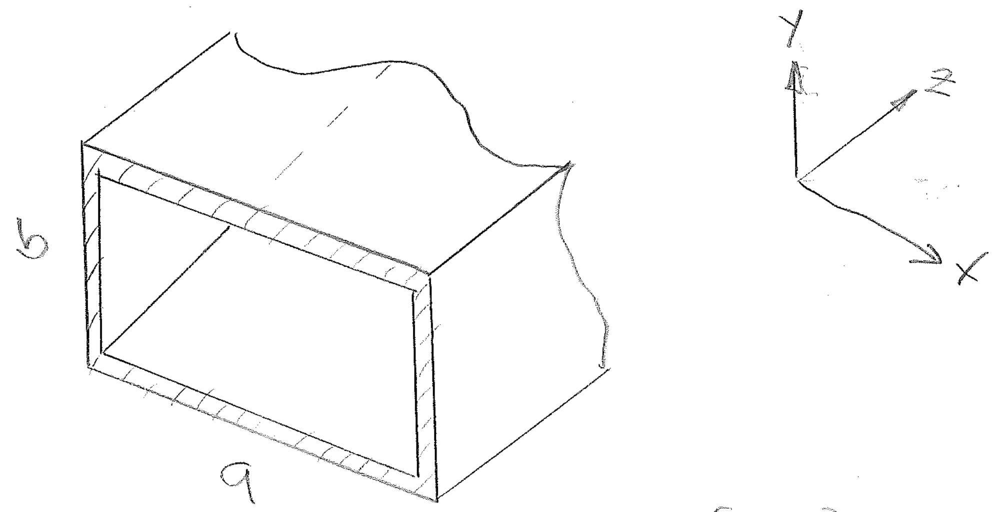
Physics 514, Electrodynamics II
Department of Physics, University of Washington
Winter quarter 2020
March 13, 2020, 11am
On-line lecture

Administrative
Last lecture today.
Homework 9 was due Friday, March 11, 11am. Slide HW under my office door C503 or scan and send it via email.
No final exam.
MRE postponed to early Spring quarter.

Lecture
Jackson Chapter 8: Waveguides & resonant cavities.
J.C.8.4 Recap: Modes in a rectangular and circular waveguide.
J.C.8.7 Resonant cavities formed from a section of waveguide with conducting end-walls.
J.C.8.1 Fields at the surface and within a conductor. This matters when the waveguide walls are lossy.
J.C.8.8 Power losses in a cavity; Q of a cavity.

RECAP:

1. RECTANGULAR WAVEGUIDE: TE MODES



$$\left[\nabla_t^2 + \left(-k^2 + \omega^2/c^2\right)\right]\begin{Bmatrix} E_z \\ B_z \end{Bmatrix} = 0$$

TE $\longrightarrow$ $E_z = 0$ EVERYWHERE.

SEPARATE VARIABLES:

$$B_z(x,y) = X(x)\,Y(y)$$

APPLY BOUNDARY CONDITION

$$\partial B_z/\partial n\big|_s = 0$$

$$\longrightarrow \partial B_z/\partial x\big|_{\substack{x=0\\x=a}} = 0$$

$$\& \ \partial B_z/\partial y\big|_{\substack{y=0\\y=b}} = 0.$$

MODE IS

$$B_{z;m,n} \sim \cos \underbrace{\frac{m\pi}{a}}_{K_x} x \; \cos \underbrace{\frac{n\pi}{b}}_{K_y} y$$

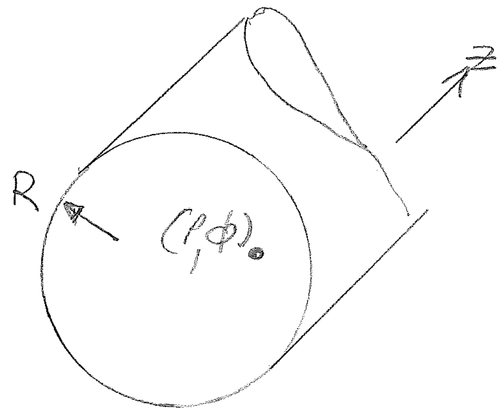
WITH $-K^2 + \omega^2/c^2 = -K_x^2 - K_y^2$.

WE FOUND A CUTOFF FREQUENCY

$$\omega_{c;mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

WHERE FOR $\omega < \omega_{c;mn}$, K IS PURE IMAGINARY.

2. ROUND WAVEGUIDE: TE MODES



SAME TRANSVERSE WAVE EQUATION,

TE $\longrightarrow$ $E_z = 0$ EVERYWHERE

SEPARATE VARIABLES

$$B_z(\rho,\phi) = R(\rho)\Phi(\phi).$$

$$\Phi(\phi) \sim e^{\pm i K_\phi \phi}$$

SINGLE-VALUED FOR $\phi \to \phi + n\phi$

IMPLIES K_ϕ IS AN INTEGER n.

$$\mathbb{R}(\rho) \sim J_n(K_c \rho),\ N_n(K_c \rho)$$

WITH $K_c^2 = -K^2 + \omega^2/c^2$

REGULAR $B_z(\rho=0) \to N_n$ ABSENT

APPLY BOUNDARY CONDITION

$$dB_z/dn\big|_s = 0 \to \frac{\partial B_z}{\partial \rho}\Big|_{\rho=R} = 0$$

REQUIRES $J_n'(K_c R) = 0$

SO $K_c = X'_{nm}/R \quad m = 1, 2, \ldots$

Roots of $J_m'(x) = 0$

$m = 0$:	$x'_{0n} = 3.832, 7.016, 10.173, \ldots$
$m = 1$:	$x'_{1n} = 1.841, 5.331, 8.536, \ldots$
$m = 2$:	$x'_{2n} = 3.054, 6.706, 9.970, \ldots$
$m = 3$:	$x'_{3n} = 4.201, 8.015, 11.336, \ldots$

