1. **The geometry is**

![Grounded Plate](image)

![Plate at \( \Phi_0 \)](image)

Assume there's an infinite source of charge at the plates.

With the simplification of no fringing field, \( \vec{J} \) is constant between the plates, as well \( \vec{J} \sim \vec{E} \).

Poisson's equation is satisfied between the plates with \( \nabla^2 \Phi \rightarrow \frac{d^2}{d^2} \Phi(2) \).

And with source \( J = \rho V \)


If the potentials aren't too high, the electron speed is non-relativistic with \( e\Phi = \frac{1}{2}mV^2 \).
Poisson's Equation then reads

\[ \frac{d^2 \Phi (z)}{dz^2} = \frac{1}{\varepsilon_0} \left\{ \frac{m}{2 \varepsilon \Phi (z)} \right\}^{1/2} \]

Solve for \( \Phi \) by substituting

\[ \Psi = \frac{d \Phi}{dz} \]

Then

\[ \Psi \frac{d \Psi}{dz} = \frac{1}{\varepsilon_0} \left\{ \frac{m}{2 \varepsilon \Phi (z)} \right\}^{1/2} \]

Having solution

\[ \frac{1}{2} \Psi^2 = 2 \frac{1}{\varepsilon_0} \left\{ \frac{m}{2 \varepsilon \Phi (z)} \right\}^{1/2} \]

At the ground terminal (source for \( e^- \)), the \( e^- \) start off at rest, so

with this boundary condition \( \frac{d \Phi}{dz} \bigg|_{z=2} = 0 \)

\[ \frac{d \Phi}{\sqrt{\Phi}} = 2 \left\{ \frac{1}{\varepsilon_0 \sqrt{2 \varepsilon}} \right\}^{1/2} \, d^2 \Phi \]

On integrating with boundary condition

\( \Phi (\text{ground}) = 0 \), \( \Phi (\text{other plate}) = \Phi_0 \)

\[ \frac{1}{3} \Phi_0^{3/4} = 2 \left\{ \frac{1}{\varepsilon_0 \sqrt{2 \varepsilon}} \right\}^{1/2} \, d \]

with \( d \) the plate spacing.
\[ J = \frac{4 \varepsilon_0}{9d^2} F_0 \left\{ \frac{2e}{m} \right\}^{\frac{1}{2}} \]

This is the "Space-Charge-Limited Current" in vacuum. At this corresponding charge density between the plates, all the E-field lines sourced at the plate get "sunk" on free charges; none are "sunk" at the other plate.

There are numerous incorrect solutions of mine to this floating around; a 513 student finally sorted it out (thanks).
2. The geometry is

![Diagram](image)

**What is the solid angle subtended by the ring at the field point?** We imagine the loop is on the surface of a sphere, the sphere being centered on the field point.

The solid angle is then

\[
\Omega (x) = \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi
\]

Each area element \(d\Omega\) of the sphere is

\[
d\Omega = (x^2 + R^2) \sin \theta \, d\theta \, d\phi.
\]

Also \(\sin \theta_0 = \frac{x}{\sqrt{x^2 + R^2}}\), hence

\[
\Omega (x) = 2\pi (\cos \theta_0 - 1) = 2\pi \left(\frac{x}{\sqrt{x^2 + R^2}} - 1\right)
\]
RECALL THE GEOMETRIC FORM OF THE MAGNETIC SCALAR POTENTIAL \( \Phi_m = \frac{\mathbf{I}}{4\pi} \mathbf{r} \cdot \mathbf{B} \).

with \( \mathbf{B} = -\mathbf{M}_0 \nabla \Phi_m \)

\[
\mathbf{B} = -\mathbf{M}_0 \frac{d}{dx} \Phi_m \quad (x^x)
\]

Hence

\[
\mathbf{B} = \frac{\mathbf{M}_0}{4\pi} I \frac{d}{dx} \left\{ 2\pi \left( \frac{x}{\sqrt{x^2 + R^2}} \right) \right\}
\]

\[
= \frac{\mathbf{M}_0}{2} I \frac{R^2}{(x^2 + R^2)^{3/2}} \quad (x^x)
\]

WE HAD SEEN THIS EARLIER FROM A DIRECT EVALUATION OF THE BIOT-SAVART LAW. THIS PARTICULAR FORMULATION IS FROM ZANEWILL.

YOU MIGHT PONDER WHY IN THIS CASE IT'S OK TO USE THE SCALAR POTENTIAL WITH TRUE CURRENTS
3.9. Via equivalent currents, there are no bulk currents $J = \nabla \times M$, but there are surface currents $\vec{J} = \nabla \times \vec{A}$.

