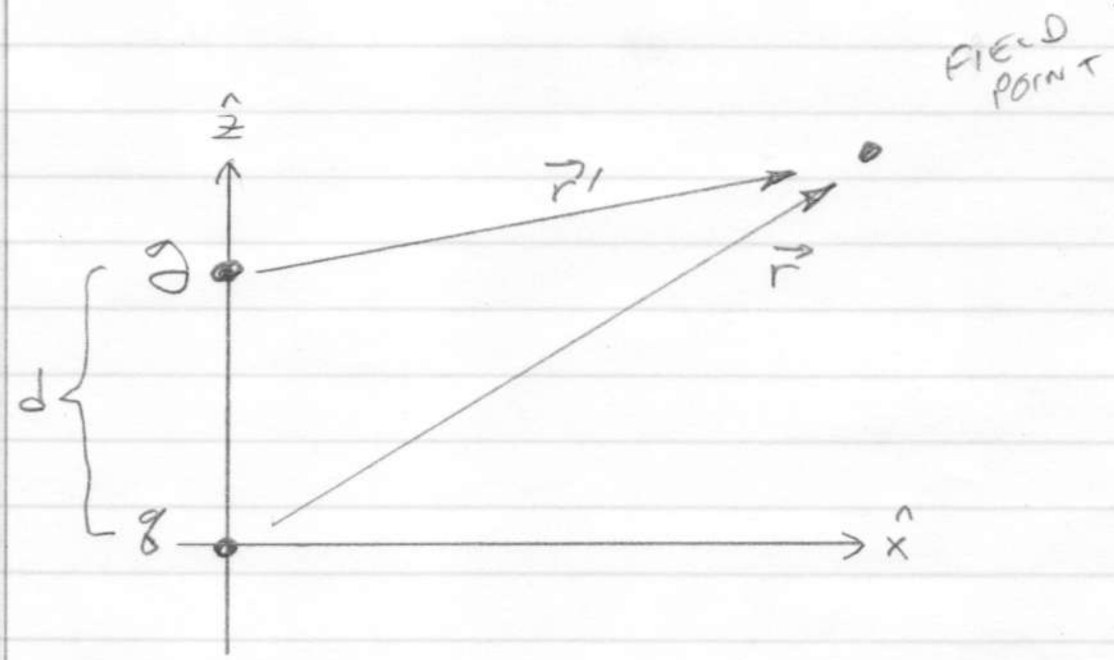
DERIVE JACKSON EQN. 6.159

EVALUATE THE PAIR OF A POINT MAGNETIC MONOPOLE g AND A POINT CHARGE q.

RECALL $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

WE THEN HAVE $\vec{B} = \frac{\mu_0}{4\pi} \frac{g}{r^2} \hat{r}$

APPLY THESE FIELDS TO THE SYSTEM:



$$\vec{B} = \frac{\mu_0}{4\pi} g \frac{\vec{r}'}{r'^3} = \frac{\mu_0}{4\pi} g \frac{\vec{r} - d\hat{z}}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} q \frac{\vec{r}}{r^3}$$

FIND THE MOMENTUM DENSITY

JACKSON E. 6.118 $\vec{\wp} = \epsilon_0 \vec{E} \times \vec{B}$

$$\vec{\wp} = \frac{\mu_0}{(4\pi)^2} q g \frac{-d\,\vec{r} \times \hat{z}}{r^3(r^2 + d^2 - 2rd\cos\theta)^{3/2}}$$

NOTICE THIS INTEGRATED VANISHES OVER ALL SPACE

FIND THE ANGULAR MOMENTUM DENSITY

$$\vec{\mathcal{L}} = \vec{r} \times \vec{\wp}$$

$$\vec{\mathcal{L}} = \frac{\mu_0}{(4\pi)^2} q g d \frac{-\vec{r} \times (\vec{r} \times \hat{z})}{r^3(r^2 + d^2 - 2rd\cos\theta)^{3/2}}$$

EVALUATE THE TRIPLE PRODUCT $\vec{r} \times (\vec{r} \times \hat{z})$

(JACKSON INSIDE FRONT COVER)

$$\vec{r} \times (\vec{r} \times \hat{z}) = \vec{r}(\vec{r} \cdot \hat{z}) - \hat{z}(\vec{r} \cdot \vec{r})$$

$$= \hat{r}\, r^2 \cos\theta - \hat{z}\, r^2$$

NOTICE IN $\vec{r} \times (\vec{r} \times \hat{z}) = \hat{r} r^2 \cos\theta - \hat{z} r^2$, THE X- AND Y- COMPONENTS ARE SPATIALLY ODD SO WILL VANISH WHEN INTEGRATED OVER ALL SPACE.

THIS LEAVES ONLY Z COMPONENTS

$$\left[\vec{r} \times (\vec{r} \times \hat{z})\right]_z \sim \hat{r}_z r^2 \cos\theta - \hat{z} r^2$$

THE TOTAL ANGULAR MOMENTUM $\vec{L}$ IS

$$\vec{L} = -\frac{\mu_0}{(4\pi)^2} q g d \cdot \iiint \frac{r^2(\cos^2\theta - 1)}{r^3(r^2 + d^2 - 2rd\cos\theta)^{3/2}} r^2 \sin\theta \, dr \, d\theta \, d\phi \; \hat{z}$$

THIS IS A PROBLEM IN JACKSON.

LET $U = \cos\theta$,

ϕ INTEGRAL

$$\vec{L} = -\frac{\mu_0}{(4\pi)^2} q g d \cdot 2\pi \cdot \int_{-1}^{+1} \int_{0}^{\infty} \frac{r(1 - U^2)}{(r^2 + d^2 - 2rdU)^{3/2}} \, dU \, dr \; \hat{z}$$

$U \quad r$

AND IN GENERAL $\vec{L} = \frac{\mu_0}{4\pi} q g \hat{r}$

FIRST, THE "r" INTEGRATION:

$$\int_{0,\,r}^{\infty} \frac{r}{(r^2+d^2-2rdU)^{3/2}}\, dr$$

$$= \left.\frac{rU-d}{d(1-U^2)(r^2+d^2-2rd)^{1/2}}\right|_0^\infty = \frac{1}{d(1-U)}$$

THEN THE "U" INTEGRATION:

$$\vec{L} = \frac{\mu_0}{(4\pi)^2} q g \cdot d \cdot 2\pi \cdot$$

$$\frac{1}{d}\int_{-1,\,U}^{+1} \frac{1-U^2}{1-U}\, dU \quad \hat{z}$$

$$= \frac{\mu_0}{(4\pi)^2} q g \cdot 2\pi \cdot \int_{-1,\,U}^{+1} (1+U)\, dU \quad \hat{z}$$

$$= \frac{\mu_0}{(4\pi)^2} q g \cdot 2\pi \cdot \left[U + \frac{U^2}{2}\right]_{-1}^{+1} \quad \hat{z}$$

$$\vec{L} = \frac{\mu_0}{4\pi} q g \ \hat{z}$$