**Physics 514, Electrodynamics II**
**Department of Physics, University of Washington**
**Winter quarter 2020**
**March 11, 2020, 11am**
**On-line lecture**

***Administrative***

**Homework 9 due Friday, March 11, 11am. Slide HW under my office door C503 or scan and send it via email.**

**No final exam.**

**MRE postponed to early Spring quarter.**

***Lecture***

**Jackson Chapter 8: Waveguides & resonant cavities**

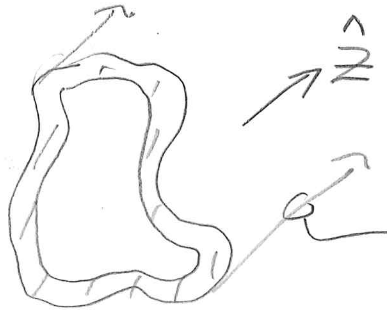
**J.C.8.1 Fields at the surface and within a conductor. We'll come back to this. This section is important when the waveguide walls are lossy.**

**J.C.8.2 Cylindrical cavities and waveguides. Actually, this is a generic discussion of how to set up a waveguide problem.**

**J.C.8.3. Waveguides. Brief discussion of TEM waveguides.**

**J.C.8.4 Modes in a rectangular waveguide. We'll also evaluate the corresponding circular waveguide.**

# WAVEGUIDES II: JACKSON Ⓒ 8.2-.4




VACUUM INSIDE, WALLS ARE PERFECT CONDUCTOR, CONSTANT CROSS-SECTION.

ASSUME HARMONIC ($\omega$) WAVES.

MAXWELL'S EQUATION LEAD TO THE HELMHOLTZ EQUATION

$$\left(\nabla^2 + \frac{1}{c^2}\omega^2\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$$

WITH SOLUTIONS

$$\begin{Bmatrix}\vec{E}(\vec{r},t)\\ \vec{B}(\vec{r},t)\end{Bmatrix} = \begin{Bmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{Bmatrix} e^{i(\pm kz-\omega t)}$$

WHERE WE ANTICIPATED PROPAGATION ALONG $\hat{z}$.

WE WILL NEED TO FIND THE WAVE NUMBER $k$. (THE "GUIDED WAVE NUMBER".)

THE $z$-DIRECTION IS SPECIAL, SO WE SEPARATE FIELDS AND EQUATIONS INTO PARTS ALONG $\hat{z}$, AND PARTS TRANSVERSE TO $\hat{z}$.

$$\vec{\nabla} = \vec{\nabla}_t + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{E} = \vec{E}_t + \vec{E}_z$$

$$\vec{B} = \vec{B}_t + \vec{B}_z$$

LET'S ALSO SEPARATE MAXWELL'S EQUATIONS, e.g.,

$$\vec{\nabla} \times \vec{B} = -i\frac{1}{c^2}\omega\vec{E} \qquad \text{BECOMES}$$

$$\left(\vec{\nabla}_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{B}_t + \vec{B}_z\right)$$
$$= -i\frac{1}{c^2}\omega\left(\vec{E}_t + \vec{E}_z\right).$$

THE ABOVE MAXWELL EQUATION EMBEDS LONGITUDINAL AND TRANSVERSE PARTS,

HERE'S THE TRANSVERSE PART:

$$\underbrace{\vec{\nabla}_t \times \vec{B}_z}_{-\hat{z}\times(-\vec{\nabla}_t B_z)} + \underbrace{\hat{z}\frac{d}{dz} \times \vec{B}_t}_{\hat{z}\times\frac{d}{dz}\vec{B}_t} = \underbrace{-i\frac{1}{c^2}\omega \vec{E}_t}_{\hat{z}\times\left(-i\frac{1}{c^2}\omega\, \hat{z}\times\vec{E}_t\right)}$$

* SEE JACKSON EQN 8.22

ALTHOUGH NOT FORMALLY CORRECT; DROP THE LEADING "$\hat{z}\times\ldots$", YIELDING

$$-\vec{\nabla}_t B_z + \frac{d}{dz}\vec{B}_t = -i\frac{1}{c^2}\omega\, \hat{z}\times\vec{E}_t$$

* JACKSON EQN. 8.24 (LEFT).

THERE ARE 3 OTHER SIMILAR EQUATIONS.

IF YOU PROVIDE $\vec{E}_z$ AND $\vec{B}_z$, THE FOUR EQUATIONS PROVIDE SOLUTIONS TO THE 4 COMPONENTS OF $\vec{E}_t$ AND $\vec{B}_t$.

