IS THERE A Z-THEOREM FOR DYNAMIC CRITICAL EXPONENTS?

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Abstract

Dynamic processes are related to ground state properties of many-body quantum systems and also to equilibrium critical phenomena. For example, KPZ type growth of a line interface, describes also the ground state properties of interacting electrons running around a ring in the presence of an electric field, and the equilibrium crystal shape at a facet-ridge endpoint. These equivalences suggest we search for generalizations of conformal field theory to classify scaling properties of time evolution operators in D=1+1 dimensions: for Tomonaga-Luttinger liquids and conformal field theory the "dynamic critical exponent" is equal to $z = 1$; for non-relativistic electrons and EW type growth $z = 2$; and for KPZ type growth $z = 1.5$. Phase diagrams of specific models contain all four dynamic universality classes and the crossover scaling is such that $z \geq 1$ typically decreases. The z-theorem explains this. The crossover from EW to KPZ type growth, and from isotropic to directed percolation are examples.

1 Introduction

One of the major challenges in contemporary physics is to understand the universal properties of strongly fluctuating systems, where fluctuations lack characteristic time and length scales. Scale invariant phenomena are realized in equilibrium statistical mechanics (STM), in quantum field theory (QFT), and also in non-equilibrium (NED) phenomena. In most cases they require the tuning of a parameter, such as temperature at gas-liquid critical points. Other systems "self-organize" themselves into a critical phase. A recent example are sandpile models to describe avalanches [1]. Older examples are the so-called floating-solid phase of equilibrium adsorbed mono-layers [2,3] and the rough phase of equilibrium crystal surfaces [4,5].

During previous decades scale invariance has been studies primarily in the context of equilibrium STM and QFT. More recently we are witnessing how the concepts and methods developed in those fields are generalized to other branches of physics, in particular to scale invariance in dynamic processes such as growing interfaces [6] and catalytic chemical reactions [7].

It is instructive to compare our present understanding of scale invariance in non-equilibrium systems with that of equilibrium critical phenomena. There are roughly three stages in such studies. The first one is the data gathering stage. The existence of scale invariance is being established and universality classes are being cataloged. For equilibrium critical phenomena this was achieved in the late sixties, for most non-equilibrium systems dynamic only a few years ago.

Several new dynamic universality classes have been identified. I will focus in this talk on two examples: KPZ type surface growth [8] and directed percolation (DP) [9,7]. In both cases the empirical evidence is mostly numerical in nature, in particular from Monte Carlo studies [6]. It is unfortunate that experimental realizations are still virtually absent. Experimental evidence was crucial in the development of the theory of equilibrium critical phenomena [10,11]. This goes beyond the fact that critical exponent have been measured with great accuracy. Experimental realizations are important in setting the agenda. For example, the development of the theory of 2D commensurate-incommensurate (C-IC) phase transitions was a direct response to their appearance in adsorbed mono-layers [3]. Disordered flat phases in crystal surfaces were discovered theoretically [12], but I did not discover a mechanism to stabilize them without step-step interactions until I was forced to look into the peculiar topological properties of Si(100) surfaces [13].

The second stage is to obtain analytical insight in those scaling properties. Exactly soluble models confirm scaling in equilibrium critical phenomena [14]. Mean field approximations and Landau-Ginzburg theory yield the existence of upper critical dimensions and of a fixed point in renormalization theory.

For KPZ type growth we reached stage-two only in D=1+1 (one spatial and one temporal dimension). A few D=1+1 models are exactly soluble [15–18]. Theoretical renormalization studies of dynamic systems go back to the seventies [19]. They emerged from the long tradition of research on Langevin and Fokker-Planck equations. However, for KPZ type growth the mere existence of an upper critical dimension is still an issue[20]. The ϵ-expansion of the KPZ equation [8,19] describes the fixed point structure close to D=2+1, associated with the reversal of stability of Edwards-Wilkinson (EW) growth with respect to the non-linear KPZ term. It does not yield a fixed point for the KPZ universality class except in D=1+1.

For directed percolation we reached stage two. There are no exactly soluble models, but the field theory is known, Reggion field theory [21]. Unfortunately the upper critical dimension is large, D_u=5. Moreover, field theory [22] suggests the wrong sign for the stability of isotropic percolation (IP) with respect to directionality [23].

The third stage is to find a description of the scaling properties in terms of a massless free field theory. In general this is not possible at all. However, for 2D equilibrium critical phenomena such a theory emerged during the last 15 years. Coulomb gas representations (with boundary charges) [24,25] and conformal field theory (CFT) [26] provide a full free field theory description of virtually all 2D equilibrium phase transitions There is no assurance that this can be generalized to dynamic processes in D=1+1, but we have a good change since the time evolution operators of master equations in D=1+1 resemble closely transfer matrices of 2D equilibrium critical phenomena.

The first aim of this talk is to illustrate these equivalences between equilibrium STM, QFT, and the master equation approach to NED phenomena, The most elementary example of this is the diffusion equation, discussed in section 2. The phase diagram of the exactly soluble XXZ quantum spin-$\frac{1}{2}$ chain is reviewed in section 3. Such equivalences invite us to generalize the concept of dynamic universality beyond stochastic processes, to all possible time evolution operators (transfer matrices) in STM. From that perspective most ordinary 2D equilibrium critical points belong to the conformal dynamic universality class, and have a dynamic critical exponent $z = 1$.

Generalizations of conformal invariance are being considered [27,28], but we did not get far yet. Simple guesses do not work. For example, the description of KPZ type growth in terms of a free field theory with a k^z-type dispersion relation [29] might be a reasonable approximation, but is not exact, because the finite size scaling (FSS) amplitudes of the mass gap do not obey the correct amplitude ratios [30]. It is probably too early to tackle the problem from such a general perspective. First, we need to gather more insight into the general structure of 1+1 dimensional time evolution operators.