JACKSON P. 370

IF $\omega^2/c^2 - K_c^2 < 0$, THEN K IS IMAGINARY AND THE WAVE DOESN'T PROPAGATE; THIS IS THE CUTOFF FREQUENCY

$$\omega_{c;nm} = c K_{c;nm}$$

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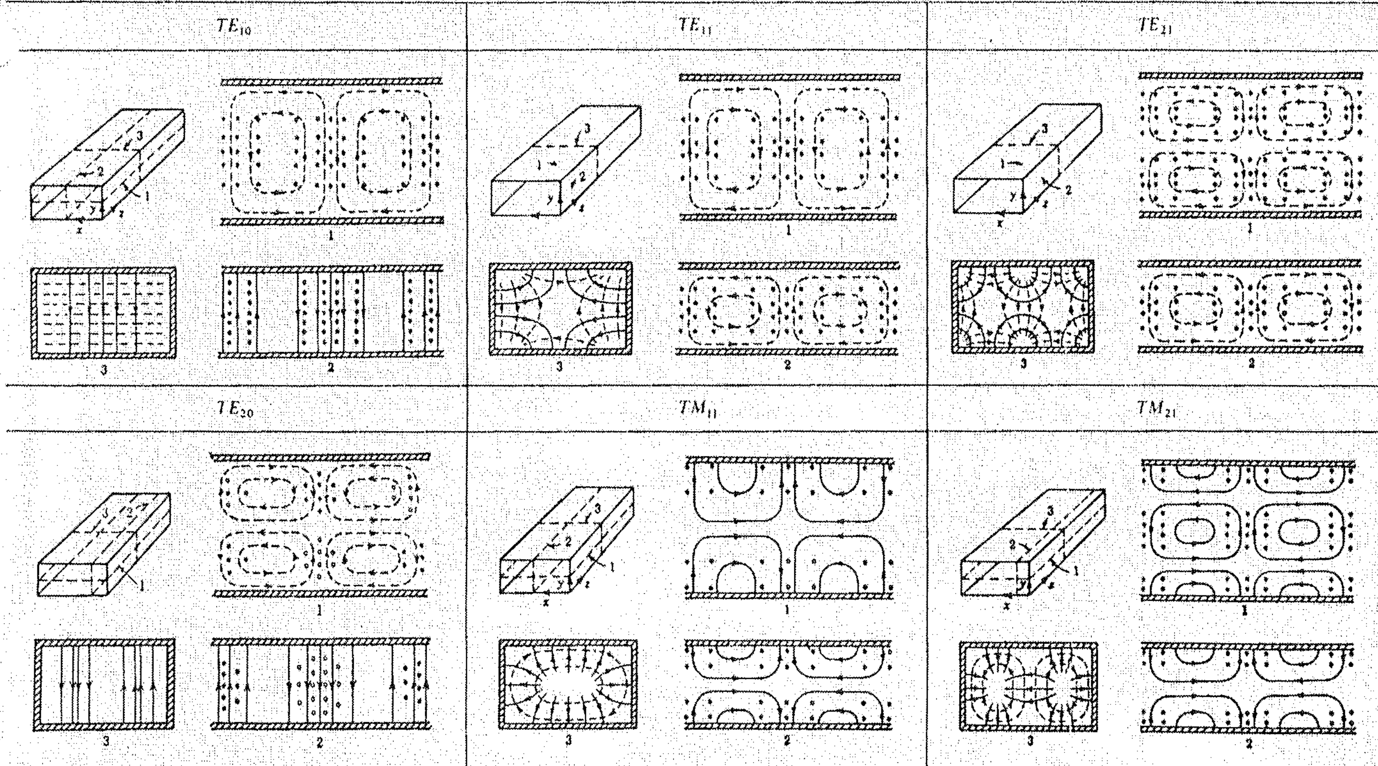
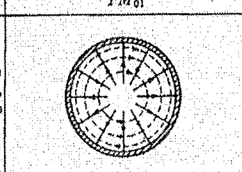
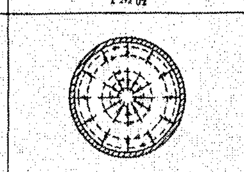
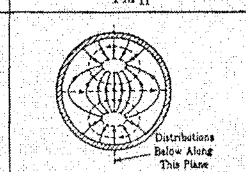
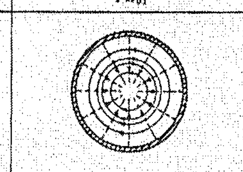
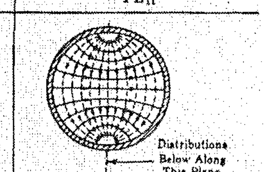
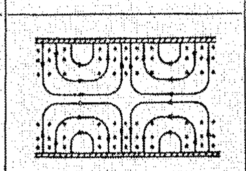
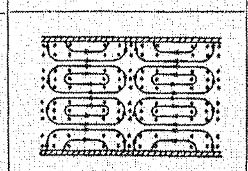
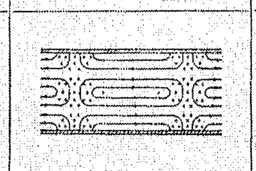
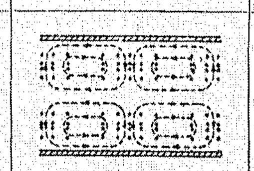
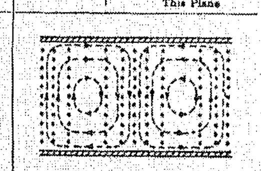
Table 8.7
Summary of wave types for rectangular guides[a]



[a] Electric field lines are shown solid and magnetic field lines are dashed.