SEE JACKSON EQUATIONS 8.26a-b.

SO, WAVEGUIDE PROBLEMS REDUCE TO FINDING $\vec{E}_z$ AND $\vec{B}_z$. BY THEMSELVES, $\vec{E}_z$ AND $\vec{B}_z$ PROVIDE INFORMATION ABOUT THE "MODE STRUCTURE", CUTOFF FREQUENCIES, etc.

COMMENT I: WITH SMOOTH WALL, GOOD CONDUCTOR, CONSTANT CROSS-SECTION, 3 TYPES OF MODES.

TE "TRANSVERSE ELECTRIC"
$E_z = 0$ EVERYWHERE.

TM "TRANSVERSE MAGNETIC"
$B_z = 0$ EVERYWHERE.

TEM "TRANSVERSE ELECTRO-MAGNETIC"
$E_z = 0$, $B_z = 0$ EVERYWHERE.

SPECIAL CASE: TEM MODES (JACKSON P. 358).

- SKIN DEPTH IS VERY SMALL FOR A GOOD CONDUCTOR, SO CURRENTS $\vec{K}$ ARE PARALLEL TO THE SURFACE.
- THE BOUNDARY CONDITION ON $\vec{H}$ IMPLIES $\vec{H}$ IS PARALLEL TO THE SURFACE, AS WELL.
- BUT FOR TEM MODES, $\vec{H}$ IS TRANSVERSE. FROM

$$\vec{H}_1 - \vec{H}_2 = \vec{K} \times \hat{n},$$

THE CURRENTS $\vec{K}$ ARE ALONG $\hat{z}$.

- SINCE $\vec{K} \sim \hat{z}$, $\vec{A} \sim \hat{z}$.
- SO THE ELECTRIC FIELD

$$\vec{E} = -\vec{\nabla}\Phi - \frac{d}{dt}\vec{A}$$

BECOMES

$$\vec{E} = -\vec{\nabla}_t \Phi - \hat{z}\frac{d}{dz}\Phi - \frac{d}{dt}\vec{A}$$

SINCE $E_z = 0$, THE LAST 2 TERMS SUM TO ZERO.

WE ARE LEFT WITH

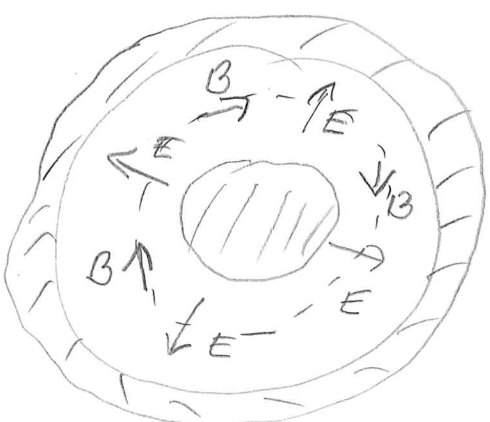
$\vec{E} = -\vec{\nabla}_t \Phi$, THIS IS 2D ELECTRO-STATICS.

WE STILL HAVE MAXWELL'S EQUATIONS, WHICH GIVE

$$\vec{B} = \frac{1}{c}\hat{z} \times \vec{E}.$$

TO GET THE SIGNS RIGHT, YOU'LL NEED $\vec{E} \times \vec{B}$ TO BE ALONG THE PROPAGATION DIRECTION. JACKSON EQN 8.28.

EXAMPLE: COAX CABLE: TEM MODE




AT A FIXED POSITION AT A FIXED TIME.

VARY POSITION:

$e^{ikz}$.

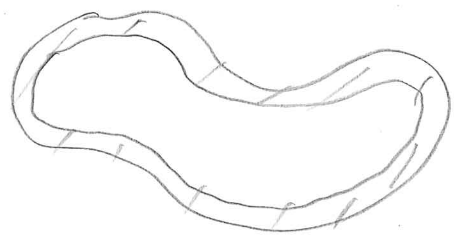
VARY TIME

$e^{i\omega t}$

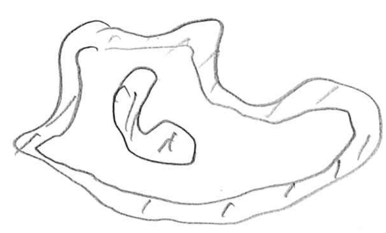
DISPERSION RELATION IS $k^2 = \frac{1}{c^2}\omega^2$! NO CUTOFF.