The Coulomb gas method, the predecessor of CFT, describes excitation in 2D equilibrium critical phenomena as spinwaves and vortices in free scalar field theory [31]. This requires a sequence of equivalences and mappings between models. The Coulomb gas description only emerged after considerable insight into the general structures of 2D equilibrium phase diagrams existed.

Crossover scaling between dynamic universality classes is an important issue. The XXZ model phase diagram illustrates that many (maybe not all) dynamic universality classes with $z \neq 1$ are located at the edge of the conformal universality class. Crossover scaling functions are severely restricted when the crossover field is a redundant non-renormalizing (RNR) type operator, like the stress tensor in CFT. This leads to the z-theorem in section 4. The change in dynamic exponent is linked to the value of the crossover scaling exponent. Sections 5-7 are devoted to applications of the z-theorem, in particular the crossover from EW to KPZ type growth (section 6), and from isotropic to

directed percolation (section 7).

2 Statistical Mechanics, Quantum Mechanics, and Master Equations

Transfer matrices in equilibrium STM, time evolution operators in QFT, and master equations for NED processes are closely related. This is true to such an extend that specific phenomena are mathematically identical. The most trivial example is very familiar. Consider a single quantum mechanical (QM) particle in a one dimensional box

$$\mathcal{H} = \frac{\hat{p}^2}{2m}. \tag{1}$$

Denote its spatial coordinate by $0 < h < L$, such that $[h, p] = i$, and its wave function as $Z(h)$, such that

$$|\Psi\rangle = \int dx \;\; Z(h) \; |h\rangle. \tag{2}$$

As we all know, the Schrödinger equation

$$i\frac{\partial}{\partial t}\mathcal{Z}(h) = -\frac{1}{2m}\frac{\partial^2}{\partial h^2}\mathcal{Z} \tag{3}$$

becomes the diffusion equation in Euclidean time $\tau = it$.

Consider a random walker. It hops during each time step, with probability $1/(2m)$ to one of its nearest neighbour sites, or stays put with probability $1-1/(m)$. The probability to find the walker at site h at time τ, $Z_\tau(h)$, obeys the recursion relation

$$Z_{\tau+1}(h) = (1 - \frac{1}{m})Z_\tau(h) + \frac{1}{2m}[Z_\tau(h-1) + Z_\tau(h+1)]. \tag{4}$$

This master equation is the discrete time-space equivalent of the above diffusion equation. The random walk problem is mathematically identical to that of the single QM particle.

Finally, consider a domain wall in 2D equilibrium STM, for example a gas-liquid interface. Assume this line spans the entire lattice in the τ direction, and (in this specific model) is not allowed to back-track in the τ direction ("overhangs" are not allowed). $h(\tau)$ parameterizes the interface in terms of the "height" of the liquid with respect to a reference line. The transfer matrix method for evaluating partition functions leads to a recursion relation identical

to the Euclidean Schrödinger equation of the QM particle. The Euclidean action of the quantum mechanical particle is identical to the energy of the interface

$$E = \int d\tau \ \ \frac{1}{2} m (\frac{\partial h}{\partial \tau})^2. \tag{5}$$

The QM wave function is identical to the equilibrium partition function, $Z_\tau(h)$. Its h-dependence represents the constraint that the interface must end at site h at the τ-edge of the lattice.

The slope-slope correlation function of the 1D interface obeys the simple relation

$$\langle (\frac{\partial h(\tau_1)}{\partial \tau})(\frac{\partial h(\tau_2)}{\partial \tau}) \rangle = \frac{1}{m} \delta(\tau_1 - \tau_2) \tag{6}$$

by virtue of Gaussian integrals (the equipartition theorem). Splitting Eq.(6) leads to a Langevin equation

$$\begin{aligned} \frac{\partial}{\partial \tau} h(\tau_1) &= \eta \\ \langle \eta(\tau_2) \eta(\tau_1) \rangle &= \frac{1}{2m} \delta(\tau_1 - \tau_2) \end{aligned} \tag{7}$$

with η the random noise. This Langevin equation describes random deposition and evaporation of particles onto a single column of particles. That process is equivalent to a random walk. The above analysis amounts to an exact derivation of a Langevin equation from a master equation. This is hardly a surprise because of the simplicity of this process. It is actually misleading, because in general master equations and Langevin equations are only approximately equivalent.

This elementary example is misleading in another aspect as well. It suggests that every problem in STM has a counter part in QFT and NED. That is not true. Time evolution operators

$$|\Psi\rangle_{\tau+1} = \mathcal{T} |\Psi\rangle_\tau = \exp(-\mathcal{H}) |\Psi\rangle_\tau \tag{8}$$

in QM and master equations are much more restricted than transfer matrices in STM. In QM, the probability to be in micro state h is equal to $|Z(h)|^2$. Conservation of probability implies that the norm, $\langle \Psi | \Psi \rangle = 1$ needs to be preserved in real time $t = -i\tau$. This requires $\mathcal{H}$ to be Hermitian. In master equations, $Z(h)$ itself represents the probability to be in micro state h. In that case, conservation of probability requires that $\langle D | \Psi \rangle = 1$ must be preserved in time. $|D\rangle = \int dx |x\rangle$ is the "disordered state". $\langle D|$ must be the left eigenvector of $\mathcal{T}$ corresponding to its largest eigenvalue, and this eigenvalue must be equal

to one. STM does not place any such restrictions on its transfer matrices, and is the most general of the three. In the space of all possible transfer matrices in D dimensional equilibrium STM, all D=d+1 dimensional QFT time evolution operators form a subset (the unitary ones), and all D=d+1 dimensional master equations form another subset (the stochastic ones). In the above free particle example these subsets accidentally coincide. From the QFT perspective, the ground state of the particle is "accidentally" the completely disordered $k = 0$ momentum state $|D\rangle$. From the master equation perspective, $\mathcal{T}$ is accidentally unitary, $\mathcal{T}^\dagger = \mathcal{T}$, because the random deposition process has particle-hole symmetry.