Table 8.9
Summary of wave types for circular guides[a]

Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	$E_z, E_r, E_\phi, H_r, H_\phi$	H_z, H_r, E_ϕ	$H_z, H_r, H_\phi, E_r, E_\phi$
p_{nl} or p'_{nl}	2.405	5.52	3.83	3.83	1.84
$(k_c)_{nl}$	$\frac{2.405}{a}$	$\frac{5.52}{a}$	$\frac{3.83}{a}$	$\frac{3.83}{a}$	$\frac{1.84}{a}$
$(\lambda_c)_{nl}$	$2.61a$	$1.14a$	$1.64a$	$1.64a$	$3.41a$
$(f_c)_{nl}$	$\frac{0.383}{a\sqrt{\mu\varepsilon}}$	$\frac{0.877}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.609}{a\sqrt{\mu\varepsilon}}$	$\frac{0.293}{a\sqrt{\mu\varepsilon}}$
Attenuation due to imperfect conductors	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{(f_c/f)^2}{\sqrt{1-(f_c/f)^2}}$	$\frac{R_s}{a\eta}\frac{1}{\sqrt{1-(f_c/f)^2}}\left[\left(\frac{f_c}{f}\right)^2+0.420\right]$

[a] Electric field lines are shown solid and magnetic field lines are dashed.

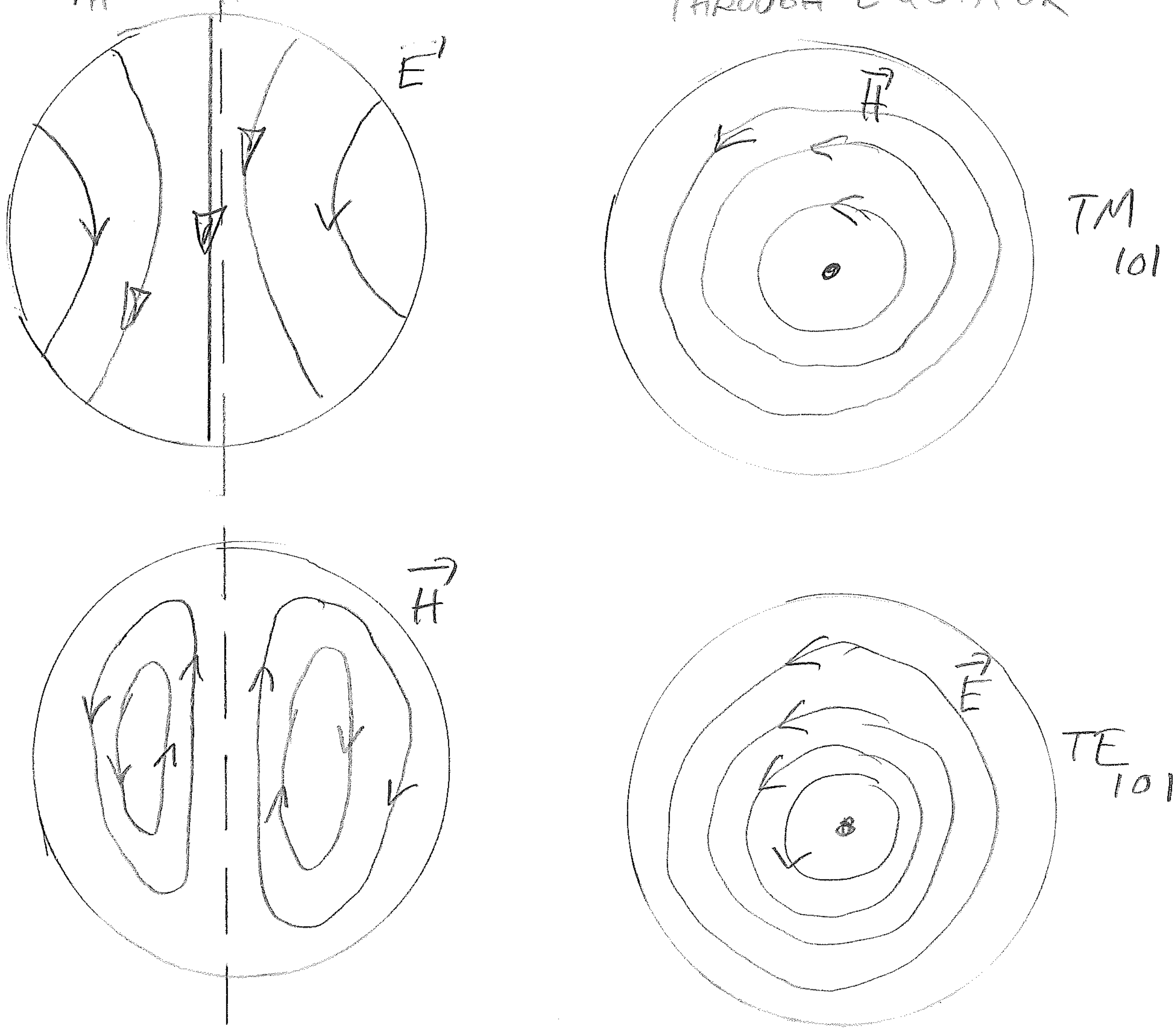
"FIELDS & WAVES IN COMMUNICATION ELECTRONICS", 2ND ED,
RAMO, WHINNERY & VAN DUZER

