1. In a conductor free of excess charge,
\[ \nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \]

**First, recall** \( \mathbf{J} = \sigma \mathbf{E} \) in \( \nabla \times \mathbf{B} \).

**Take the curl of \( \nabla \times \mathbf{E} \) and \( \nabla \times \mathbf{B} \) :**
\[ \nabla \times (\nabla \times \mathbf{E}) = \nabla \times ( -\frac{\partial \mathbf{B}}{\partial t}) \]
\[ \nabla \times (\nabla \times \mathbf{B}) = \nabla \times ( \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} ) \]

or
\[ -\nabla^2 \mathbf{E} + \nabla \left( \nabla \cdot \mathbf{E} \right) = -\frac{\partial}{\partial t} \left\{ \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \mathbf{E} \right\} \]
\[ -\nabla^2 \mathbf{B} + \nabla \left( \nabla \cdot \mathbf{B} \right) = \frac{\partial}{\partial t} \left( \varepsilon_0 \mathbf{E} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \frac{\partial \mathbf{B}}{\partial t} \right) \]

**Giving**
\[ \nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{ \partial^2 \mathbf{E} }{ \partial t^2 } - \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} = 0 
\]
\[ \nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \frac{ \partial^2 \mathbf{B} }{ \partial t^2 } - \mu_0 \sigma \frac{\partial \mathbf{B}}{\partial t} = 0. \]

**These are sometimes called** "**Modified Wave Equations**".
2. It's interesting that, at this early stage, we can find the Thomson cross section. This characterizes the "size" of an unbound electron subject to incident plane waves.

In lecture, and Jackson Eqn. 7.49, we found for the model of unbound electrons, there's an associated dipole moment.

\[
\begin{align*}
\vec{y} &= \frac{eE}{m}, \\
\vec{y} &= \frac{eE}{m} \left( \frac{1}{\omega^2} \right)
\end{align*}
\]

For incident plane waves at angular frequency \( \omega \), the dipole moment \( \vec{p} \) is \( \vec{p} = -\vec{y} \cdot \vec{E} = -\frac{1}{\omega^2} \frac{e^2}{m} \vec{E} \cdot \vec{y} \).

The strategy is to find the total radiated power, then divide by the plane-wave's power density. This ratio has units of area.

Recall the radiated fields:

\[
\begin{align*}
E_r &= 0 \\
E_\theta &= \frac{\vec{p}}{4\pi \varepsilon_0} \sin \theta \left( -\frac{\omega^2}{c^2} \right) \\
&= -\frac{1}{\omega} \frac{e^2}{m} \frac{E}{\varepsilon_0} \sin \theta \left( -\frac{\omega^2}{c^2} \right) \\
&= \frac{e^2}{m} \frac{E}{4\pi \varepsilon_0} \frac{1}{c^2} \sin \theta
\end{align*}
\]
The time-averaged Poynting vector gives the radiated power per area:

\[ \langle \mathbf{S} \rangle = \frac{1}{\hbar} \left\{ \frac{E^2}{m c^2} \sin^2 \theta \right\} \]

\[ = \frac{E^2}{m c^2} \sin^2 \theta \left( \frac{I_0}{r^2} \right) \]

where \( I_0 = \sqrt{\frac{e_0}{m c^2}} E^2 \) is the incident plane-wave intensity and \( r_0 = \frac{E^2}{(4\pi \varepsilon_0 m c^2)} \) is the classical electron radius. You won't often see this in MKSA units.

The total radiated power is an integral over the sphere with the dipole at the center:

\[ -P = \iiint r^2 \langle \mathbf{S} \rangle \, dV \]

\[ = 2\pi \left( \iiint r^2 r_0^2 \sin^2 \theta \left( \frac{I_0}{r^2} \right) \sin \theta \, d\omega \right) \]

\[ = \frac{8}{3} \pi I_0 r_0^2 \]

This is the radiated power in terms of \( I_0 \), the power per area. Hence, dividing \( P \) by \( I_0 \) is an area: the Thomson cross section

\[ \sigma = \frac{8}{3} \pi r_0^2. \]
3. My earlier solution is clumsy, I’m here reproducing that of Lorraine & Carson. They were inspired by Jackson Eq. N. 7.61

\[ n = \sqrt{1 - \left(\frac{\nu_p}{\nu_n}\right)^2} \quad n < 1. \]

a. An incident ray gradually bends, eventually escaping the plasma. This appears as:

![Diagram of plasma and vacuum with an incident angle and ray trajectory.]

The incident angle is \( \Theta_i \). Along the ray's trajectory there's an incident angle \( \Theta(R) \) which is the ray's distance along its trajectory.

b. Nothing in this problem invalidates Snell's law of refraction. We imagine this slowly-varying medium consists of thin layers with \( n(R) \).
The key thing to notice is that $n(r) \sin \theta(r)$ is a constant of the motion. That is,

$$n(r) \sin \theta(r) = n(r') \sin \theta(r')$$

at any two positions $r$ and $r'$ along the ray's trajectory.

In particular $n(r) \sin \theta(r) = \sin \theta_i$.

Differentiate this with respect to an arc-length $dr$:

$$\frac{d\theta(r)}{dr} = \frac{1}{n(r)} \frac{dn(r)}{dr} \tan \theta.$$

Observe that, as the ion density increases with $r$, $n(r)$ decreases with $r$ so $dn/dr$ is negative. Hence $\theta(r)$ increases with $r$.

As $r$ continues to increase, eventually $\theta(r) = \pi/2$. Then as $r$ continues to increase $dn/dr$ is positive until the ray returns to the vacuum.

C. At the deepest penetration of the ray, $\sin \theta = 1$, and from

$$n \sin \theta_1 = \sin \theta_i,$$

the turnaround point is $n(\text{turn around}) = \sin \theta_i$. 
For an incident ray at normal incidence, $\psi(\text{ around }) = 0$. This occurs where $\psi = \sqrt{1 - (\mu_2/\mu_1)^2}$ or $\mu_2 = \mu_1$.

Phase velocity. At normal incidence, $\psi(\theta) = 0$ where the phase velocity is infinite.

4) Due on HW #6.