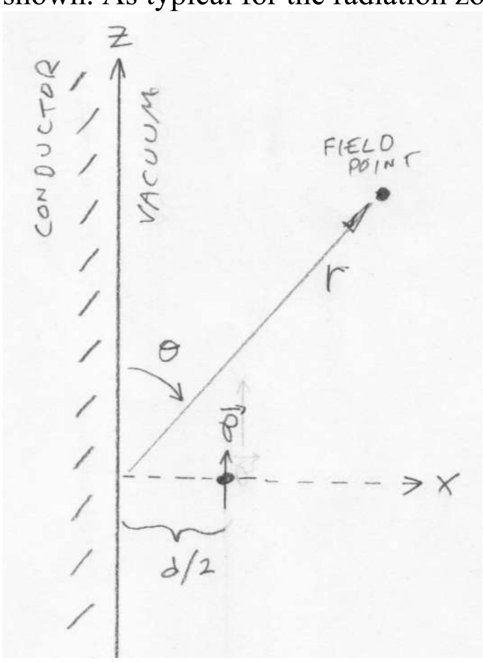
III. (40 points) Hertz vectors. A small electric dipole (small relative to the wavelength) $\mathbf{p} = \mathbf{p_0} e^{-i\omega t}$ is located a distance *d/2* from an infinite conducting plane. The dipole vector is parallel to the plane.

a. (10 points) What is the electric Hertz vector $\mathbf{\Pi_e}$ for field points not inside the conductor?

b. (15 points) For field points not inside the conductor, far from the plane (that is, $r \gg d$), for short wavelengths (that is, $r \gg \lambda$), find the vector potential **A**. Use polar coordinates (r, θ, ϕ) shown.

c. (15 points) For field points not inside the conductor, far from the plane (that is, $r \gg d$), for short wavelengths (that is, $r \gg \lambda$), (these are conditions to be in the "radiation zone"), find the electric and magnetic fields **E** and **B**. Use polar coordinates (r, θ, ϕ) shown. As typical for the radiation zone, ignore terms in **E** and **B** that fall faster than $1/r$.

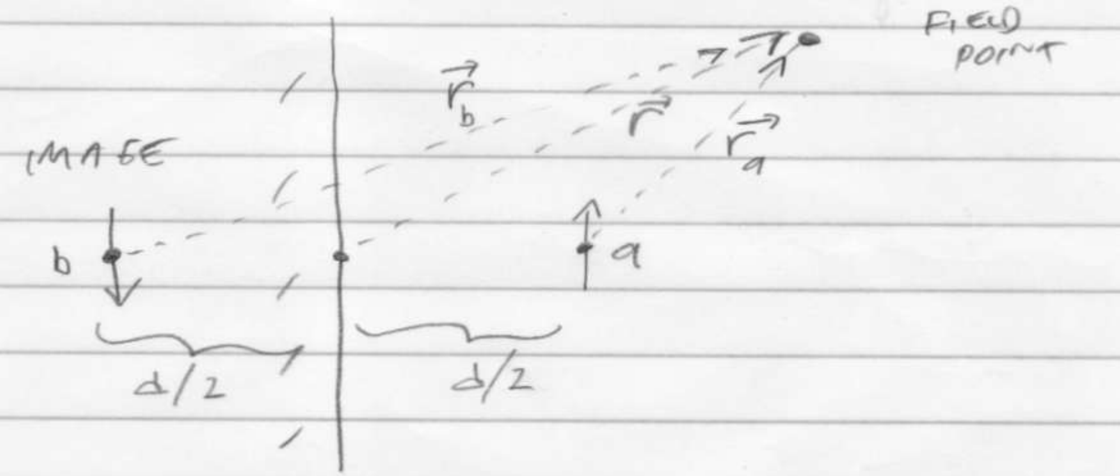


3a. THE PROBLEM OF THE SINGLE "POINT" DIPOLE WAS DONE IN CLASS AND DETAILED IN THE HOMEWORK:

$$\vec{\Pi}_e = \frac{1}{4\pi\varepsilon_0} \frac{e^{i\omega/c\, R}}{R} \vec{p}$$

WITH $\vec{R} = \vec{r} - \vec{r}\,'$;
FIELD POINT $\vec{r}$, DIPOLE AT $\vec{r}\,'$.