CAVITIES J, C, 8.7

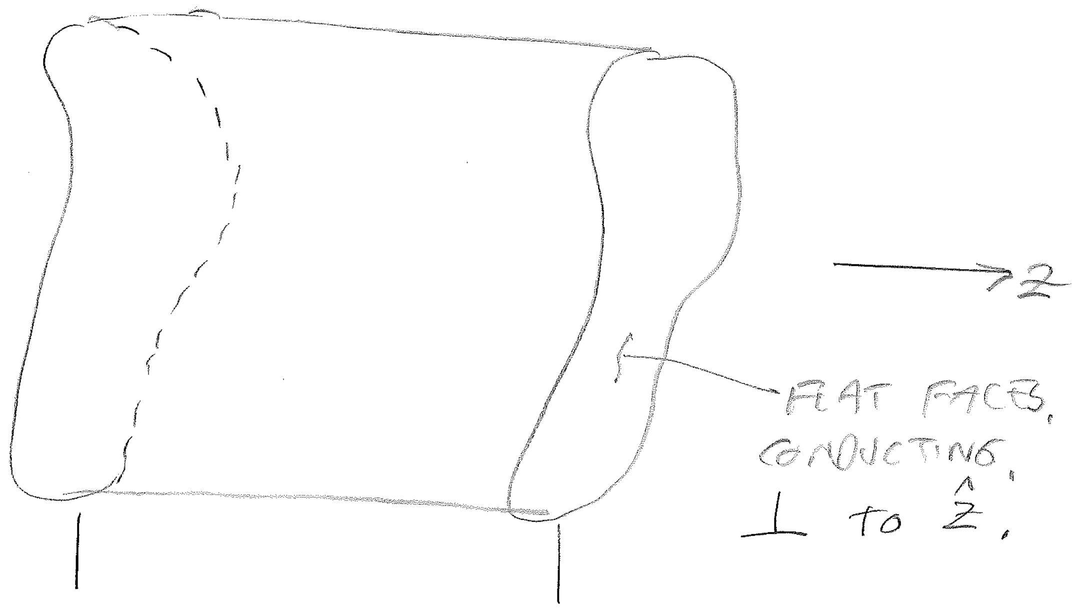
THE KIND OF CAVITIES DESCRIBED IN JACKSON ARE FORMED BY ADDING CONDUCTING FACES TO THE ENDS OF A LENGTH OF WAVEGUIDE; THIS DESCRIBES A BROAD CLASS OF CAVITIES, BUT, NOT, e.g., A SPHERICAL CAVITY

SECTIONS THROUGH AXIS

SECTIONS THROUGH EQUATOR



BACK TO WAVEGUIDE-CAVITIES



THE MAIN EFFECT OF ADDING END-FACES IS TO INTRODUCE STANDING WAVES ALONG $\hat{z}$.

$$\sim \sin kz, \quad \sim \cos kz.$$

THE $\vec{E}$-FIELD ON THE END-FACES SATISFY BOUNDARY CONDITIONS

- TM: $\vec{E}_t\big|_{z=0} = \vec{E}_t\big|_{z=d} = 0.$

RECALL INSIDE $\vec{\nabla}\cdot\vec{E} = 0$. JUST INSIDE THE END-FACES

$$\left[\vec{\nabla}_t \cdot \vec{E}_t + \frac{\partial}{\partial z} E_z\right]\Big|_{\substack{z=0\\z=d}} = 0.$$

WITH $\vec{\nabla}_t\cdot\vec{E}_t\big|_{\substack{z=0\\z=d}}$,

$$\frac{\partial}{\partial z} E_z \Big|_{z=0} = \frac{\partial}{\partial z} E_z \Big|_{z=d} = 0.$$

THE SIN KZ TERM IS THEREFORE MISSING, AND

$$E_z(x,y,z) \sim \cos \frac{\ell\pi}{d} z \; ; \; \ell = 0, 1, \cdots$$

WE HAVE A THIRD MODE INDEX.

- WHAT ABOUT TE MODES? AND EVERYWHERE INSIDE $B_z = 0$, AND IN PARTICULAR $B_z \big|_{z=0} = B_z \big|_{z=d} = 0$,

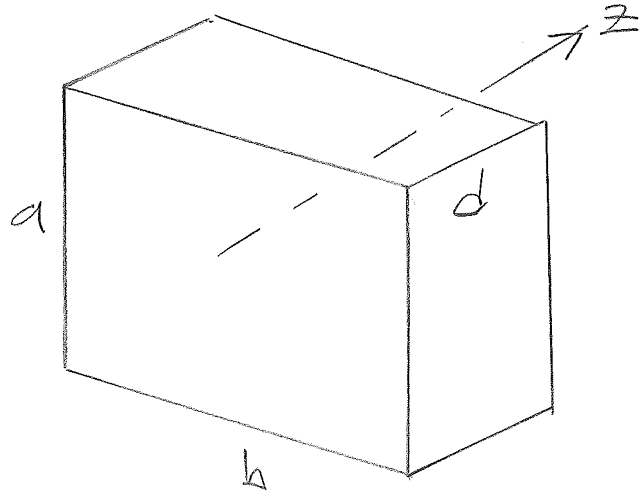
  THE COS KX TERM IS THEREFORE MISSING, AND