This elementary example displays "self-organized" criticality. It is a "massless theory", which lacks a characteristic time scale, and displays "critical slowing down". Wave packets of type

$$Z(x) \sim \frac{1}{\sqrt{\tau}} \exp[-\frac{mx^2}{2\tau}] \tag{9}$$

broaden in time only as a power law. Their width scales as $l_x \sim \tau^{1/z}$ with $z = 2$ the so-called dynamic exponent. The stationary state, $|D\rangle$, can never be reached (critical slowing down). The dynamic exponent determines the rate at which the energy (mass) gap, $m(L)$, closes with system size. Consider periodic boundary conditions, $Z(h + L) = Z(h)$. The energy spectrum, $E_k = \frac{1}{2m}k^2$ is discrete, since the eigen states are separated by $\Delta k = 2\pi/L$. Therefore, the gap between the ground state and the first excited state scales as $m = E_1 - E_0 = \frac{1}{2m}k^2 \sim L^{-z}$ with $z = 2$.

There is no reason to limit the concept of dynamic universality to stochastic transfer matrices. It applies to all STM transfer matrices. The stochastic ones form a subset. Dynamic universality classes are defined by the power with which the energy gap closes, $m \sim L^{-z}$.

A typical phase diagram, like the one for the XXZ quantum spin-$\frac{1}{2}$ chain discussed in section 4, contains several phases and dynamic universality classes. Most phases lack scale invariance. The energy gap is non-zero, and the fluctuations are bound by characteristic time and spatial correlation lengths. These are ordinary disordered phases, (quantum) liquids, solids, insulators, e.t.c. The most common scale invariant type phase is the one with dynamic exponent $z = 1$. This includes Fermi-liquids (metals, the dispersion relation is linear and therefore $z = 1$) and relativistic massless bosons (Lorentz invariance). It also includes all critical points in equilibrium STM with isotropic scaling (rotational invariance). D=2 dimensions is special, the $z = 1$ dynamic universality class contains all processes described by conformal invariance.

Non-relativistic free fermions represent the simplest dynamic universality

class with $z \neq 1$. The energy gap scales with $z = 2$. Anisotropic scaling with $z = 2$, is realized in 2D critical phenomena at Pokrovksy-Talapov (PT) type C-IC transitions [3].

The above two examples are special, in the sense that for both we have a free field theory description in D=1+1. KPZ type growth and DP type processes are examples of dynamic universality classes with a non-trivial $z \neq 1$ for which we have no free field theory description (yet). Both lie in the subspace of stochastic time evolution operators. Dynamic universality classes with $z \neq 1$ represent equilibrium critical points with anisotropic scaling. In D=2 equilibrium critical phenomena anisotropic scaling is typically associated with so-called Lifshitz points and C-IC phase transitions. Unfortunately, such Lifshitz points have never been established firmly, except for the above PT-type transitions with $z = 2$ [3].

Finally, two general remarks: Most conventional equilibrium critical phenomena universality classes are contained within the $z = 1$ dynamic conformal universality class. There is no intrinsic reason why other dynamic universality classes should lack a similar sub-structure. Secondly, in general dynamic universality classes are characterized by D-1 independent z-type exponents instead of only one. There is no intrinsic reason why the correlation length in only one of the D directions should scale differently from the others.

3 The XXZ Quantum Spin-$\frac{1}{2}$ Chain

The phase diagram of the XXZ quantum spin-$\frac{1}{2}$ chain, (the 6-vertex model in 2D STM) is a nice example of these close relations between STM, QFT, and master equations. In the fermion representation its Hamiltonian takes the form

$$\begin{aligned} \mathcal{H} &= \sum_n \frac{1}{4}[\ (1-\lambda) + 4\lambda a_n^+ a_n^- - 4\lambda a_n^+ a_n^- a_{n+1}^+ a_{n+1}^- \\ &- 2(a_n^+ a_{n+1}^- + a_{n+1}^+ a_n^-) - 2s(a_n^+ a_{n+1}^- - a_{n+1}^+ a_n^-)\ ]. \end{aligned} \tag{10}$$

In this formulation the model represents spinless fermions which hop around a closed chain in the presence of an electric field s pointing along the wire, and a nearest neighbour interaction λ. Notice that $\mathcal{H}$ is not Hermitian at $s \neq 0$. This model is exactly soluble by the Bethe Ansatz [32]. It played a central role in the development of the theory of 2D critical phenomena and also in recent work on Fermi-liquid theory.

Fig.1 shows the phase diagram. For $\lambda \geq 0$ the fermions attract each other at nearest neighbour sites. Along the line $\lambda = 0$, where they are free and

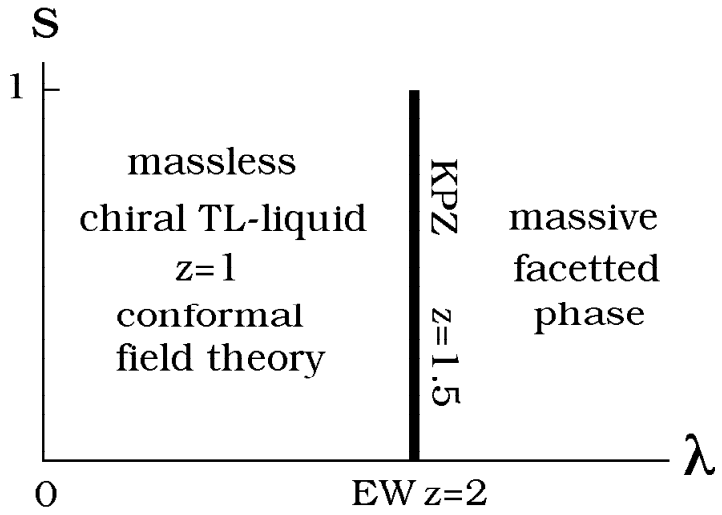

Figure 1: Phase diagram of the XXZ model with attractive interactions between fermions.

non-interacting, the model is trivially soluble by a Fourier transform. At $s = 0$ it describes a conventional d=1 metal with a half-filled conduction band. The dispersion relation is linear at the Fermi surface and therefore the dynamic exponent is equal to $z = 1$. The electric field s creates a persistent current around the wire for $s \neq 0$.