TEM CONTINUED.
HOW MANY CONDUCTORS?

START WITH 1 CONDUCTOR:



- ON THE INTERIOR SURFACE, $\vec{E}_{\parallel} = 0$ (THE $\vec{E}$ FIELD IS "NORMAL").
- HENCE, A LINE INTEGRAL BETWEEN ANY TWO POINTS $a$, $b$ ON THE CROSS-SECTION SURFACE IS ZERO. HENCE, THE SURFACE CROSS-SECTION IS AN EQUIPOTENTIAL. HENCE $\vec{E} = 0$ EVERYWHERE.
- YOU NEED 2 OR MORE SURFACES TO HAVE A TEM WAVEGUIDE:



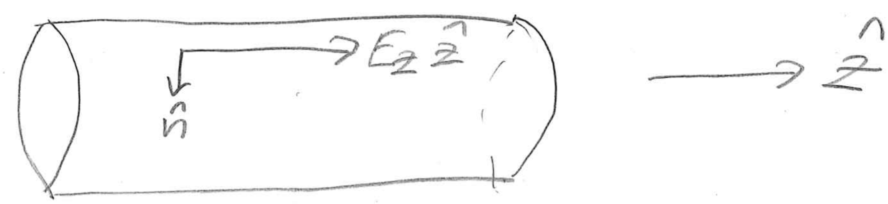
OR

BACK TO GENERAL COMMENTS ON TE, TM AND TEM MODES.



WE WISH TO FIND THE BOUNDARY CONDITIONS IN TERMS OF $E_z$ AND $B_z$.

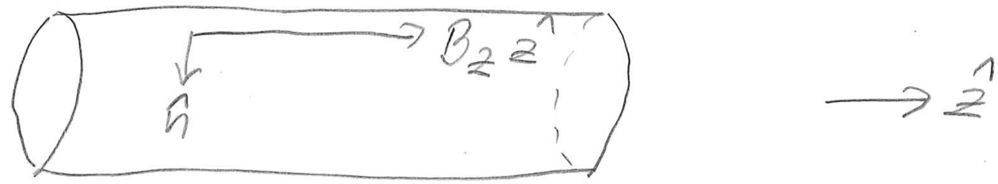
i. $E_z$ BOUNDARY CONDITION.




$$\vec{E}\big|_s \sim \hat{n}, \quad \text{so } E_z\big|_s = 0$$

JACKSON EQN. 8.29.

ii. $B_z$ BOUNDARY CONDITION.




THIS IS TRICKIER. RECALL

$$-\vec{\nabla}_t B_z + \frac{d}{dz} \vec{B}_t = -i \frac{1}{c^2} \omega \hat{z} \times \vec{E}_t .$$

- PICK A POINT ON THE SURFACE.
- TAKE THE PROJECTION OF EACH TERM ALONG $\hat{n}$.

$$\left(-\vec{\nabla}_t B_z\right) \cdot \hat{n}\Big|_s + \left(\frac{d}{dz} \vec{B}_t\right) \cdot \hat{n}\Big|_s$$