Start with the on-axis magnetic induction of a circular current loop of radius $R$ carrying current $I$ (cf., Problem 2):

$$B_r(2) = \frac{\mu_0 I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}}$$

We can model the magnetization surface current as composed of many loops. This has geometry:

$$d B_r(2) = \frac{\mu_0}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} \, d \theta$$

$$= \frac{\mu_0}{2} K \, d \cos \theta$$

so

$$B_r(\theta_e) = \frac{\mu_0}{2} K \int_0^1 d \cos \theta = \frac{\mu_0}{2} K \left\{ 1 - \cos \theta_e \right\} \cos \theta_e$$

Since the two pole faces contribute equally

$$B_r(\theta_e) = M_0 M \left\{ 1 - \cos \theta_e \right\}^2$$
b. The equivalent charge density on each pole face is $\mathbf{M} \cdot \mathbf{A}$. Looking end-on to a pole face, we have geometry.

The surface charge is a Coulomb-like source of $\mathbf{H}!$ For a ring of charge shown

$$dH = \frac{M 2\pi rd\pi}{(r^2 + l^2)^{3/2}}$$

Again integrate out to $R$:

$$H = \frac{M}{2} \int_0^R \frac{r^2 dr}{(r^2 + l^2)^{3/2}} = \frac{M}{2} \left(1 - \cos \theta \right)$$

With two pole faces, multiply by 2.

For $B$, multiply by $\pi a^2$.

This is the same as (g)
4. PER THE HINT, THERE ARE TWO SETS OF SOLUTIONS: $r > R$ AND $r < R$. THE $r > R$ SOLUTION SATISFIES LAPLACE’S EQUATION, THE $r < R$ SOLUTION POISSON’S EQUATION.

With $\frac{dB}{dz} = 0$, THE MAGNETIC INDUCTION IS

$$B_x = \frac{d}{dy} A_z, \quad B_y = -\frac{d}{dx} A_z.\]$$

LAPLACE’S EQUATION REDUCES TO $\nabla^2 A_z = 0$,
POISSON’S EQUATION REDUCES TO $\nabla^2 A_z = -MJ$.

$r < R$. THERE IS ADDITION THE PARTICULAR SOLUTION $A_z = -MJr^2/4$.

THE SOLUTION IS REGULAR AT $r = 0$, SO NEGATIVE POWERS OF $r^n$ ARE ABSENT;
THE FORM OF THE SOLUTION IS THEN

$$A_z = -\frac{MJr^2}{4} + \sum_{k=0}^{\infty} (a_k \cos \theta + b_k \sin \theta) r^k \tag{RC R}$$

$r > R$. AT LARGE DISTANCES THE MAGNETIC FIELD APPROACHES THE CONSTANT FIELD $B_0$, SO $B_0 y = B_0 \sin \theta$ AND THE THE CURRENT DISTRIBUTION HAS THE EFFECT OF A LINE CURRENT.
\[ A_2 \sim \frac{m_0 I}{2\pi} \ln \frac{1}{r}. \] For \( r > R \), positive powers of \( r^n \) are absent. The form of the solution is

\[ A_2 = \frac{m_0 I}{2\pi} \ln \frac{1}{r} + B_0 r \sin \theta \]

\[ + \int_0^\infty \left( \cos \phi + \frac{1}{2} \sin \phi \right) \frac{1}{r^2} \, d\phi \]

with \( I = 4\pi R^2 \).

(Some care needs to be paid to the \( \ln \frac{1}{r} \) term, there is an implied length scale \( R_0 \) for \( \ln \frac{R_0}{r} \).)

**Boundary conditions:**

- Parallel components are the same.
- \( A_2 \) is the same on both \( 500 \).

- Parallel components of \( \vec{H} \) are the same on both \( 500 \):

\[ \frac{1}{m} \frac{d}{dr} A_2(r=R) \bigg|_R = \frac{1}{m} \frac{d}{dr} A_2(r=R) \bigg|_R \]
This has the same structure as the similar electrostatics problem.
Applying boundary conditions

\[ A_2 (r < R) = -\frac{\mu J r^2}{4} + \frac{2\mu}{r} B_0 r \sin \theta \]

\[ A_2 (r > R) = \mu_0 \frac{4\pi r^2 J}{2\pi} N_0 \left( \frac{R}{r} \right) \]

\[ -\frac{\mu J R^2}{4} \]

\[ + \left( 1 + \frac{\mu - \mu_0}{\mu + \mu_0} \frac{(R/r)^2}{3} \right) B_0 r \sin \theta \]