The fermions remain massless until $\lambda = 1$ for all s. The dynamic exponent remains equal to $z = 1$. One would expect this, because perturbation theory in λ gives rise in dimensions d>1 to a Fermi-liquid. Naive Fermi-liquid theory breaks down however in d=1, since the λ operator is marginal [a]. The resulting massless interacting fermion phase for $\lambda < 1$ is nowadays known as the Tomonaga-Luttinger (TL) liquid [34]. At $s \neq 0$ it is a chiral TL-liquid. In d=1, fermion are equivalent to bosons. The low lying plasmon excitations in the TL-liquid act as bosons. Therefore the entire $\lambda < 1$ phase is equivalent free scalar field theory. The detail of this are rather technical [35,36,31], but important because these equivalences are at the hearth of CFT and the Coulomb gas descriptions of 2D equilibrium critical phenomena. For example, the Baxter line in the 8-vertex model, and also the critical line of the q-state Potts model map onto the $s = 0$ line [37]. Moreover, the 6-vertex model describes equilib-

[a] Each creation and annihilation operator has a critical dimension equal to $x_F = \frac{1}{2}$. Power counting yields therefore that the interaction operator scales with $x_\lambda = 2$, i.e., has a critical dimension equal to that of the embedded (space-time) dimension D.

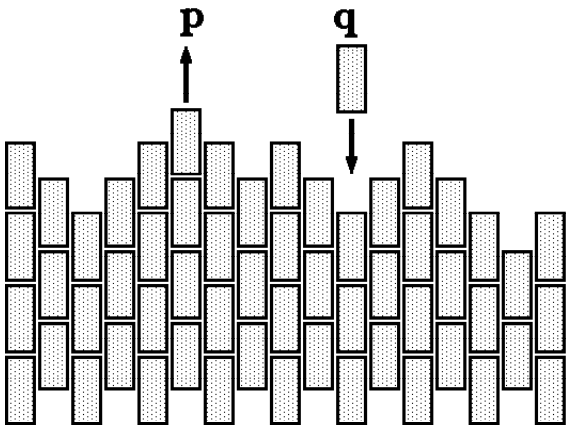

Figure 2: Growth rule in the D=1+1 dimensional BCSOS model.

rium crystal surfaces. In that representation the TL-liquid is identical to the so-called rough equilibrium crystal surface phase [4].

At $\lambda = 1$ the interaction become strong enough compared to the zero point motion that the fermions coalesce into droplets and thus localize. The TL-liquid transforms into an insulator. The persistent current stops. Surprisingly this transition in "meso-scopic metallic rings" maps exactly onto the master equation for KPZ type growth[30,16]. This is related to the asymmetric exclusion model representation of KPZ type growth models[17], and adds to other relations established before, in particular the equivalence of KPZ type growth to the Burgers equation for randomly stirred fluids, and the directed polymer problem [6,19].

Consider a wall built from rectangular shaped bricks as shown in Fig.2. Nearest neighbour columns differ in height by one half-unit. The interface can be characterized in terms of steps, by introducing a spin variable $S_n^z = \pm 1$ at each bond. The growth rule is as follows: Choose one of the columns at random. A particle adsorbs with probability p (or nothing happens with probability $1-p$) if this column is at the bottom of a local valley, $S_n^z = -1$ and $S_{n+1}^z = +1$. A particle desorbs with probability q (or nothing happens with probability $1-q$) if it is at the top of a local hill, $S_n^z = +1$ and $S_{n+1}^z = -1$. Nothing happens if it is part of a local slope, $S_n^z = S_{n+1}^z$.

This model has been studied extensively in the literature, first by Monte Carlo simulations [6], and more recently it was realized it can be solved exactly [15,16]. The Master equation, Eq.(8), describes the time evolution of the

probability distribution

$$|\Psi\rangle = \sum_{\{S_n^z\}} \mathcal{Z}(\{S_n^z\})\ |\{S_n^z\}\rangle. \tag{11}$$

The time evolution operator has the familiar form

$$\mathcal{T} = 1 - \epsilon N^{-1} \sum_n \mathcal{H}_{n,n+1} \tag{12}$$

with

$$\begin{aligned} \mathcal{H}_{n,n+1} &= p(1 - S_n^+ S_{n+1}^-)\delta(S_n^z, -1)\delta(S_{n+1}^z, 1) \\ &+ q(1 - S_n^- S_{n+1}^+)\delta(S_n^z, 1)\delta(S_{n+1}^z, -1) \end{aligned} \tag{13}$$

and is equivalent to Eq. (10) for $\lambda = 1$ and $s = (p-q)/(p+q)$. The XXZ model is stochastic only along the line $\lambda = 1$. $\mathcal{T}$ must be applied N times to evolve the surface by one unit of time. Without loss of generality we can set $\epsilon = p + q = 1$.

For $p = q$, the dynamic rule is equivalent to a Monte Carlo process to simulate the equilibrium non-growing d=1 surface. From that perspective it belongs to the EW dynamic universality class. It is easy to demonstrate, that the time evolution equations for all correlation functions at $\lambda = 1$ and $s = 0$ are linear-diffusion equations. This EW point is known as the KDP point in the XXZ and 6-vertex model literature. Slater introduced it to describe the ferroelectric transition in KH_2PO_4 [32].