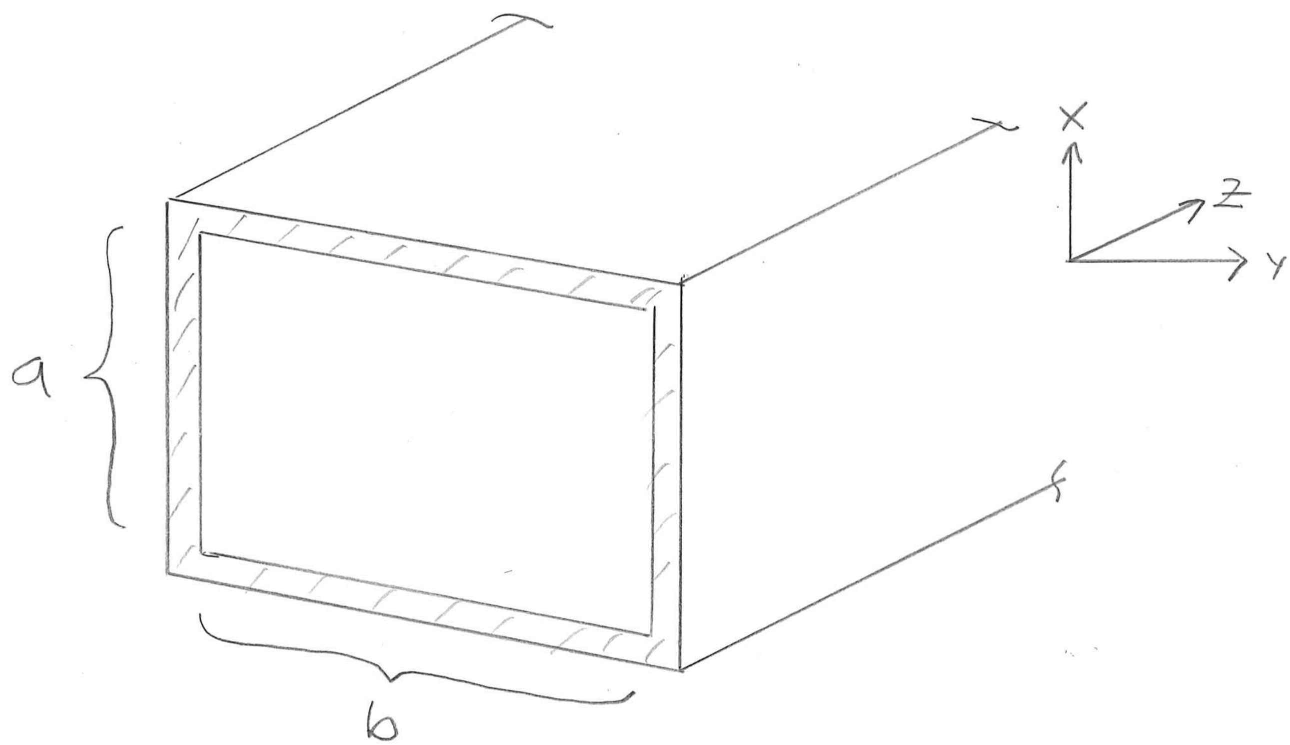
$$= \left(-i \frac{1}{c^2} \omega \hat{z} \times \vec{E}_t\right) \cdot \hat{n}\Big|_s$$

- THE SECOND TERM $(\vec{B}_t \cdot \hat{n})$ VANISHES (WHY?)
- FOR THE LAST TERM, RECALL $\vec{E}_{\parallel}\big|_s = 0$. SO $\vec{E}_t \sim \hat{n}$. SO $\hat{z} \times \vec{E}_t$ IS ORTHOGONAL TO $\hat{n}$. SO THE LAST TERM VANISHES, LEAVING

$$\vec{\nabla}_t B_z \cdot \hat{n}\Big|_s = 0, \quad \text{OR} \quad \frac{d}{dn} B_z\Big|_s = 0$$

JACKSON EQN. 8.30.

EXAMPLE: RECTANGULAR WAVEGUIDE, TE MODES.




THE HELMHOLTZ EQUATION

$$\left(\nabla^2 + \frac{1}{c^2}\omega^2\right)\begin{Bmatrix}\vec{E} \\ \vec{B}\end{Bmatrix} = 0$$

BECOMES, FOR $E_z$ AND $B_z$:

$$\left(\nabla_t^2 + \left(-k^2 + \frac{1}{c^2}\omega^2\right)\right)\begin{Bmatrix}E_z \\ B_z\end{Bmatrix} = 0$$

FOR TE MODES, $E_z = 0$.

IN CARTESIAN COORDINATES, THE THE HELMHOLTZ EQUATION IS

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \left(-k^2 + \frac{\omega^2}{c^2}\right)\right) B_z = 0$$

AS USUAL, WE SEPARATE VARIABLES! ASSUME $B_z(x,y) = X(x)\,Y(y)$. IF WE FIND A SOLUTION, THE SEPARATION IS JUSTIFIED (AS WE KNOW IS THE CASE).

$$\frac{1}{X}\frac{d^2}{dx^2}X + \frac{1}{Y}\frac{d^2}{dy^2}Y + \left(-k^2 + \frac{\omega^2}{c^2}\right) = 0$$

EACH O.D.E. IS SEPARATELY CONSTANT:

$$\frac{1}{X}\frac{d^2}{dx^2}X = -k_x^2\,; \quad \frac{1}{Y}\frac{d^2}{dy^2}Y = -k_y^2$$

$$\text{WITH } -k_x^2 - k_y^2 + \left(-k^2 + \omega^2/c^2\right) = 0$$

[THE TERM IN PARENTHESES IS THE "CUTOFF WAVE NUMBER" $k_c$, FOR REASONS THAT WILL BECOME CLEAR:

$$k_c^2 \equiv -k^2 + \omega^2/c^2.$$ ]

WE GET THE USUAL SOLUTIONS

$$\mathbb{X}(x) = X_S \sin k_x x + X_c \cos k_x x .$$

APPLY BOUNDARY CONDITIONS:

$$\frac{d}{dn} B_z \Big|_S = 0 \quad \text{BECOMES}$$

$$\frac{d}{dx} B_z \Big|_{\substack{x=0 \\ x=a}} = 0$$

THE $x=0$ BOUNDARY CONDITION REQUIRES $X_S = 0$.

