency of oscillation.



d. (10 points) Shape perturbation. The resonator is restored to length L. Now suppose the upper plate of the resonator is perturbed in shape by adding a conducting rectangular bar of length ΔL and height Δd across the full width w at the center of the resonator as shown. Find the corresponding change in the angular frequency of oscillation. (You can assume $\Delta d << \Delta L << d$, L.)



II. (30 points total) Radiation. A free point charge *e* at the origin having mass *m* is subject to a linearly-polarized plane wave of angular frequency ω with electric field amplitude $\mathbf{E}_0$ as shown.



a (10 points). Find the radiated (asymptotic) E and B fields. Assume the charge oscillates at a low enough speed where you can ignore the effects of the incident-wave's magnetic field. Recall the radiation fields emitted by an electric dipole **P** are

$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\frac{1}{rc}\hat{\boldsymbol{r}}\times\frac{\partial^2}{\partial t^2}[\mathbf{P}]_{ret}$ and $\mathbf{E}(\mathbf{r},t) = -c\hat{\boldsymbol{r}}\times\mathbf{B}(\mathbf{r},t)$

b (10 points). At a field point a distance *r* from the origin in the x-z plane and at a polar angle θ near π/4, sketch in the figure above the directions of radiation **E** and **B** fields at the field point. Describe the polarization of the radiation fields.

c (10 points) Find the intensity of the radiation fields (the time-average of the Poynting vector) in terms of the polar angle θ and the azimuthal angle ϕ.

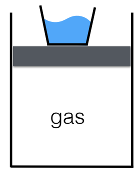
III. (35 Points total) Reflection. A plane wave of angular frequency ω is normally incident on a conductor having permittivity and permeability that of free space and real conductivity σ. The frequency and conductivity are such that within the conductor the magnitude of conduction and displacement currents are equal.

a (10 points). Within the conductor, find the relation between conduction (true currents) and displacement currents.

b (10 points). Find the (complex) index of refraction n within the conductor.

c (15 points). Find the reflection coefficient r (the ratio of time-average incident and reflected powers).

1 Classical ideal gas (30 points total)



A cylinder with volume V is filled with a (classical) ideal gas and closed off with a freely moving piston of mass M (subject to gravitation); the system is in mechanical equilibrium (i.e. nothing is moving) and in thermal equilibrium at temperature T. Now an amount of liquid of mass M is slowly poured into a container of negligible mass sitting on top of the piston, adiabatically compressing the gas. What is the final temperature of the gas? (Assume the gas is thermally isolated from the liquid and from everything else, and outside is vacuum.)

2 Ideal Bose gas in harmonic potential (40 points total)

Consider an ideal gas of non-relativistic spinless bosons of mass m moving in a harmonic potential

$$U(x, y, z) = \frac{1}{2}\alpha(x^2 + y^2 + z^2). \tag{1}$$

A. [**5** points] First consider a single particle moving in this potential. The Hamiltonian for this single particle is equivalent to that of three independent simple harmonic oscillators, corresponding to independent motions in the x, y, and z direction respectively, so each **single particle** state is specified by a triple of non-negative integers (n_x, n_y, n_z), with corresponding **single particle** energy $(\frac{3}{2} + n_x + n_y + n_z)\hbar\omega$. Find ω in terms of α and m. What are the degeneracies of the 3 lowest **single particle** energy states?

B. [**5** points] Show that for large n, the degeneracy of the single particle state with energy $(\frac{3}{2} + n)\hbar\omega$ is approximately equal to $\frac{1}{2}n^2$.

C. [**10** points] Using part B, write down the single particle density of states $g(E)$. Here $g(E)$ is defined as follows: $g(E)dE$ is equal to the number of single particle states with energies between E and $E + dE$, valid when $dE \gg \hbar\omega$.