KPZ type growth maps exactly onto the transition point of the persistent current problem. At $\lambda = 1$ the persistent current in the "meso-scopic metallic ring" stops. The scaling properties of the fermions at this point are quite intriguing. The energy mass gap scales as $m \sim N^{-z}$ with $z = 1.5$.

The phase diagram of the spin-$\frac{1}{2}$ chain provides us also with a nice example of the close relationship between dynamic processes and equilibrium critical phenomena (in one higher dimension). John Neergaard and I found that KPZ type growth describes the equilibrium crystal shape properties at facet-ridge endpoints [30]. The time evolution operator of the BCSOS growth model becomes identical to the transfer matrix of the 6-vertex model (in the subspace where $\mathcal{T}_{6v}$ is stochastic), when the sites are being updated sequentially instead of at random. The XXZ chain, Eq. (10), represents the so-called time continuum limit of the 6-vertex model. The latter describes the temperature evolution of equilibrium 2D crystal surfaces. The anti-ferro electric side of the 6-vertex model describes conventional surface roughening transitions [4].

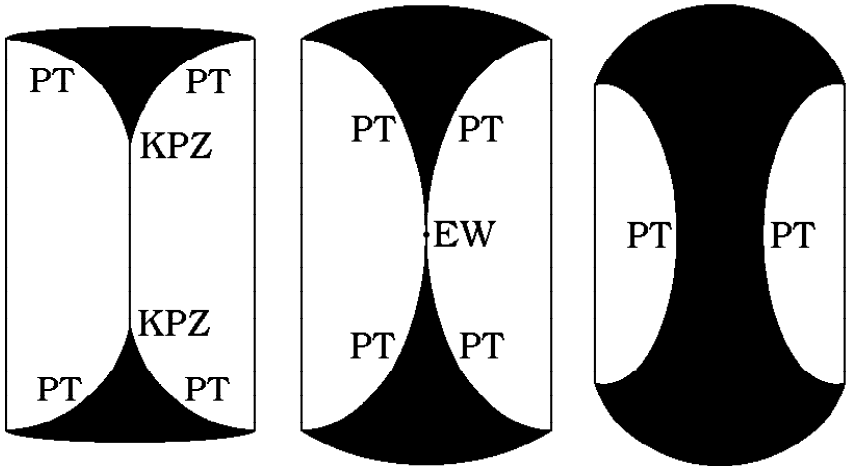

Figure 3: KPZ and EW type critical points in equilibrium shapes of crystals

The ferro electric side is less well known. The interactions favour facetting. At low T two facets meet at a facet-ridge, as shown in Fig.3. KPZ type growth maps exactly onto the endpoint of this facet-ridge where the two facets separate into two PT transition lines with rough rounded surface in-between. The TL-liquid represents the rough phase. The vertical spatial direction represents time in the fermion model and master equation. A tilt in the vertical direction is equivalent to a persistent current and finite growth rate. A tilt in the horizontal direction requires an additional chemical potential term in Eq.(10). λ couples to the temperature such that lines of constant temperature are circles in Fig.1. The facet ridge shortens with temperature. The point where it vanishes (the KDP point) maps onto EW type growth [30].

At PT transitions the surface rounds smoothly [4]. At facet-ridge endpoints it changes discontinuously since in KPZ growth the surface has a finite slope in the time-like direction (it grows). The dynamic exponent $z = 1.5$ translates into anisotropic scaling. The correlation lengths in the horizontal and vertical direction diverge with different exponents.

Why did we not know about KPZ type scaling for a long time already? After all, Lieb solved the 6-vertex model in 1967 [32]. Many aspects of the Bethe Ansatz solution have only been worked out very recently. In retrospect, the "KPZ" critical point is visible in Fig. 38 of the famous review paper by Lieb

and Wu [32], but it took van Beijeren and Nolden [4], Dhar[15], Gwa and Spohn [16], Bukman and Shore [33], Kim [18], and many others, to work out the details.

4 Is there a z-Theorem for Dynamic Processes in D=1+1?

Crossover scaling plays an important role in our quest towards a free field description of $z \neq 1$ dynamic scaling. The crossover towards $z = 1$ conformal field theory class is of particular interest, because phase diagrams like Fig.1 suggest that generically $z \neq 1$ theories lie at the edge of the TL-liquid. The following almost trivial example suggests that crossover scaling properties are severely restricted.

Consider the 1D free fermion model, i.e., eq. (10) at $\lambda = s = 0$ but in the presence of a chemical potential term $\mu a_n^+ a_n^-$. The fermions are non-relativistic when the chemical potential coincides with the bottom of the energy band, $E(k) \simeq k^2$. There the massgap scales as $m \simeq (2\pi/N)^2$ because the momenta are spaced as $\Delta k = 2\pi/N$. This implies that $z_0 = 2$. The fermions are relativistic at $k_F > 0$. There the massgap scales as $m \simeq 4\pi k_F/N$ and $z_1 = 1$. The crossover scaling function $m = (k_F + 2\pi/N)^2 - k_F^2 = 4\pi k_F/N + (2\pi/N)^2$ has two important properties. Firstly, the crossover exponent y_s at $k_F = 0$ (defined as $dm/dk_F \sim N^{-z_0+y_s}$) is equal to the change in the dynamic exponent: $y_c = z_0 - z_1 = 1$. Secondly, the FSS amplitude of the massgap is not a constant in the metal, but a universal number multiplied with the rapidity k_F.

This property has deep roots. The metal phase belongs to the conformal $z = 1$ dynamic universality class. The rapidity couples to the stress tensor, which lies at the core of CFT. This is a redundant operator which does not renormalize under scale transformations (a RNR-type operator). The rapidity does not change under renormalization. Systems with different rapidity are equivalent apart from a scale factor k_F.

