SLATER PERTURBATION FORMULA: SHAPE
(J. SLATER, "MICROWAVE ELECTRONICS", VAN NOSTRAND, 1950, P81.)

HOW DOES THE RESONANT FREQUENCY CHANGE ON SMALL CHANGES TO THE CAVITY SHAPE? SUPPOSE $\vec{E}_0$, $\vec{H}_0$ & ω_0 REFER TO THE UNPERTURBED CAVITY. SUPPOSE $\vec{E}$, $\vec{H}$ & ω REFER TO THE PERTURBED CAVITY. MAXWELL'S EQUATIONS ARE

$$\vec{\nabla}\times\vec{E}_0 = -i\omega_0\mu\vec{H}_0, \quad \vec{\nabla}\times\vec{H}_0 = i\omega_0\varepsilon\vec{E}_0,$$

$$\vec{\nabla}\times\vec{E} = -i\omega\mu\vec{H}, \quad \vec{\nabla}\times\vec{H} = i\omega\varepsilon\vec{E}.$$

MULTIPLY $(\vec{\nabla}\times\vec{E}_0)^*$ BY $\vec{H}$,
MULTIPLY $(\vec{\nabla}\times\vec{H})^*$ BY $\vec{E}_0$, GIVING

$$\vec{H}\cdot(\vec{\nabla}\times\vec{E}_0^*) = i\omega_0\mu\vec{H}\cdot\vec{H}_0^*, \quad \text{AND}$$

$$\vec{E}_0^*\cdot(\vec{\nabla}\times\vec{H}) = i\omega\varepsilon\vec{E}_0^*\cdot\vec{E}.$$

NOW SUBTRACT ONE FROM THE OTHER AND APPLY VECTOR IDENTITY $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) - \vec{c}\cdot(\vec{a}\times\vec{b})$ TO GIVE

$$\vec{\nabla}\cdot(\vec{E}_0^*\times\vec{H}) = i\omega_0\mu\vec{H}\cdot\vec{H}_0^* - i\omega\varepsilon\vec{E}_0^*\cdot\vec{E}$$

NOW DO THE SAME TO $(\vec{\nabla}\times\vec{H}_0)^*$ AND $(\vec{\nabla}\times\vec{E})^*$ TO GIVE.

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}_0^*) = -i\omega\mu\vec{H}_0^*\cdot\vec{H} + i\omega_0\varepsilon\vec{E}\cdot\vec{E}_0^*$$

ADD THE TWO TOTAL DIVERGENCES, INTEGRATE OVER THE ENTIRE CAVITY VOLUME, AND APPLY THE DIVERGENCE THEOREM TO GIVE

$$\oiint_S (\vec{E} \times \vec{H}_0^* + \vec{E}_0^* \times \vec{H}) \cdot \hat{n}\, dS$$

$$= \oiint_S (\vec{E}_0^* \times \vec{H}) \cdot \hat{n}\, dS \quad \left(\text{SINCE } \hat{n} \times \vec{E}\big|_S = 0\right)$$

$$= -i(\omega - \omega_0) \iiint_V \left(\varepsilon \vec{E} \times \vec{E}_0^* + \mu_0 \vec{H} \cdot \vec{H}_0^*\right) dV$$

WE DECOMPOSE THE PERTURBED SURFACE S INTO THE UNPERTURBED SURFACE S_0 PLUS A PERTURBATION ΔS, SO

$$\oiint_S \vec{E}_0^* \vec{H} \cdot \hat{n}\, dS = \oiint_{S_0} \vec{E}_0^* \times \vec{H} \cdot \hat{n}\, dS_0 - \oiint_{\Delta S} \vec{E}_0^* \times \vec{H} \cdot \hat{n}\, d\Delta S$$

BUT $\vec{E}_0 \times \hat{n}\big|_{S_0} = 0$, SO

$$\oiint_S \vec{E}_0^* \times \vec{H} \cdot \hat{n}\, dS = -\oiint_{\Delta S} \vec{E}_0^* \times \vec{H} \cdot \hat{n}\, dS.$$ HENCE

$$\omega - \omega_0 = \frac{-i \oiint_{\Delta S} \vec{E}_0^* \times \vec{H} \cdot n\, d\Delta S}{\iiint_V \left(\varepsilon \vec{E} \cdot \vec{E}_0^* + \mu \vec{H} \cdot \vec{H}_0\right) dV}$$

THIS IS AN EXACT FREQUENCY SHIFT, BUT NOT TOO USEFUL SINCE $\vec{E}$, $\vec{H}$, ω AREN'T KNOWN. WE'LL DERIVE A MORE USEFUL EXPRESSION IF WE ASSUME $\Delta S/S \ll 1$.

WITH $\Delta S/S$ SMALL AND $\vec{E}, \vec{H}$ ARE THEREFORE CLOSE TO $\vec{E}_0, \vec{H}_0$, THEN THE NUMERATOR OF THE FREQUENCY SHIFT IS

$$\oiint_{\Delta S} \vec{E}_0^* \times \vec{H} \cdot \hat{n} \, d\Delta S$$

$$\approx \oiint_{\Delta S} \vec{E}_0^* \times \vec{H}_0 \cdot \hat{n} \, d\Delta S$$

THIS LAST TERM CAN BE BROUGHT INTO A VOLUME INTEGRAL

$$\oiint_{\Delta S} \vec{E}_0^* \times \vec{H}_0 \cdot \hat{n} \, d\Delta S = \iiint_{\Delta V} \vec{\nabla} \cdot \left(\vec{E}_0^* \times \vec{H}_0 \right) dV$$

USING THE SAME VECTOR IDENTITY AS BEFORE

$$\oiint_{\Delta S} \vec{E}_0^* \times \vec{H}_0 \cdot \hat{n} \, d\Delta S$$

$$= \iiint_{\Delta V} \vec{H}_0 \cdot (\vec{\nabla} \times \vec{E}_0^*) \, dV - \iiint_{\Delta V} \vec{E}_0^* \cdot (\vec{\nabla} \times \vec{H}_0) \, dV$$

$$= -i\omega_0 \iiint_{\Delta V} \left(\varepsilon |\vec{E}_0|^2 - \mu |\vec{H}_0|^2 \right) dV$$

(WHERE WE SET $\sigma = 0$, $\vec{J} = 0$),

HENCE

$$\frac{\omega - \omega_0}{\omega_0} \approx \frac{\iiint_{\Delta V} \left(\mu |\vec{H}_0|^2 - \varepsilon |\vec{E}_0|^2 \right) dV}{\iiint_{V_0} \left(\mu |\vec{H}_0|^2 + \varepsilon |\vec{E}_0|^2 \right) dV}$$

THIS LAST EXPRESSION SAYS THE FRACTIONAL SHIFT IN FREQUENCY DUE TO A SMALL CHANGE IN CAVITY SHAPE IS GIVEN BY THE RATIO OF THE CHANGE IN THE DIFFERENCE OF STORED ELECTRIC AND MAGNETIC ENERGY TO THE TOTAL STORED ENERGY. THE INTEGRATION IN THE NUMERATOR IS OVER THE PERTURBED VOLUME, NOT THE ENTIRE VOLUME. THE TOTAL ENERGY IN THE DENOMINATOR ASSUMES THE PERTURBED TOTAL ENERGY IS THE SAME AS THE UNPERTURBED TOTAL ENERGY.

THIS IS A VERY USEFUL FORMULA