$$B_z(x,y,z) \sim \sin \frac{\ell\pi}{d} z \; ; \; \ell = 0, 1, 2 \cdots$$

- WITH K DISCRETIZED, ONLY CERTAIN RESONANT-FREQUENCIES ARE PRESENT.
- THE TRANSVERSE WAVE EQUATION IS

$$\left[\nabla_t^{\,2} + \Big(\underbrace{-\left(\frac{\ell\pi}{d}\right)^2 + \omega^2/c^2}_{K^2}\Big)\right] \begin{Bmatrix} \vec{E} \\ \vec{B} \end{Bmatrix} = 0$$

EXAMPLE: RECTANGULAR RESONATOR, TE MODES.



THIS IS SIMILAR TO THE PROBLEM OF THE RECTANGULAR WAVEGUIDE, WITH THE SUBSTITUTION

$$K^2 \longrightarrow \left(\frac{l\pi}{d}\right)^2.$$

RECALL THE SOLUTIONS TO THE RECTANGULAR WAVEGUIDE

$$K_x^2 + K_y^2 = \left(\frac{\omega}{c}\right)^2 - K^2,$$

BECOMING

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2 = \left(\frac{\omega_{mnl}}{c}\right)^2.$$

WE'VE FOUND THE RESONANT FREQUENCIES.

SUPPOSE WE WISH TO FIND ALL THE FIELD COMPONENTS.

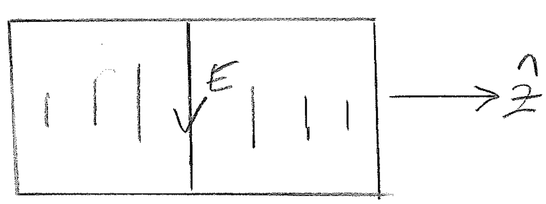
STEP 1: FIND $B_z(x,y)$ FROM THE RELATED WAVEGUIDE PROBLEM.

2. CONSTRUCT $B_z(x,y,z)$ VIA

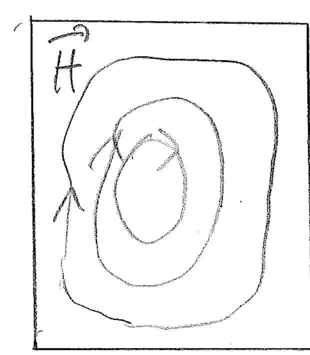
$$B_z(x,y,z) = B_z(x,y) \sin \frac{\ell\pi}{d} z .$$

3. $E_z = 0$, FIND $\vec{E}_t$ AND $\vec{B}_t$ FROM JACKSON EQNS 8.31, 8.33, AS USUAL.

A TYPICAL FIELD: e.g., TE_{101}



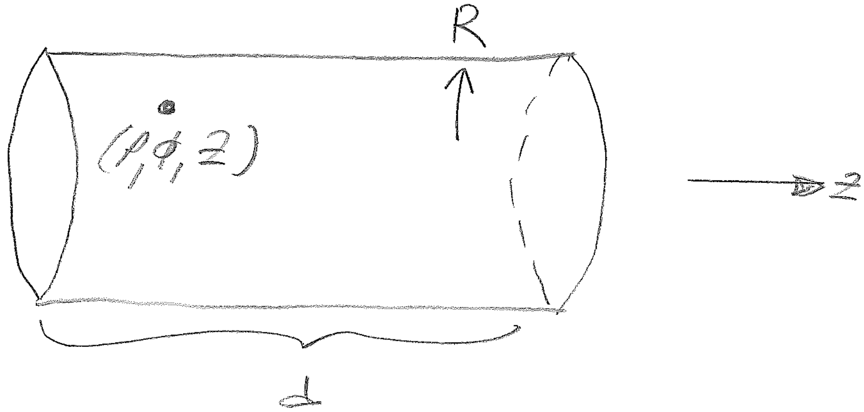
SIDE VIEW



TOP VIEW

EXAMPLE CIRCULAR RESONATOR,

TM MODES.



RECALL THE RELATED WAVEGUIDE PROBLEM

$$E_z(\rho,\phi) \sim J_n(k_{c;nm}\,\rho)\, e^{\pm i m \phi}$$

WITH $k_{c;nm} = X_{nm}/R$

ALSO $k^2_{c;nm} = (\omega/c)^2 - \left(\frac{\ell\pi}{d}\right)^2$

WE CAN SOLVE FOR THE RESONANT FREQUENCIES

$$\left(\frac{\omega_{nm\ell}}{c}\right)^2 = \left(\frac{X_{nm}}{R}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2$$