The rapidity represents lattice anisotropy in ordinary critical phenomena. Ising critical points on a square lattice and a rectangular lattice are identical by a rescaling the unit of length in one direction. Contours of constant correlation are ellipsoids instead of spheres. ξ_x and ξ_τ diverge with the same exponent, $\xi_x \sim \xi_\tau \sim |T - T_c|^{-1/y_T}$. Many 2D critical points can be mapped into the $z = 1$ TL-liquid in Fig.1 [3,25]. Changing lattice anisotropy corresponds to changing the slope of the Fermi surface. The scaling properties inside the TL-liquid are strictly proportional to the rapidity.

Suppose that energy gap crossover scaling functions are of the generic form

$$m \simeq A/N^{z_0} + B\ \lambda/N^{z_1} \tag{14}$$

with A and B universal numbers and λ the rapidity scaling field associated with the RNR-type crossover operator. If this is true, dynamic processes with larger z should be unstable. This is somewhat similar to multi-critical points in CFT, which have a larger central charge c according to the "c-theorem" [39]. Moreover, the dynamic exponents of non-trivial theories would then follow from the $y_c = z_0 - z_1$ relation by calculating the crossover exponent in the unstable theory with larger z.

Eq.(14) is too simplistic, but many of it properties are valid in general. Consider the crossover scaling of the energy gap between an unstable dynamic multi-critical point, at $s = 0$, and a dynamic critical line, at $s > 0$,

$$m(N^{-1}, s) = b^{-z_0} m(bN^{-1}, b^{y_s} s) \sim N^{-z_0} F(N^{y_s} s). \tag{15}$$

The multi-critical point has dynamic exponent z_0, and y_s is its crossover scaling exponent in the s-direction. $F(u)$ is a scaling function. Along paths with fixed $s \neq 0$ the massgap must scale with dynamic exponent z_1, $m \sim N^{-z_1}$. This implies that the scaling function $F(u)$ must behave in the large u limit as $F(u) \sim u^{(z_0 - z_1)/y_s}$. This determines the power with which the amplitude of the mass gap vanishes or diverges on approach of the multi-critical point:

$$m \sim N^{-z_1} s^{(z_0 - z_1)/y_s}. \tag{16}$$

This is nothing unusual. The same analysis applies to any expectation value $\mathcal{O}$ (for example magnetization) at the crossover from e.g., a multicritical point to a critical line in conventional equilibrium critical phenomena

$$\mathcal{O}(N^{-1}, s) = b^{-x_0} \mathcal{O}(bN^{-1}, b^{y_s} s) \tag{17}$$

with x_0 the critical dimension of the operator $\mathcal{O}$. The amplitude of the massgap vanishes or diverges on approach of the multicritical point, depending on the sign of $x_0 - x_1$. This simply reflects the details of how the renormalization flow lines emerge from the multicritical point.

Suppose that from the perspective of the critical line at $s > 0$ the crossover field is an exact scaling field, a RNR-type operator, such that s does not renormalize. The naive choice to parameterize the crossover field, s, might not be the natural one. Let λ be the underlying rapidity-type parameter along the critical line in terms of which all scaling amplitudes are strictly proportional (or inverse proportional as explained below). Assume that close to the multicritical point, $\lambda \sim s^x$. In the above crossover from non-relativistic to relativistic fermions, the chemical potential μ is the naive choice for s, but the rapidity is the proper one, $\lambda \sim k_F \sim s^{1/2}$, i.e., $x = \frac{1}{2}$.

Assume that all scaling amplitudes along the critical line are strictly proportional to λ. This condition, $m \sim N^{z_1} s^x$, severely limits the crossover scaling properties. Eq. (16) implies the exponent equality $z_1 = z_0 - xy_s$. The critical exponents, z_0 and y_s, of the unstable dynamic universality class completely determine the dynamic exponent, z_1, of the stable one provided we know the value of x.

In the free fermion example we reach an edge of $z = 1$ CFT where the rapidity vanishes. In such cases $x > 0$ and the dynamic exponent always decreases, $z_1 < z_0$. It is also possible that the rapidity diverges at the edge. In those cases $x < 0$ and the dynamic exponent always increases, $z_1 > z_0$. These two possibilities are equivalent by the following space-time duality.

Consider the STM representation, and reformulate the problem by running the transfer matrix horizontally instead of vertically (turn the lattice over 90 degrees). This interchanges the role of time and space. $l_t \sim l_x^z/s$ transforms under $l_x \leftrightarrow l_t$ into $l_t \sim (l_x s)^{1/z}$. Each dynamic universality class with dynamic exponent z is therefore equivalent to a dual one with $z' = 1/z$. An example of this is discussed by Doochul Kim [38].

Eq.(15) describes the scaling of the inverse correlation length in the time-like direction, $m = \xi_\tau^{-1}$, for a semi-infinite lattice directed in the time-like direction. In the dual representation the lattice is semi-infinite in the spatial direction. The dual energy gap m' represents the inverse correlation length in that set-up. Therefore, Eq.(15) transforms as

$$m'(N^{-1}, s) = b^{-1/z_0} m'(bN^{-1}, b^{y_s/z_0} s) \sim N^{-1/z_0} F(N^{y_s/z_0} s). \qquad (18)$$

Along paths with fixed $s \neq 0$, the dual massgap must scale as $m' \sim N^{-1/z_1}$. This determines (as before) the power with which the amplitude vanishes,

$$m' \sim N^{\frac{-1}{z_1}} s^{\frac{z_0}{y_s}(\frac{1}{z_0} - \frac{1}{z_1})} \sim N^{-\frac{1}{z_1}} s^{-\frac{x}{z_1}} \qquad (19)$$

using the relation $z_1 = z_0 - xy_s$. Under space-time duality the crossover scaling exponents transforms therefore as: $z_0' = 1/z_0$, $z_1' = 1/z_1$, $y_s' = y_s/z_0$, and $x' = -x/z_1$. The dual exponents still obey the exponent equality $z_1' = z_0' - x'y_s'$. Therefore, for each theory with $z > 1$ and a vanishing λ there exists an equivalent dual one with $z' = 1/z$ and a diverging λ'. In the dual formulation the dynamic exponent of the unstable theory is always smaller, $z_1' > z_0'$.