D. [**20** points] Suppose you have N such spinless bosons. Below a certain critical temperature T_c a Bose-Einstein condensate forms. Find T_c as a function of N, α, m, and fundamental physical constants. As for the numerical prefactor, answers correct within 10 percent will be accepted. You will need

$$\int_0^\infty \frac{x^2\, dx}{e^x - 1} \approx 1.202 \tag{2}$$

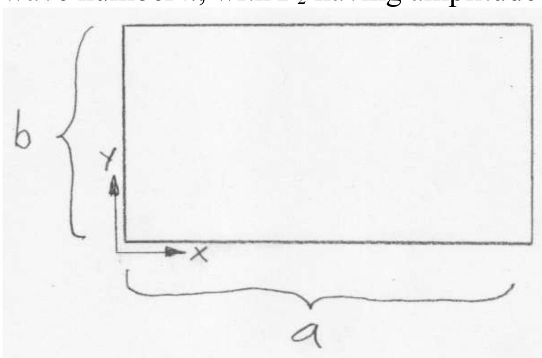
3 Fermi gas [30 points]

Consider a gas of non-relativistic fermions cooled far below the Fermi temperature in an isolated box of volume V. The Fermi energy is E_F. Now suppose the box is adiabatically expanded to volume $2V$.

A. [**10** points] What is the new Fermi energy of the system, in terms of the original E_F? Explain your answer.

B. [**20** points] If the original pressure was p, what is the new pressure? Again, explain your answer.

I. (30 Points total) Rectangular waveguide. Consider a rectangular waveguide with cross-section shown below, with sides of length a and b, perfectly-conducting walls and vacuum inside. In the following consider TE_{nm} waves of angular frequency ω and guided wave number k, with B_z having amplitude B_{0z}.



a. (5 points). Show that the longitudinal component of the magnetic field $B_z(x, y) = B_{0z} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$ satisfies the boundary conditions. Then, find all the other field components. (Group all components of the **E** and **B** fields into a single table for ease of grading).

b. (5 points). Find the time-average Poynting vector everywhere within the guide.

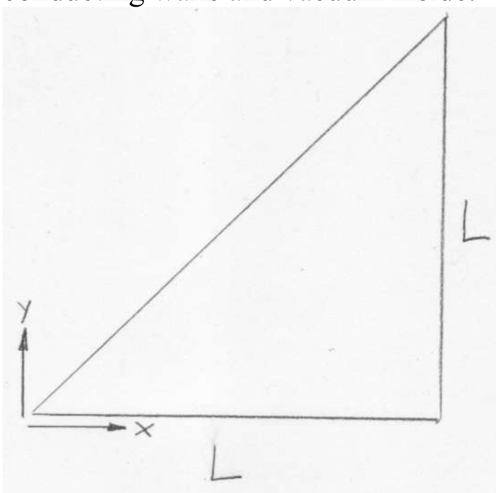
c. (5 points). Find the time-average power flowing down the guide in terms of a, b, the amplitude B_{0z} and the cutoff angular frequency ω_{mn}.

d. (5 points). Find the time-average energy density everywhere within the guide.

e. (5 points). Find the time-average energy per unit length (in the longitudinal direction) within the guide in terms of a, b, the amplitude B_{0z} and the cutoff angular frequency ω_{mn}.

f. (5 points). Using the results of (c) and (e), find the energy-flow velocity down the guide.

II. (35 points total) Triangular waveguide.
A waveguide cross-section is a right triangle of sides L, shown below, with perfectly-conducting walls and vacuum inside.



a. (7 points) TM fields. In terms of the mode indices (n,m) for the related square waveguide of side L, find the longitudinal TM fields.

b. (7 points) TM modes. In terms of the (n,m) modes for the related square waveguide of side L, which TM modes in the triangle guide are absent?

c (7 points) TE fields. In terms of the mode indices (n,m) for the related square waveguide of side L, find the longitudinal TE fields.

d. (7 points) TE modes. In terms of the (n,m) modes for the related square waveguide of side L, which TE modes in the triangle guide are absent?

e. (7 points) Guided-wave properties.
i. Find the guided wave numbers k_{nm} for TM and TE modes.
ii. Find the phase velocities $v_{p;nm}$ for TM and TE modes.
iii. Find the group velocities $v_{g;nm}$ for TM and TE modes.
iv. Find the cutoff angular frequencies ω_{nm} for TM and TE modes.