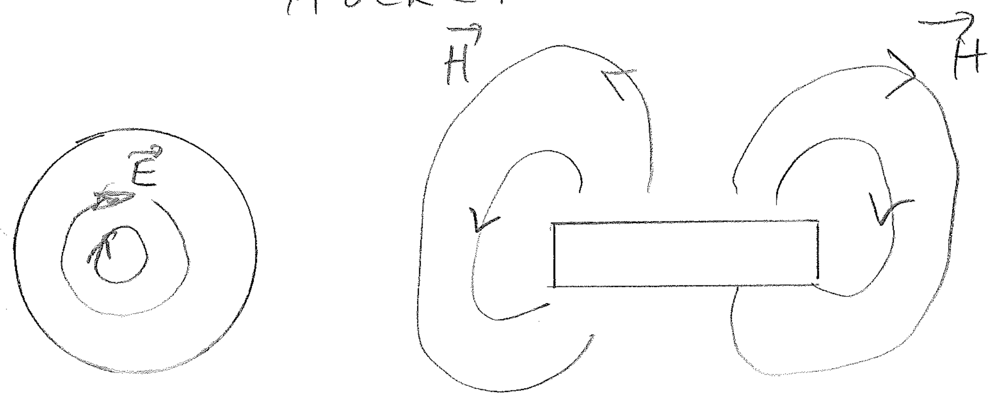
Q: WHAT ABOUT THE RESONANT FREQUENCIES FOR THE TE MODES?

A: $X_{nm} \rightarrow X'_{nm}$

THERE ARE A HUGE NUMBER OF RESONATOR TYPES.

e.g., THE SPHERICAL RESONATOR WE SAW EARLIER. IT HAPPENS TO HAVE THE HIGHEST STORED ENERGY PER SURFACE AREA. IT'S SOMEWHAT OF AN ODDITY.

e.g., DIELECTRIC RESONATOR "HOCKEY PUCK"

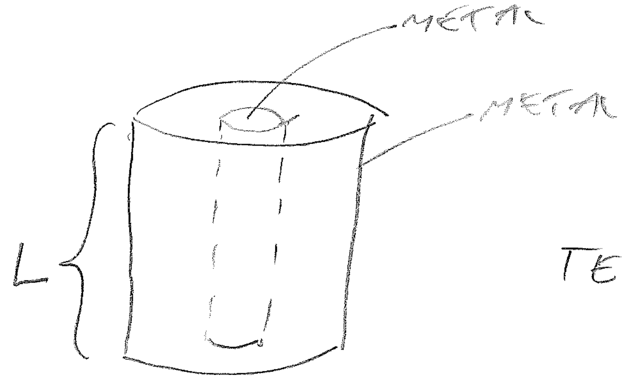


NO CONDUCTOR.

COMMENT ON TUNING RESONATORS

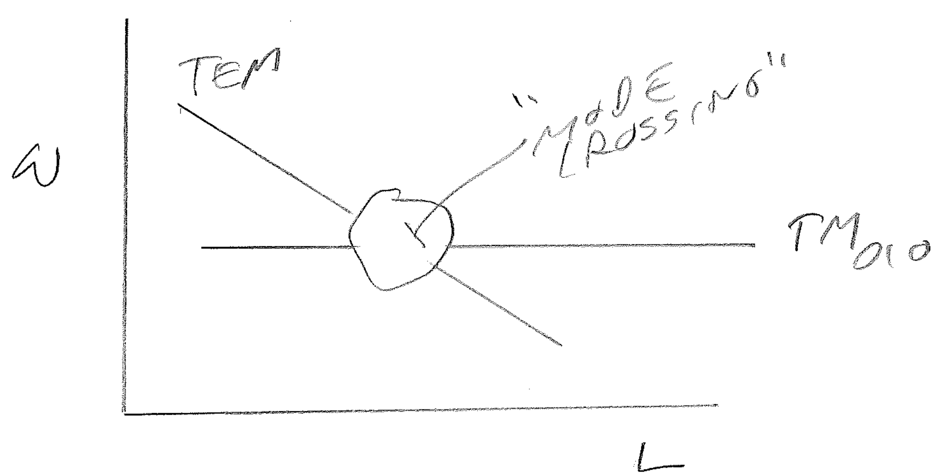
JUST ABOUT ANYTHING YOU DO TO THE RESONATOR EFFECTS ITS RESONANT FREQUENCIES.

e.g., CHANGE LENGTH.

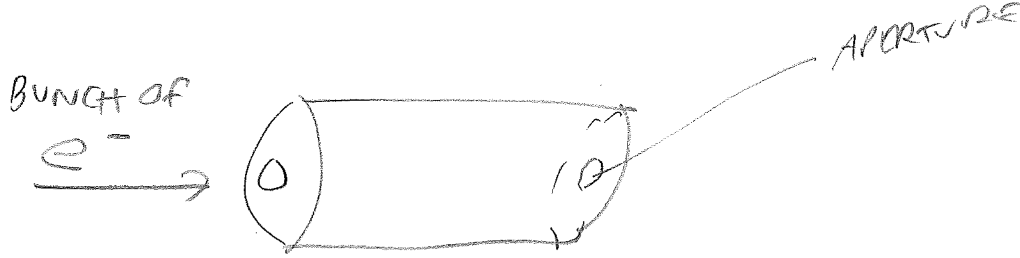


TEM RESONATOR

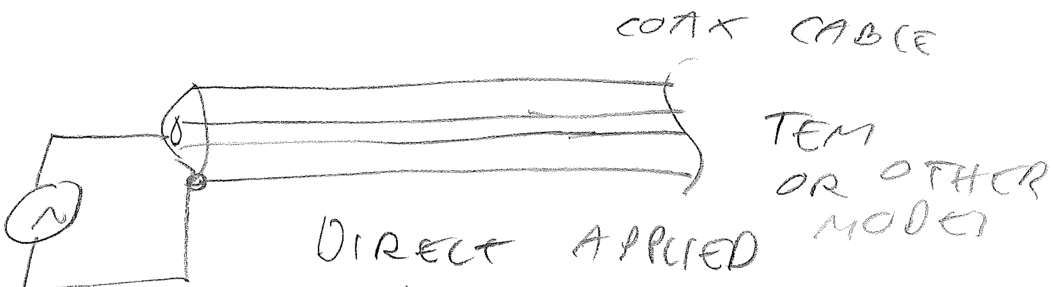
THE TEM FREQUENCY DEPENDS ON THE CAVITY LENGTH L. THE LOWEST TM MODE DOES NOT.



