SPECIAL LECTURE: KRAMERS-KRONIG RELATIONS

THIS IS THE RELATION BETWEEN THE REAL AND IMAGINARY PARTS OF THE COMPLEX PERMITTIVITY; THAT IS, THE RELATION BETWEEN REFRACTION AND ATTENUATION. I'M USING KRAMERS'S ARGUMENT FROM H.A. KRAMERS, "COLLECTED SCIENTIFIC PAPERS," NORTH HOLLAND 1956.

SOME SYSTEM FEELS AN EXTERNAL E FIELD:

$$\vec{E}(t) = \int_{-\infty}^{\infty} \vec{E}_\omega e^{-i\omega t}\, d\omega, \quad \vec{E}(t) \text{ REAL.}$$

THIS INDUCES THE DIPOLE MOMENT

$$\vec{P}(t) = \frac{1}{N}\int_{-\infty}^{\infty} \varepsilon_0 \left[\frac{\varepsilon(\omega)}{\varepsilon_0} - 1\right] \vec{E}_\omega e^{-i\omega t}\, d\omega$$

WITH N THE DENSITY OF PARTICLES AND $\varepsilon(\omega)$ THE COMPLEX PERMITTIVITY.

FOR $\vec{P}(t)$ TO BE REAL (RELATIONS FOR LATER)

$$\varepsilon(-\omega) = \varepsilon^*(\omega),$$

THAT IS $\mathrm{Re}\,\varepsilon(-\omega) = \mathrm{Re}\,\varepsilon(\omega)$ AND

$$\mathrm{Im}\,\varepsilon(-\omega) = -\mathrm{Im}\,\varepsilon(\omega).$$

KRAMERS ADDED ANOTHER CONDITION: THE POLARIZATION $\vec{P}$ CANNOT BE ADVANCED OF THE $\vec{E}$-FIELD PRODUCING IT. KRAMERS ARGUED THAT $\varepsilon(\omega)$ HAVE NO SINGULARITIES

IN THE UPPER-HALF COMPLEX FREQUENCY PLANE. HE ARGUED THIS BY CONSIDERING A VERY NARROW PULSE IN TIME AT $t=0$, GIVING A FOURIER INVERSION,

$$\vec{E}_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(t) e^{+i\omega t} dt \quad \text{NEARLY CONSTANT.}$$

THE POLARIZATION CONSTRAINT IS THEN

$$\int_{-\infty}^{\infty} \left[\varepsilon(\omega)/\varepsilon_0 - 1\right] e^{-i\omega t} d\omega = 0 \quad \text{FOR } t < 0$$

RECALL THAT $\varepsilon(\omega) \to 1$ AS $\omega \to \infty$ IN OUR OSCILLATOR MODEL. THIS IS GENERALLY TRUE IN GENERAL WHERE THE DRIVING FREQUENCY ARE MUCH GREATER THAN ANY BINDING FREQUENCY.

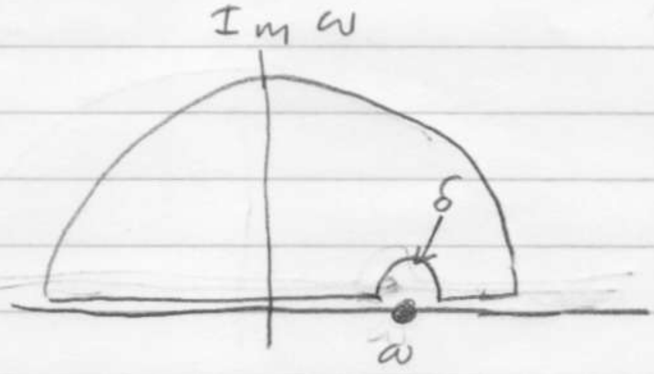
NOTICE THE ω-INTEGRAL ABOVE CAN BE EXTENDED TO THE INFINITE SEMICIRCLE IN THE UPPER-HALF PLANE, SINCE $e^{-i\omega t} = e^{-i\,\mathrm{Re}\,\omega t} e^{\mathrm{Im}\,\omega t}$ VANISHES FOR t NEGATIVE AND $|\omega| \to \infty$. (THIS IS KRAMER'S CONVENTION FOR THE FOURIER AND INVERSE-FOURIER TRANSFORM. JACKSON'S CONVENTION CLOSES THE INTEGRATION IN THE LOWER-HALF PLANE.).