III. (35 points total) Dipole Radiation. Two perfectly-conducting hemispheres of radius R are separated at the equator by a thin insulating gap. An oscillating potential $V(t) = V_0 e^{i\omega t}$ is applied between the hemispheres. In the following, the free-space wavelength of the resulting radiation is much longer than the radius of the sphere. In your answers below, use polar coordinates with the equator of the sphere at $\theta=\pi/2$.

a. (7 points) Possible electric dipole moment. Find the electric dipole moment, if any, of the sphere.

b. (7 points) Possible magnetic dipole moment. Find the magnetic dipole moment, if any, of the sphere.

c. (7 points) Find the **E** and **H** radiation fields (far fields).

d. (7 points) Find the angular distribution of the time-average radiated power.

e. (7 points) Find the total time-average radiated power.

Possibly useful equations, in the right context:

$$J^{\pm}|j,m\rangle = \sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle$$

$$r=\sqrt{x^2+y^2+z^2},\quad x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta$$

$$[J_i,J_j]=i\hbar\epsilon_{ijk}J_k$$

$$\left\{\frac{-\hbar^2}{2mr}\frac{d^2}{dr^2}r+\frac{\hbar^2\ell(\ell+1)}{2mr^2}+V(r)\right\}\chi(r)=E_{\ell n}\chi(r)$$

$$Y_{00}=\frac{1}{\sqrt{4\pi}},\quad Y_{11}=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi},\quad Y_{10}=\sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{\ell -m}=(-1)^m Y^*_{\ell m}$$

$$Y_{22}=\sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi},\quad Y_{21}=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi},\quad Y_{20}=\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$$

$$i\hbar\frac{dc_m}{dt}=\lambda\sum_n V_{mn}e^{i\omega_{mn}t}c_n,\quad \omega_{mn}\equiv\frac{E_m-E_n}{\hbar},\quad V_{mn}\equiv\langle m|V|n\rangle$$

$$c_f^{(1)}=-\frac{i}{\hbar}\int_0^t\langle f|V|i\rangle e^{i\omega_{fi}t}\quad c_n=c_n^{(0)}+\lambda c_n^{(1)}+\cdots$$

$$a=\left(\frac{m\omega}{2\hbar}\right)^{1/2}x+\frac{i}{(2m\hbar\omega)^{1/2}}p,\quad a^\dagger=\left(\frac{m\omega}{2\hbar}\right)^{1/2}x-\frac{i}{(2m\hbar\omega)^{1/2}}p,\quad [a,a^\dagger]=1$$

$$\sin(\omega t)=\frac{e^{i\omega t}-e^{-i\omega t}}{2i},\quad \int dte^{i\omega t}=\frac{e^{i\omega t}}{i\omega}$$

1 Spherical symmetry and angular momentum

[**20** points] Consider the three dimensional spherically symmetric harmonic oscillator, with Hamiltonian

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{m\omega^2}{2}(x^2 + y^2 + z^2)\ .$$

A. Find the energy of the ground state and the first excited states, and show that the ground state is unique and the first excited states are triply degenerate.

B. What is the total angular momentum quantum number of the ground state? Explain your reasoning.

C. What is the total angular momentum quantum number of the first excited states? Explain your reasoning.

2 Compatible observables and angular momentum operators [**20** points]

Definition: a complete set of compatible operators (CSCO) is a set of commuting operators whose eigenvalues completely specify the state of a system. Consider a system consisting of the spin states of 3 spin 1 particles–particle A, particle B, and particle C. Their spin operators satisfy the usual commutation relations for angular momentum operators. The Hamiltonian for the system is

$$H = -\gamma \vec{S}_A \cdot \left(\vec{S}_B + \vec{S}_C \right)$$

where γ is a numerical constant. One possible CSCO for this system is the set S_{Az}, S_{Bz}, S_{Cz}. Which of the other following sets of operators are CSCOs for this system? Answer yes or no with a brief explanation.