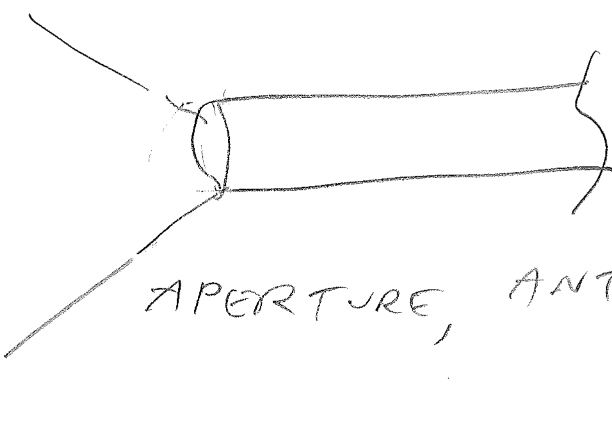
SOME EXAMPLES OF EXCITING WAVEGUIDES AND RESONATORS



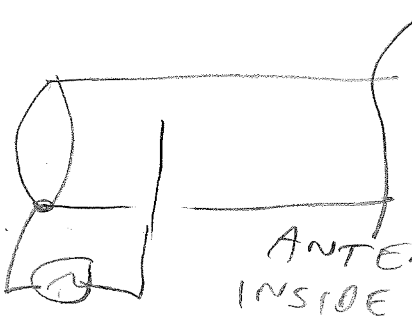
SOME FIELDS FROM THE e^- ARE "STRIPPED OFF" (WEIZACHER-WILLIAMS PICTURE; DISCUSSED IN 515.).



DIRECT APPLIED VOLTAGE



APERTURE, ANTENNA



ANTENNA INSIDE GUIDE

BACK TO J.C. 8.1: FINITE CONDUCTIVITY
THIS IS A VERY TRICKY TOPIC: ONLY
OUTLINED HERE.

RECALL THE BOUNDARY CONDITION ON
$\vec{H}$; $\vec{H}_{2_{\parallel}} - \vec{H}_{1_{\parallel}} = \hat{n} \times \vec{K}$; $\vec{K}$ SURFACE CURRENTS.

IF THE CONDUCTIVITY σ ISN'T ∞,
FIELDS AND CURRENTS EXTEND INTO
THE MATERIAL.

RECALL FOR A GOOD CONDUCTOR,

$$\mathrm{Re}^2 K = \underbrace{\mathrm{Im}^2 K}_{1/\delta^2} = \frac{2}{\mu \omega \sigma}.$$

HENCE, AT A DISTANCE z
INTO THE CONDUCTOR, THE
FIELD IS

$$\vec{H}_{c\parallel} = \vec{H}_{0\parallel} e^{-z/\delta} e^{iz/\delta}$$

YOU STILL HAVE AN $\vec{E}$ FIELD
$\hat{n} \times \vec{H}$, SO INTERESTINGLY
$\vec{E}$ HAS A $\parallel$ COMPONENT

(RECALL FOR A GOOD
CONDUCTOR $\vec{E}$ & $\vec{H}$ ARE
OUT-OF-PHASE BY $45°$).

HENCE, THERES A $[\vec{S}]_z$ COMPONENT.
(SEE JACKSON EQN 8.10).

WE CAN FIND THE EFFECTIVE SURFACE CURRENT $\vec{K}$:

$$\vec{K} = \int_0^\infty \vec{J}\, dz = \hat{n} \times \vec{H}_{o\parallel}$$

INTO CONDUCTOR

WITH $\vec{J}$ FROM $\vec{E}_{c\parallel}$.

- THE CONCEPT OF "SURFACE RESISTANCE".

FROM CIRCUIT THEORY, THE OHMIC POWER LOSS PER LENGTH IS

$$\left\langle \frac{dP}{dl} \right\rangle = \frac{1}{2} I^2 \mathcal{R}$$

I IS IN AMPÉRES.

THIS $\mathcal{R}$ IS RESISTANCE PER LENGTH,

FOR SHEET CURRENT, THE POWER LOSS PER UNIT AREA IS

$$\left\langle \frac{dP}{da} \right\rangle = \frac{1}{2} K^2 \mathcal{R}$$

K IS IN AMPÉRES PER LENGTH

THIS $\mathcal{R}$ IS A RESISTANCE

ALTHOUGH THIS $\mathcal{R}$ HAS UNITS OF OHMS, IT IS NOT THE SAME AS A TWO-TERMINAL RESISTOR

$\mathcal{R}$ IS SOMETIMES STATED AS HAVING UNITS "OHMS PER SQUARE".

THIS $\mathcal{R}$ IS CALLED "SURFACE "RESISTANCE", USUALY CALLED R_S.