SINCE THIS IS TRUE FOR ALL $t<0$, THE SUMMED RESIDUES OF $[\mathrm{Re}\,\varepsilon + i\,\mathrm{Im}\,\varepsilon - 1]\,e^{-i\omega t}$ IN THE UPPER-HALF PLANE VANISHES.
A SUBTLE INFERENCE IS THAT $\mathrm{Re}\,\varepsilon + i\,\mathrm{Im}\,\varepsilon - 1$ IS ANALYTIC IN THE UPPER-HALF PLANE. (THIS IS FIRMED-UP IN JAUCH & ROHRLICH, "THEORY OF PHOTONS & ELECTRONS", APPENDIX A7.)

LET'S CHOOSE ONE, REAL ω. CAUCHY'S THEOREM READS

$$\oint \frac{\mathrm{Re}\,\varepsilon + i\,\mathrm{Im}\,\varepsilon - 1}{\omega' - \omega}\, d\omega' = 0$$

SO LONG AS THE CONTOUR EXCLUDES THE POINT $\omega = \omega'$. WE'LL USE THIS CONTOUR:



THIS GIVES

$$\int_{-\infty}^{\infty} \frac{\varepsilon(\omega') - 1}{\omega' - \omega}\, d\omega' = 0$$

WHERE THE INTEGRAL IS ALONG THE REAL AXIS EXCEPT FOR A SMALL

SEMICIRCLE OF RADIUS δ. THE INTEGRAL IS THEN

$$\int_{-\infty}^{0} \frac{\varepsilon(\omega')-1}{\omega'-\omega}\,d\omega' + \int_{0}^{\infty} \frac{\varepsilon(\omega')-1}{\omega'-\omega}\,d\omega' + \int_{C} \frac{\varepsilon(\omega')-1}{\omega'-\omega}\,d\omega'$$

WHERE THE INTEGRAL FROM 0 TO ∞ EXCLUDES THE SEMI-CIRCLE, AND THE LAST INTEGRATION IS AROUND THE SEMI-CIRCLE.

FROM THE PAGE 1 - RELATIONS AND $\delta \to 0$, THE ABOVE INTEGRALS BECOME

$$\int_{0}^{\infty} \frac{\mathrm{Re}\,\varepsilon(\omega') - i\,\mathrm{Im}\,\varepsilon(\omega') - 1}{-\omega' - \omega}\,d\omega'$$

$$+ \int_{0}^{\infty} \frac{\mathrm{Re}\,\varepsilon(\omega') + i\,\mathrm{Im}\,\varepsilon(\omega') - 1}{\omega' - \omega}\,d\omega'$$

$$+ \left[\mathrm{Re}\,\varepsilon(\omega) + i\,\mathrm{Im}\,\varepsilon(\omega) - 1\right] \int_{C} \frac{d\omega'}{\omega' - \omega}$$

$$= 2\int_{0}^{\infty} \frac{\omega\left[\mathrm{Re}\,\varepsilon(\omega') - 1\right] + i\omega'\,\mathrm{Im}\,\varepsilon(\omega')}{\omega'^2 - \omega^2}\,d\omega'$$

$$-\pi i\left[\mathrm{Re}\,\varepsilon(\omega) + i\,\mathrm{Im}\,\varepsilon(\omega) - 1\right] = 0$$

WHERE WE USED $\int \frac{d\omega'}{\omega'-\omega} = -\pi i$ FROM CAUCHY'S THEOREM.

BOTH REAL AND IMAGINARY PARTS OF THE EXPRESSION AT THE BOTTOM OF PAGE 4 VANISH:

$$\mathrm{Re}\,\varepsilon(\omega) - 1 = \frac{2}{\pi} \int_0^\infty \frac{\omega' \,\mathrm{Im}\,\varepsilon(\omega')}{\omega'^2 - \omega^2} \, d\omega'$$

$$\mathrm{Im}\,\varepsilon(\omega) = -\frac{2\omega}{\pi} \int_0^\infty \frac{\mathrm{Re}\,\varepsilon(\omega') - 1}{\omega'^2 - \omega^2} \, d\omega'$$

THESE ARE KNOWN AS DISPERSION RELATIONS (OR KRAMERS - KRONIG RELATIONS). SEE JACKSON EQNS 7.119.