THE $x=a$ BOUNDARY CONDITION REQUIRES $k_x = m\pi/a$; $m = 0, 1, \ldots$.

SIMILARLY

$$\mathbb{Y}(y) = Y_c \cos k_y y \quad \text{WITH}$$

$$k_y = n\pi/b; \quad n = 0, 1, \ldots .$$

MODES ARE IDENTIFIED WITH MODE NUMBERS $m$ AND $n$.

THE GENERIC MODE LOOKS LIKE

$$B_{z;m,n}(x,y) = B_{0,z;m,n} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y .$$

WHAT ARE THE PROPAGATION PROPERTIES (PHASE VELOCITY, CUTOFF FREQUENCIES, etc.) OF THE $TE_{mn}$ MODE?

FROM $-k_x^2 - k_y^2 = -k^2 + \omega^2/c^2$,

$$k^2 = \omega^2/c^2 - \frac{\pi^2 m^2}{a^2} - \frac{\pi^2 n^2}{b^2}.$$

WRITE THIS AS

$$k^2 = \omega^2/c^2 - k_c^2.$$

FOR $\omega^2/c^2 < k_c^2$, THE WAVE NO LONGER PROPAGATES. THIS DEFINES A CUTOFF FREQUENCY FOR EACH MODE $\omega_{c;\,mn}$.

$k^2$ LEADS TO THE PHASE VELOCITY (THE "GUIDED WAVE VELOCITY $v_g$) IN THE USUAL WAY:

$$v_g^2 = \omega^2/k^2 > 1 \text{ FOR } \omega^2 > c^2 k_c^2.$$

THE FORM OF K

$$K^2 = \omega^2/c^2 - \pi^2 \frac{m^2}{a} - \pi \frac{^2 n^2}{b}$$

IS OFTEN WRITTEN

$$K^2 = \frac{1}{c^2}\left(\omega^2 - \omega_{c;\,m,\,n}^2\right)$$

SO AS TO RESEMBLE THE FORM OF THE DISPERSION RELATION OF PLASMAS, etc.

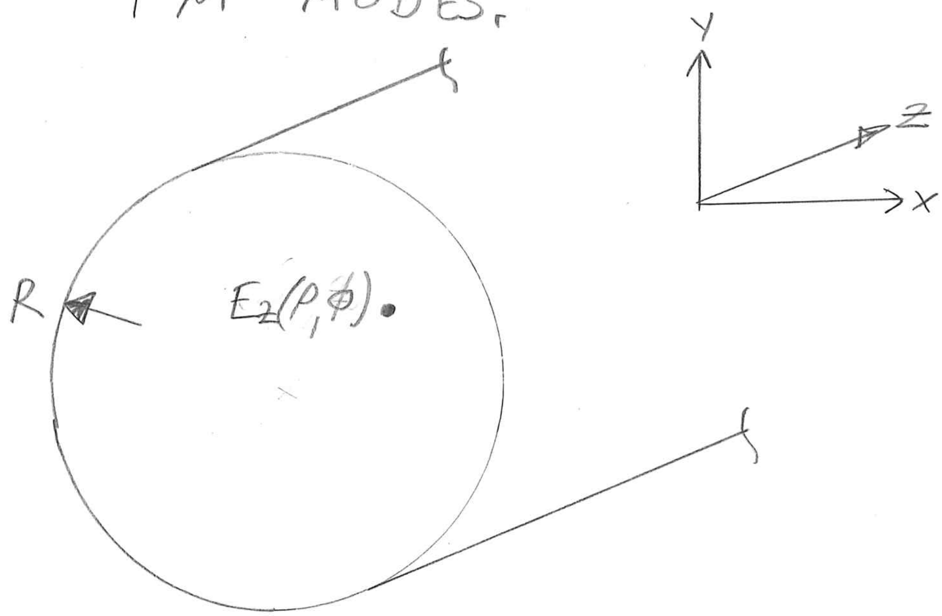
IF YOU WISH, YOU COULD FIND THE GROUP VELOCITY $\frac{1}{dk/d\omega}$ AND FIND ITS $< c$.