The z-theorem can be summarized as follows: At the crossover between dynamic universality classes the dynamic exponent z_0 of the unstable theory and z_1 of the stable one, are relates by the exponent equality

$$z_0 = z_1 + xy_s \qquad (20)$$

provided that the crossover parameter s represents an exact scaling field which does not renormalize, and that the amplitude of the massgap along the critical line is strictly proportional to a parameter which scales as $\lambda \sim s^x$ close to the multicritical point.

5 Crossover into the Conformal z=1 Dynamic Universality Class

One of the most eye catching features of Fig.1 is that the TL-liquid phase is surrounded at its edge by dynamic critical points with $z \neq 1$. This includes PT transitions (non-relativistic fermions), KPZ and EW type growth, and also directed percolation as discussed below in section 7. Kosterlitz-Thouless transitions are the only $z = 1$ type critical points on the TL-liquid edge, as far as I know. They describe roughening transitions from a flat non-tilted phase into the rough (TL-liquid) phase. The crossover exponent is marginal, $y_s = 0$. KT transitions are weak infinite-order transitions because of this marginality.

The notion that many, if not all dynamics universality classes with $z \neq 1$ live on the edge of the $z = 1$ conformal theory (TL-liquids) is exciting. It should be helpful in constructing free field theories for $z \neq 1$ type scaling, but how is still an open question. The z-theorem implies a relation between the critical exponents z_0 and $z_1 = 1$, the crossover exponent y_s and the precise power x with which the rapidity of the TL-liquid vanishes or diverges. Unfortunately it does not predict the value of z_0 from the scaling properties of the TL-liquid only.

In Fig.1, the crossover scaling cascade between dynamic universality classes is such that z always decreases until it reaches its "natural lower bound" $z = 1$. This is due to the fact that all dynamic universality classes in Fig.1 have a dynamic exponent $z \geq 1$ and the rapidity of the TL-liquid always vanishes, $x > 0$. By space-time duality, there exists also a cascade where the dynamic exponent always increases until it reaches $z = 1$, in which all dynamic exponents are smaller than one, $z < 1$, and the rapidity in the TL-liquid always diverges, $x < 0$.

Are there any realizations of crossover scaling with $z > 1$ and $x < 0$? The crossover from KPZ type growth to free non-relativistic electron theory (PT transition) might be one of them. Fig.1 becomes three dimensional when we include the chemical potential term $\mu a_n^+ a_n^-$ into Eq.(10). The TL-liquid is bound in the μ direction by two PT transition planes. See also Fig.3. One plane corresponds to a metal-insulator transition to the state with zero fermion density and the other to the opposite state where all sites are occupied. These planes merge along the KPZ line in Fig.1. In these planes $z = 2$ while along the KPZ line $z = 1.5$. The unstable theory (the KPZ growth line) has a

smaller dynamic exponent than the stable one (the PT planes). Inside the PT planes, the curvature of the dispersion relation $E_k = \lambda k^2$ plays the same role as the rapidity for relativistic fermions. The amplitude of the energy gap is proportional to λ, $m = \lambda(2\pi/N)^2$. The z-theorem requires that λ diverges on approach of the KPZ line. This is plausible, because the merging of the two PT planes represents a collapse of the band width. It would be useful to check this in detail.

In section 3 it was assumed that the rapidity type parameter vanishes or diverges as a powerlaw at the multicritical point in terms of the naive crossover scaling field,$s \sim \lambda^x$. In some cases this function will be more singular. KPZ$\rightarrow$ PT type crossover might be an example of this. The naive crossover parameter should be abolished in favour of the natural one as early as possible.

6 Crossover from EW to KPZ type Growth

The z-theorem is much more powerful in the reverse mode, when the properties of the multicritical universality class are known, but those of the z_1 critical line are not. Suppose we know the exact values of z_0 and the crossover exponent y_s. Moreover, assume that on general grounds it is clear that the crossover operator is a RNR-type operator along the z_1 critical line in term of a certain λ parameterization, such that we know the value of x. The dynamic exponent z_1 along the critical line is then fully determined.

An example of this is the crossover from EW to KPZ type growth in D=1+1 (the line $\lambda = 1$ in Fig.1). EW type growth is trivial. The Langevin equation is the linear diffusion equation with noise. Its exponents follow from power counting: $z_0 = 2$ and $y_s = \frac{1}{2}$. The non-linear term in the KPZ Langevin equation is expected to be a RNR field. The KPZ equation is equivalent to the Burger's equation for randomly stirred fluids. In that context the scaling properties must be Galilean invariant. This implies that systems at a different s are equivalent apart from a scale factor s; i.e., that s itself is the proper scaling field, $x = 1$, and must play the same type of role in KPZ type growth as the stress tensor does in TL-liquids.

The z-theorem, $z_{KPZ} = z_{EW} - y_s$, immediately reproduces the well known value $z_{KPZ} = 1.5$. It is amazing that the dynamic exponent of the non-linear KPZ Langevin equation follows from simple power counting in the linear EW theory.

One of the conventional derivations of z_{KPZ}, employs the exponent equality $\alpha + z = 2$ [6], between the dynamic exponent z and the roughness exponent α of the stationary state. This equality follows from Galilean invariance. The dynamic exponent must be equal to $z_{KPZ} = 1.5$ if the roughness exponent

is equal to $\alpha = \frac{1}{2}$. In the stationary state surface height differences scale as $\delta h \sim l^{\alpha}$) with $\alpha = \frac{1}{2}$ if beyond a certain correlation length scale the probability to go up or down along the surface is random. This is a reasonable assumption (in D=1+1) and consistent with all numerical evidence. The z-theorem derivation seem more general, however. It does not make any reference to the stationary state.