A. S_{Az}^2, S_{Bz}, S_{Cz}

B. H

C. $(\vec{S}_A + \vec{S}_B + \vec{S}_C) \cdot (\vec{S}_A + \vec{S}_B + \vec{S}_C),\ S_{Az} + S_{Bz} + S_{Cz},\ (\vec{S}_B + \vec{S}_C) \cdot (\vec{S}_B + \vec{S}_C)$

D. $(\vec{S}_A + \vec{S}_B + \vec{S}_C) \cdot (\vec{S}_A + \vec{S}_B + \vec{S}_C),\ (S_{Az} + S_{Bz} + S_{Cz}),\ H$

E. $\vec{S}_A \cdot \vec{S}_B,\ \vec{S}_A \cdot \vec{S}_C, S_{Az} + S_{Bz} + S_{Cz}$

3 Time dependent perturbation theory [**30** points]

A particle is in the ground state of a two dimensional harmonic oscillator at time $t = 0$. For $t \leq 0$ the Hamiltonian is

$$H_0 = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}(x^2 + y^2) \ .$$

A weak field is turned on at time $t = 0$ and turned off at time $t = T$. The Hamiltonian is

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}(x^2 + y^2) + \lambda f(t) m\omega^2 xy$$

with

$$f(t) = 0, t < 0 \text{ or } t > T$$

$$f(t) = \sin(\alpha t), \ 0 < t < T \ .$$

λ is a small dimensionless number, and α is a constant.

Use first order time dependent perturbation theory to compute the probability $P(t)$ that the particle is in any of the excited states at time $t > T$, for all the excited states for which this probability is non zero.

Physics 505 FINAL Spring 2019 6/12/19

Name: ____________________ **Signature:** ____________________

100 pts — "1 point per minute" + 10 bonus minutes

Exam procedures.

- Please sit away from other students
- Please write your name above.
- If you have a question about the exam, please ask.
- This is a closed book exam.
- Write your answers on the exam. I have tried to leave ample space, but if you need more, use additional paper and be sure to write "505 Final" and your name on the top of each page.
- Unless stated otherwise, it is necessary for full credit that you explain the logic of your calculation, deriving any results that you use.

Some useful formulas:

Newton's law:

$$\vec{F} = m\vec{a}$$

Euler-Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) = \frac{\partial L}{\partial q_a}.$$

Rotating Frame:

$$\ddot{\vec{r}} = -\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \dot{\vec{r}} - \vec{F}_{ext}/m$$

Moment of Inertia Tensor:

$$T_{ij} = \int d^3x\, \rho(x^2\delta_{ij} - x_i x_j).$$

Euler Equations (i, j, k cyclic, not summed over):

$$I_i\dot{\omega}_i - (I_j - I_k)\omega_j\omega_k = \tau_i.$$

Euler-angles:

$$R = R_3(\psi)R_1(\theta)R_3(\phi).$$

Hamilton's equations:

$$H(p_i, q_i) = p_i\dot{q}_i - L, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson brackets:

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}, \qquad \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

1. `(25 pts)` **Hamiltonians:** The Hamiltonian for a simple harmonic oscillator in units where $m = \omega = 1$ is given by

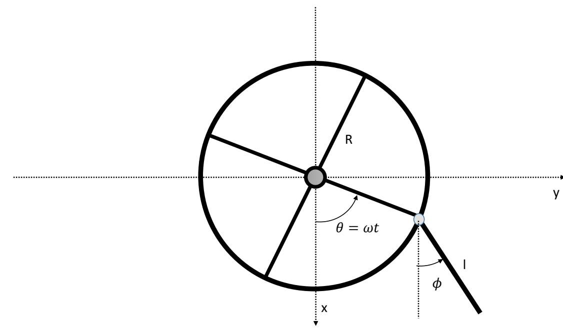
$$H = \frac{p^2}{2} + \frac{q^2}{2}.$$

To simplify the dynamics we want to change variables to

$$P(p,q) = H(p,q), \quad Q(p,q) = \tan^{-1}(q/p).$$

Confirm that this change of variables is a canonical transformation. Write down the Hamiltonian in the new coordinates and solve the equations of motion for P and Q. Confirm that this correctly reproduces the expected dynamics of a single harmonic oscillator.