THIS PROBLEM ALSO HAS AN IMAGE DIPOLE!



HENCE

$$\vec{\Pi}_e = \frac{1}{4\pi\varepsilon_0}\left\{ \frac{e^{-i\omega/c\, r_a}}{r_a} - \frac{e^{-i\omega/c\, r_b}}{r_b} \right\} \vec{p}$$

b. THE VECTOR POTENTIAL IS

$$\vec{A} = \frac{1}{c^2}\frac{d}{dt}\vec{\Pi}_e = \frac{1}{c^2}(-i\omega)\left\{ \frac{e^{-i\omega/c\, r_a}}{r_a} - \frac{e^{-\omega/c\, r_b}}{r_b} \right\} \vec{p}$$

SINCE FOR THE FIELD POINT $r \gg d$,

$$r_a \approx r - \frac{d}{2}\sin\theta\cos\phi$$

$$r_b \approx r + \frac{d}{2}\sin\theta\cos\phi$$

ALSO, $1/r_a \approx 1/r$; $\frac{1}{r_b} \approx 1/r$, HENCE

$$\vec{A} = i\omega \frac{1}{4\pi\varepsilon_0}\frac{1}{r}$$

$$\cdot \left\{ e^{+i\frac{\omega}{c}\frac{d}{2}\sin\theta\cos\phi} - e^{-i\frac{\omega}{c}\frac{d}{2}\sin\theta\sin\phi} \right\}$$

$$\cdot \sin\left(\frac{\omega}{c}\frac{d}{2}\sin\theta\cos\theta\right) e^{i(\frac{\omega}{c}r - \omega t)}\, p_0\, \hat{z}$$

WITH $\vec{p} \sim \hat{z}$, OR

$$\vec{A} = (-\omega)\frac{1}{2\pi\varepsilon_0}\frac{1}{r} e^{i(\frac{\omega}{c}r - \omega t)}$$

$$\cdot \sin\left(\frac{\omega}{c}\frac{d}{2}\sin\theta\cos\phi\right) p_0\, \hat{z}.$$

A SUBLETY, WE ARE GOING TO IGNORE A $\hat{r}$ PART OF $\vec{A}$ (WE'LL SEE WHY). SINCE $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$, WE'LL PICK UP AN EXTRA FACTOR OF $\sin\theta$ IN THE $\hat{\theta}$ COMPONENT.

C. WE'LL NEED TO TAKE $\vec{\nabla}\times\vec{A}$ TO GET $\vec{B}$. A SUBLETY, THE RADIATION

FIELDS DONT HAVE A $\hat{r}$ COMPONENT. ANOTHER SUBTLETY IS THAT THE DISTANT $\vec{B}$ FIELD ONLY HAS $\hat{\phi}$ COMPONENTS. THE $\frac{1}{r}\frac{\partial A_r}{\partial \theta}$ TERM

FALLS AT LEAST AS FAST AS $1/r^2$, SO WE'LL IGNORE IT'S CONTRIBUTION TO $\vec{B}$. THE OTHER TERM IN THE CURL $[\vec{\nabla}\times\vec{A}]_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)$ HAS LOTS OF TERMS; WE'LL ONLY KEEP TERMS THAT FALL AS $1/r$. (WE DID THIS ON THE HOMEWORK.) HENCE

$$\vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) = -\frac{1}{r}\frac{\partial}{\partial r}(r\sin\theta A)$$

$$= (-\omega)\frac{1}{2\pi\epsilon_0}(-i\omega)\frac{1}{r}e^{i(\omega/c\, r - \omega t)}$$

$$\cdot \sin\theta \sin\left(\omega/c\ d/2\ \sin\theta\cos\phi\right) p_0 \hat{\phi}.$$

(TERMS $1/r^2$ DROPPED).

IN THE FAR (RADIATION ZONE)

$$\vec{E} = c\vec{B}\times\hat{r}$$

$\{$ THIS IS IN THE $\hat{\theta}$ DIRECTION $\}$