$K^2$ ALSO LEADS TO THE GUIDED WAVELENGTH: $\lambda = 2\pi/k$

$$\lambda = \frac{1}{1/c\,\omega\sqrt{1 - \frac{\omega_{c,m,n}^2}{\omega^2}}} \quad \text{OR}$$

$$\lambda = \lambda_0 \frac{V_g}{c}$$ WITH $\lambda_0$ THE FREE-SPACE WAVELENGTH.

EXAMPLE: ROUND WAVEGUIDE, TM MODES.




$$\left(\nabla_t^2 + \left(-k^2 + \omega^2/c^2\right)\right) E_z(\rho,\phi) = 0$$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \left(-k^2 + \omega^2/c^2\right)\right] \times E_z(\rho,\phi) = 0$$

SEPARATE VARIABLES WITH

$$E_z = \mathbb{R}(\rho)\Phi(\phi)$$

$$\frac{1}{\mathbb{R}}\frac{d^2\mathbb{R}}{d\rho^2} + \frac{1}{\rho\mathbb{R}}\frac{d\mathbb{R}}{d\rho} + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} + \left(-k^2 + \frac{\omega^2}{c^2}\right) = 0.$$

Now, MULTIPLY BY $\rho^2$!

$$\frac{\rho^2}{\mathbb{R}}\frac{d^2\mathbb{R}}{d\rho^2} + \frac{\rho}{\mathbb{R}}\frac{d\mathbb{R}}{d\rho}$$

$$+ \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + (k^2 - \omega^2/c^2)\rho^2 = 0$$

i. THE $\Phi$ TERM IS SEPARATELY CONSTANT: $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -k_\phi^2$

WITH SOLUTIONS

$$\Phi(\phi) = \phi_s \sin k_\phi \phi + \phi_c \cos k_\phi \phi$$

WE REQUIRE $\Phi(\phi)$ BE SINGLE-VALUED:

$$\Phi(\phi) = \Phi(\phi \pm m 2\pi).$$

HENCE $k_\phi$ IS AN INTEGER $(n)$.

ii THE RADIAL PIECE IS BESSEL'S EQUATION, BUT THE SOLUTION IS SLIGHTLY MORE COMPLICATED DUE TO $k^2 - \omega^2/c^2$:

$$\mathbb{R}(\rho) = C_{J,n} J_n\left(\sqrt{-k^2 + \omega^2/c^2}\,\rho\right)$$

$$+ C_{J,n} N_n\left(\sqrt{-k^2 + \omega^2/c^2}\,\rho\right)$$

WITH $J_n$ AND $N_n$ BESSEL FUNCTIONS OF THE 1ST AND 2ND KIND.

THE $N_n$ SOLUTIONS ARE SINGULAR FOR $\rho = 0$, SO ARE ABSENT.

SO FAR, WE HAVE A MODE

$$E_{z;n}(\rho,\phi) = (A_{s;n} \sin n\phi + A_{c;n} \cos n\phi)$$

$$\times J_n\left(\sqrt{-k^2 + \omega^2/c^2}\,\rho\right).$$

NOW, APPLY BOUNDARY CONDITION

$$E_{z;n}(\rho,\phi)\Big|_{\rho=R} = 0, \quad \text{SO}$$

$$J_n\left(\sqrt{-k^2 + \omega^2/c^2}\,R\right) = J_n(k_c R) = 0,$$

$$\text{OR } k_{c;n,m} = \frac{x_{n,m}}{R},$$

WITH $x_{n,m}$ THE mth ZERO OF $J_n$.

WE NOW HAVE THE SHAPE OF THE $TM_{mn}$ MODES.

LET'S FIND THE PROPAGATION PROPERTIES, WE HAVE

$$K^2 = \omega^2/c^2 - k_c^2$$

WITH CUTOFF FREQUENCY

$$\frac{\omega_{c,m,n}^2}{c^2} = k_c^2 \quad \left( = \frac{X_{m,n}^2}{R^2 c^2} \right)$$

THE PHASE VELOCITY $v_g$ ("GUIDED") HAS THE USUAL FORM

$$v_g = \frac{\omega}{K} = c \frac{1}{\sqrt{1 - \left(\frac{\omega_{c,m,n}}{\omega}\right)^2}}$$