FOR BULK CURRENTS $\vec{J}$, THE POWER LOSS PER VOLUME IS

$$\left\langle \frac{dP}{dv} \right\rangle = \frac{1}{2} J^2 \mathcal{R}$$

J IS AMPÉRES PER AREA,

THIS $\mathcal{R}$ HAS UNITS OF RESISTANCE • LENGTH

$(= \rho = 1/\sigma)$.

HOW TO FIND R_s?

RECALL FOR A GOOD CONDUCTOR

$$\mathrm{Im}^2 k = \mathrm{Re}^2 k = \frac{2}{\mu\omega\sigma}$$

WITH $\delta = 1/\mathrm{Im}\, k$.

HENCE, SINCE δ IS NON-ZERO, FIELDS PENETRATE INTO THE CONDUCTOR.

THERE ARE PARALLEL COMPONENTS OF $\vec{H}$ IN THE CONDUCTOR (RECALL $\vec{H}_{2_{\parallel}} - \vec{H}_{1_{\parallel}} = \hat{n} \times \vec{K}$).

IF THERE ARE PARALLEL COMPONENTS OF $\vec{H}$ IN THE CONDUCTOR, THERE ARE ALSO PARALLEL COMPONENTS OF $\vec{E}$ IN THE CONDUCTOR FROM $\hat{n} \times \vec{H}_{\parallel}$. (CAVEAT! $\vec{E}$ & $\vec{H}$ ARE OUT OF PHASE BY 45°.)

HENCE, THERES A POYNTING FLUX POINTING INTO THE CONDUCTOR.

$$\langle \vec{S} \rangle_z = \frac{1}{2} \mathrm{Re}\, \hat{n} \cdot (\vec{E} \times \vec{H}^*);$$

AFTER SOME WORK

$$\langle \vec{S} \rangle_z = \frac{\mu\omega\delta}{4} |\vec{H}_{\parallel}|^2$$

IN TERMS OF THE BOUNDARY CONDITION ON $\vec{H}$:

$$\hat{K} = \hat{n} \times \vec{H}_{0\parallel},$$

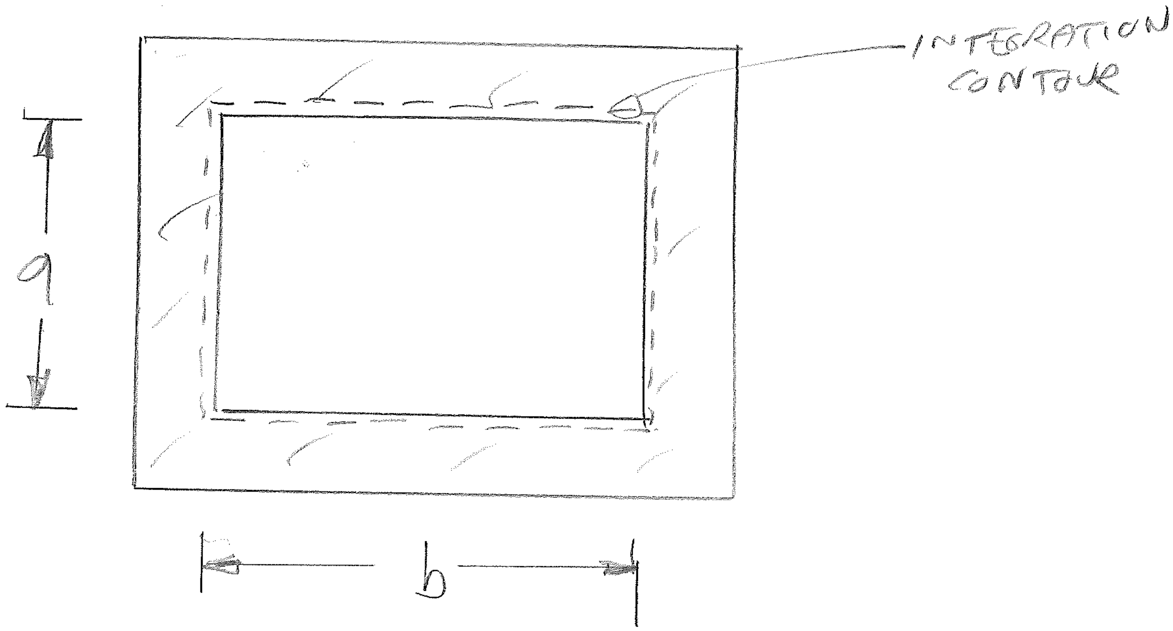
$$\langle \vec{S} \rangle_s = \left\langle \frac{dP}{da} \right\rangle = K^2 R_s,$$

SO YOU CAN READ OFF

$$R_s = \frac{1}{2\sigma\delta}$$

AND $\frac{dP}{da} = R_s |\vec{K}_{0\parallel}|^2$

EXAMPLE LOSSES IN RECTANGULAR WAVEGUIDE, TE MODES.



$$\left\langle \frac{dP}{dz} \right\rangle = \frac{1}{2} R_s \oint |\vec{K}_{011}|^2 \, d\ell$$

WE SOLVED EARLIER FOR $\vec{H}(x,y)$.
HENCE WE KNOW $\vec{K}_{011}$ FROM $\hat{n} \times \vec{H}\big|_s$
THE RESULTING INTEGRATION GIVES

$$\left\langle \frac{dP}{dz} \right\rangle \sim R_s \times \text{GEOMETRIC FACTORS.}$$

THIS IS A GENERIC EXPRESSION.