John Neergaard and I studied EW→KPZ crossover in more detail in the context of the BCSOS growth model of section 3 [30]. The (simplest) form of the crossover scaling function Eq.(14) does not apply, because a special symmetry in the BCSOS model requires that the energy gap is even in the growth field. Not too many simple formula's obey both $y_c = z_0 - z_1$ and $m \simeq As/N^{z_1}$. One of them is

$$m = \frac{A}{N^{z_0}}[1 + Bs^2 N]^{y_s}. \tag{21}$$

We tested this ansatz numerically for systems sizes $N \leq 18$ [30]. It is accurate within 7% for all values of s. More recently Doochul Kim studied this issue analytically, using the Bethe Ansatz [18]. Eq.(21) is a good approximation, but not exact.

The z-theorem is not limited to D=1+1. What does it predict for KPZ scaling in D>2? Galilean invariance of the Burger's equation is valid in all dimensions. It tells us that s is a RNR-type operator, and suggests that all scaling amplitudes are proportional to s in all dimensions. This seems to be at odds with the fact that the EW fixed point changes stability in D=2+1. The crossover exponent is equal to $y_s = (3 - D)/2$. In the KPZ ϵ-expansion a new unstable fixed point appears at finite s in D>3 [8]. It is plausible that the "strong coupling" KPZ fixed point has no relation whatsoever to the EW fixed point in D>3.

7 Crossover from Isotropic to Directed Percolation

Directed percolation (DP) has emerged in recent years as one of the most common dynamic universality classes. It applies to a wide array of dynamic processes, ranging from flow through a porous medium in an gravitational field, forest fires and epidemic growth, to surface chemical reactions [7]. In 1+1 dimensions the DP critical exponents are known accurately from numerical studies in the early eighties [9], $z_{DP} = 1.58$, but analytic insight in these values is still lacking.

At first, it seems likely that the z-theorem will give useful information about DP, when applied to the crossover from isotropic percolation (IP) to DP. IP is described by $z = 1$ TL-liquid theory and therefore completely understood.

Field theory [22] suggests that IP (the known theory) is unstable towards DP, just like in EW $\rightarrow$ KPZ type growth.

The result is somewhat disappointing. Per Fröjdh and I studied the crossover from IP to DP by representing the combined problem as a random cluster model. A parameter r controls the spontaneous birth of new forest fires. We obtained the exact crossover exponent at IP ($r = 1$) using Coulomb gas methods in D=2: $y_{DP} = y_T - 1$. IP is stable since in 2D $y_T = 3/4$ [41].

We checked this numerically with FSS. The results confirm that at all intermediate values of $0 < r < 1$ the scaling properties of the model are the same as at IP. The correlation length in the time-like direction diverges with the same power as the spatial one $\xi_\tau \sim \xi_x \sim |p - p_c|^{1/y_T}$. r only affects their amplitude ratio. So the scaling properties of forest fires in which new fires can ignite are different from those where spontaneous ignition is forbidden.

Our formula $y_{DP} = y_T - 1$ suggests that IP changes stability between $2 < D < 3$; in accordance with the field theory result close to the upper critical dimension. This is hard to swallow. We developed an intuitive argument [23] which suggests that DP at $r = 0$ is unstable not only in 2D but in all dimensions (that the scaling properties of forest fires at intermediate values r are in the same universality class as IP).

This intuitive argument is not necessarily at odds with our exact result. The crossover operator at IP is a gradient, and therefore suspect. Such operators can be integrated-up into a surface term and sometimes vanish from the theory altogether. How to reconcile this with the field theoretical RT study by Frey *et. al* [22] is yet unclear. They find in dimensions close to D_u=5, a RG flow from IP towards DP. Our model is microscopic and well defined, while their field theory seems somewhat difficult to interpret in the intermediate regime.

The crossover from IP to DP turns out to be yet another example of a crossover from the $z = 1$ conformal class to a $z \neq 1$ dynamic universality class, with the stress tensor of CFT the crossover operator. Like KPZ growth, DP lives at the edge of $c = 1$ CFT. To check this explicitly, Per and I are studying the crossover scaling at the DP point numerically in more detail. The rate at which the CFT rapidity vanishes (the exponent x) must agree with the crossover exponent y_s. These are difficult quantities to evaluate numerically. Our current results are in agreement with the z-theorem, but the numerical convergence is not good enough to clearly demonstrate this.

8 Conclusions

The first purpose of this talk was to illustrate that master equations of non-equilibrium processes are equivalent to transfer matrices of equilibrium statis-

tical mechanics (in one higher spatial dimension), and also to the ground state properties of quantum field theories. Equilibrium STM transfer matrices form the larger class. Dynamic universality classes should be defined in that more general context, and not be limited to stochastic time evolution operators.

In D=1+1 dimensions, many (if not all) $z \neq 1$ dynamic universality classes are located at the edge of the $z = 1$ conformal class. This suggests that free field theory descriptions for $z \neq 1$ scaling phenomena might exist in D=1+1; yet unknown generalizations of CFT. The study of crossover scaling properties is a first step into this direction. In section 4, I formulate the z-theorem, i.e., a condition between the change in dynamic exponent and the crossover exponent. The crossover from EW to KPZ type growth, and from isotropic to directed percolation are examples where the z-theorem is satisfied.

In the title of this talk the existence of the z-theorem is presented as a question. Its validity requires that the crossover parameter does not renormalize under scale transformations, a RNR-type operator. There is no intrinsic reason why the crossover operator must always be of this type. The z-theorem helps us to identify such RNR-operators in each dynamic universality class. This is one of the essential tasks in our quest towards a free field representation. In $z = 1$ CFT that operator is the stress tensor and it properties lie at the core of conformal invariance.

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