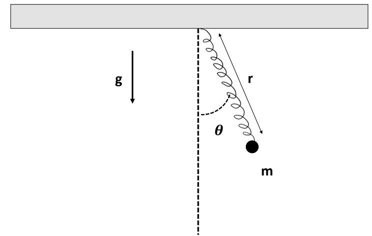
2. `(35 pts)` **Ferris Wheel:** A rod of length l and mass m hangs at the edge of a vertical wheel of radius R (like a gondola on a ferris wheel). The wheel rotates with a constant given angular velocity ω about a horizontal axis through the midpoint.



(a) `(20 pts)` Find the Lagrangian and the equation of motion of the rod.

(b) `(15 pts)` Consider the two limiting cases $R\omega^2 \ll g$ and $R\omega^2 \gg g$. In each case find a stationary solution in which $\ddot{\phi}$ vanishes and then find the angular frequency of small amplitude motions of the rod around those stationary solution.

3. (40 pts) **Spring Pendulum:** A massless spring of rest length l_0 (with no tension) has a point mass m connected to one end and the other end fixed so the spring hangs in the gravity field as shown. The motion of the system is only in one vertical plane.



(a) (10 pts) Write down the Lagrangian.

(b) `(10 pts)` Find the equations of motion for θ and $\lambda = (r - r_0)/r_0$, where r_0 is the rest length of the spring hanging with the mass m.

(c) `(10 pts)` Discuss the motion for small θ and λ with initial conditions $\theta = 0$, $\dot{\lambda} = 0$, $\lambda = A$, $\dot{\theta} = \sqrt{g/r_0}B$. A and B are constants. Express your answer in terms of the frequencies of the spring and the pendulum by themselves, $\omega_s^2 = k/m$ and $\omega_p^2 = g/r_0$.

(d) `(10 pts)` Still working with small θ and λ include the leading order effect of the θ oscillation on the λ equation of motion. Show that the λ motion now corresponds to a driven harmonic oscillator. Under what condition on ω_s and ω_p does the driving frequency equal the natural oscillation frequency, that is the system is in resonance?

4 Stationary perturbation theory

[**30** points] Consider a system of 2 spin 1/2 particles, with unperturbed Hamiltonian

$$H_0 = a(S_{1z} + S_{2z})$$

where a is a constant. We perturb this system by adding a term

$$H_1 = bS_{1x}S_{2x} \ ,$$

with $b \ll a\hbar$.

A. Find the energy eigenvalues of the eigenstates of H_0.

B. For the full Hamiltonian, $H = H_0 + H_1$, find the energy eigenvalues, to first order in b, using degenerate perturbation theory if necessary.

1 Classical gas [40 points total]

A sphere of radius R centered at $\vec{r} = 0$ is filled with a classical monoatomic ideal gas at temperature T. The atoms of the gas are subject to a potential $V(\vec{r}) = a|\vec{r}|^3$.

A. [**20** points] Compute the pressure that the gas exerts on the sphere. Check your answer by considering the limit of large temperature.

B. [**20** points] Now consider the same gas in the same potential but without the sphere. Suppose the gas is in equilibrium at $k_B T = \frac{aR^3}{30}$, and suppose there are $N = e^{30}$ atoms in the gas. At any given instant, what is the probability that all atoms are located at $|\vec{r}| < R$, i.e. inside where the sphere used to be?

2 Low temperature ideal Fermi gas [30 points]

Consider an ideal spinless Fermi gas at zero temperature in a volume V. Denote its Fermi temperature by T_F. Now suppose the volume is instantaneously increased from V to $(1+\epsilon)V$, where $\epsilon \ll 1$.

A. [**10** points] Compute the new Fermi temperature.

B. [**20** points] Write a formula for the new temperature once the gas comes to equilibrium in this larger volume. You do not need to evaluate the constants of order 1, but you do need to have the correct dependence on ϵ and T_F.

3 Low temperature ideal Bose gas [30 points]

Now consider an ideal Bose gas in a volume V at temperature $T = 50\,\text{nK} \ll \text{T}_\text{c}$, where T_c is the critical temperature at which Bose-Einstein condensation occurs. Again, suppose the volume is instantaneously increased from V to $(1+\epsilon)V$. Take $\epsilon = 0.05$. What is the new temperature of the gas once it equilibrates in the larger volume?